Towards $SL_3(\mathbb{C})$ Spherical Functions

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In hopes of deriving the expression for $SL_3(\mathbb{C})$ spherical functions by hand, I did a few backgound computations: Haar measure in Cartan coordinates, Casimir on principal series, Casimir on bi-K-invariant functions, a PDE for the spherical function. Very rough. Perhaps the principal value of the explicit computations is in providing motivation for a more structural viewpoint . . .

0. Premliminaries

The Cartan decomposition of $GL_3(\mathbb{C})$ is $G = KAK$ with $K = U(3)$ the maximal compact subgroup and A the standard maximal split torus, diagonal matrices with nonzero real entries. (In fact we can write $G = KA^+K$ where $A^+ = \exp(\mathfrak{a}_+$.) We parametrize A by a multi-radius $r = (r_1, r_2, r_3) \in \mathbb{R}^3$ by

$$a_r = \begin{pmatrix} e^{r_1} & & \\ & e^{r_2} & \\ & & e^{r_3} \end{pmatrix}$$

Certainly the Lie algebra $\mathfrak{a} \approx \mathbb{R}^3$ consists of

$$\begin{pmatrix} r_1 & & \\ & r_2 & \\ & & r_3 \end{pmatrix}$$

The Lie algebra $\mathfrak{gl}_3(\mathbb{C})$ is the complexification of $\mathfrak{gl}_3(\mathbb{R})$ and of $\mathfrak{su}(3)$, so

$$\mathfrak{gl}_3(\mathbb{C}) \approx \mathfrak{gl}_3(\mathbb{R}) \oplus i \cdot \mathfrak{gl}_3(\mathbb{R}) \approx \mathfrak{su}(3) \oplus i \cdot \mathfrak{su}(3)$$

We will often work things out for $\mathfrak{gl}_3(\mathbb{R})$ and then extrapolate for $\mathfrak{gl}_3(\mathbb{C})$. We also exploit the fact that there are three copies of $\mathfrak{gl}_2$ inside $\mathfrak{gl}_3$. Consider the following elements

$$h_\alpha = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \quad h_\beta = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad h_{\alpha+\beta} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

Together with $z = 1_{3\times 3}$ these span $\mathfrak{a}$ where $\exp \mathfrak{a} = A^+$. (Maybe these plus ih_α, ih_β, and $ih_{\alpha+\beta}$ span $\mathfrak{h}$ the Cartan subalgebra, with $\exp \mathfrak{h} = H$ the (connected component of the identity of) the Cartan subgroup (diagonal elements)?) Notice that $h_{\alpha+\beta} = h_\alpha + h_\beta$ so these do not form a *basis*, but this way of writing things is convenient because it allows us to identify elements of $\mathfrak{h}$ with the positive roots. Consider

$$x_\alpha = \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad x_\beta = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{pmatrix}, \quad x_{\alpha+\beta} = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ & & 0 \end{pmatrix}$$

These form a basis for $\mathfrak{n}_+$ where $\exp \mathfrak{n}_+ = N$ the unipotent radical of the standard minimal parabolic (upper triangular matrices). Consider

$$y_\alpha = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, \quad y_\beta = \begin{pmatrix} 0 & & \\ & 0 & \\ & 1 & 0 \end{pmatrix}, \quad y_{\alpha+\beta} = \begin{pmatrix} 0 & & \\ & 0 & \\ 1 & & 0 \end{pmatrix}$$

These form a basis for $\mathfrak{n}_-$ where $\exp \mathfrak{n}_- = \overline{N}$, ($N$, flipped over the diagonal). Notice that h_α, x_α, and y_α generate a copy of $\mathfrak{sl}_2(\mathbb{R})$ as do the β and $(\alpha + \beta)$ analogues.

We may prefer different coordinates on the Lie algebra. Consider

$$\sigma_\alpha = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, \quad \sigma_\beta = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}, \quad \sigma_{\alpha+\beta} = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ 1 & & 0 \end{pmatrix}$$

and

$$\theta_\alpha = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{pmatrix}, \quad \theta_\beta = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix}, \quad \theta_{\alpha+\beta} = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ -1 & & 0 \end{pmatrix}$$

Then the h's, σ's, and θ's give a Cartan decomposition (roughly). For $\mathfrak{sl}_2(\mathbb{R})$,

$$\mathfrak{p} = \mathbb{R}\cdot\sigma + \mathbb{R}\cdot h \qquad \text{and} \qquad \mathbb{R}\cdot\theta = \mathfrak{k}$$

1. Haar measure in Cartan coordinates

For $GL_3(\mathbb{R})$ (perhaps modulo the center) consider

$$\int_G f(g)\,dg = \int_K \int_A \int_K f(kak')\,\Phi(k,a,k')\,dk\,da\,dk'$$

What is the change of measure Φ? It is actually not dependent on k, k', because G and K are unimodular. We determine $\Phi(a)$ by computing a Jacobian.

Note that $\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{k}$ is *not* $\mathfrak{gl}_3$. But if we act on $\mathfrak{k}$ by $\mathrm{Ad}(a)$ for some $a \in A^+$ that moves things around sufficiently to span the whole Lie algebra. So to compute the Haar measure in Cartan coordinates we would like to find the Jacobian of the map

$$\mathfrak{k}_a + \mathfrak{a} + \mathfrak{k} \longrightarrow \mathfrak{k}_{a'} + \mathfrak{a} + \mathfrak{k}$$

This is a map from the tangent space at a to the tangent space at a', i.e. a change of coordinates for $\mathfrak{gl}_3$. To do this, we will be slightly clever and instead use

$$\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{k} \longrightarrow \mathfrak{k}_a + \mathfrak{a} + \mathfrak{k}$$

The θ's span $\mathfrak{k} = \mathfrak{so}(3)$.

For a fixed $a_r \in A^+$, consider the action of $\mathrm{Ad}(a_r)$ on $\mathfrak{k}$.

$$\begin{aligned}
\mathrm{Ad}(a_r)\,\theta_\alpha &= \begin{pmatrix} 0 & e^{-(r_1-r_2)} & \\ -e^{(r_1-r_2)} & 0 & \\ & & 0 \end{pmatrix} \\
\mathrm{Ad}(a_r)\,\theta_\beta &= \begin{pmatrix} 0 & & \\ & 0 & e^{-(r_2-r_3)} \\ & -e^{(r_2-r_3)} & 0 \end{pmatrix} \\
\mathrm{Ad}(a_r)\,\theta_{\alpha+\beta} &= \begin{pmatrix} 0 & & e^{-(r_1-r_3)} \\ & 0 & \\ -e^{(r_1-r_3)} & & 0 \end{pmatrix}
\end{aligned}$$

We put these in coordinates.

$$\begin{aligned}
\mathrm{Ad}(a_r)\,\theta_\alpha &= \sinh(r_1-r_2)\,\sigma_\alpha - \cosh(r_1-r_2)\,\theta_\alpha \\
\mathrm{Ad}(a_r)\,\theta_\beta &= \sinh(r_2-r_3)\,\sigma_\beta - \cosh(r_2-r_3)\,\theta_\beta \\
\mathrm{Ad}(a_r)\,\theta_{\alpha+\beta} &= \sinh(r_1-r_3)\,\sigma_{\alpha+\beta} - \cosh(r_1-r_3)\,\theta_{\alpha+\beta}
\end{aligned}$$

So the map is

$$\begin{aligned}
a\cdot\theta_\alpha \,\oplus\, b\cdot h_\alpha \,\oplus\, c\cdot\theta_\alpha &\longrightarrow a\sinh(r_1-r_2)\cdot\sigma_\alpha \,\oplus\, b\cdot h_\alpha \,\oplus\, (c - a\cosh(r_1-r_2))\cdot\theta_\alpha \\
a'\cdot\theta_\beta \,\oplus\, b'\cdot h_\beta \,\oplus\, c'\cdot\theta_\beta &\longrightarrow a'\sinh(r_2-r_3)\cdot\sigma_\beta \,\oplus\, b'\cdot h_\beta \,\oplus\, (c' - a'\cosh(r_2-r_3))\cdot\theta_\beta \\
a''\cdot\theta_{\alpha+\beta} \,\oplus\, b''\cdot h_{\alpha+\beta} \,\oplus\, c''\cdot\theta_{\alpha+\beta} &\longrightarrow a''\sinh(r_1-r_3)\cdot\sigma_{\alpha+\beta} \,\oplus\, b''\cdot h_{\alpha+\beta} \\
&\qquad \oplus\, (c'' - a''\cosh(r_1-r_3))\cdot\theta_{\alpha+\beta}
\end{aligned}$$

So the (9×9) Jacobian has three 3×3 blocks on the diagonal, each of which is just like the Jacobian for the $\mathfrak{gl}_2$ case. For example the first block is

$$\begin{pmatrix} \sinh(r_1 - r_2) & & \\ & 1 & \\ \cosh(r_1 - r_2) & & 1 \end{pmatrix}$$

The determinant of the 9×9 Jacobian is the product of the determinants of the 3×3 blocks, so

$$\Phi(a) = |\sinh(r_1 - r_2) \cdot \sinh(r_2 - r_3) \cdot \sinh(r_1 - r_3)|$$

For $GL_3(\mathbb{C})$,

$$\Phi(a) = \sinh^2(r_1 - r_2) \cdot \sinh^2(r_2 - r_3) \cdot \sinh^2(r_1 - r_3)$$

so the Haar measure in Cartan coordinates is

$$dg = dk \cdot \sinh^2(r_1 - r_2) \cdot \sinh^2(r_2 - r_3) \cdot \sinh^2(r_1 - r_3)\, dr \cdot dk'$$

2. The Casimir Operator

For any basis $\{x_i\}$ of $\mathfrak{g}$, the Casimir operator can be written in coordinates as

$$\Omega = \sum_i x_i\, x_i^*$$

We consider the restriction of Casimir to right K-invariant functions, i.e. the Laplace-Beltrami operator Δ on the symmetric space $X = G/K$. Then, for any basis $\{x_i\}$ of $\mathfrak{p}$,

$$\Delta = \sum_i x_i x_i^*$$

since $\mathfrak{k}$ acts by zero.

Since Casimir preserves left P-equivariance and right K-invariance, it acts on principal series. On principal series, $\mathfrak{n}_+$ acts on the left by zero, and since we can use the bracket relations to rewrite the action of $\mathfrak{n}_-$ in terms of the action of $\mathfrak{n}_+$ and $\mathfrak{h}$, the action of Casimir is completely determined by the action of $\mathfrak{h}$.

Since Casimir preserves bi-K-invariance, it acts on the space of bi-K-invariant functions. On bi-K-invariant functions, $\mathfrak{k}$ acts on the left and right by zero, $\mathfrak{k}_r$ acts on the right by zero. Again using commutator relations, we can see that the action of Casimir is determined by the action of $\mathfrak{h}$.

2.1 Casimir on Principal Series

Use the Iwasawa coordinates $\mathfrak{h}$ (h's), $\mathfrak{n}_+$ (x's), $\mathfrak{n}_-$ (y's). We need to compute a dual basis with respect to the trace pairing,

$$\langle \gamma, \gamma' \rangle = \mathrm{Re}\left(\mathrm{tr}(\gamma\gamma')\right)$$

Unfortunately I don't know a faster way for computing the dual basis than to solve the equations given by the condition

$$\langle x_j, x_i^* \rangle = \delta_{i,j} \quad \text{(Kronecker delta)}$$

One might hope that, by analogy with the GL_2 case, for $\gamma = \alpha, \beta$, $h_\gamma^* = \frac{1}{2} h_\gamma$ but this is not true. To assume that h_γ^* is a multiple of h_γ implicitly assumes the orthogonality of the roots, which we certainly not *not* have. In fact,

$$h_\alpha^* = \tfrac{1}{3}\begin{pmatrix} 2 & & \\ & 1 & \\ & & -1 \end{pmatrix} \qquad h_\beta^* = \tfrac{1}{3}\begin{pmatrix} -1 & & \\ & 2 & \\ & & 1 \end{pmatrix}$$

Fortunately, we can compute the action of Casimir on principal series without using the exact expression for h_γ^*, and as in the GL_2 case, we do have $x_\gamma^* = y_\gamma$ and $y_\gamma^* = x_\gamma$.

$$\begin{aligned} \Omega \;=\;& h_\alpha h_\alpha^* + h_\beta h_\beta^* \\ &+ x_\alpha\, y_\alpha + y_\alpha\, x_\alpha \\ &+ x_\beta\, y_\beta + y_\beta\, x_\beta \\ &+ x_{\alpha+\beta}\, y_{\alpha+\beta} + y_{\alpha+\beta}\, x_{\alpha+\beta} \end{aligned}$$

Using the commutator relations,

$$x_\gamma\, y_\gamma = [x_\gamma, y_\gamma] + y_\gamma\, x_\gamma = h_\gamma + y_\gamma\, x_\gamma$$

for $\gamma = \alpha, \beta, \alpha+\beta$. So we can rewrite Casimir

$$\begin{aligned} \Omega \;&=\; h_\alpha h_\alpha^* + h_\beta h_\beta^* \\ &\quad + h_\alpha + 2 y_\alpha\, x_\alpha \\ &\quad + h_\beta + 2 y_\beta\, x_\beta \\ &\quad + h_{\alpha+\beta} + 2 y_{\alpha+\beta}\, x_{\alpha+\beta} \end{aligned}$$

Since, on the left, x_α, x_β, and $x_{\alpha+\beta}$ act by zero on principal series,

$$\Omega = h_\alpha h_\alpha^* + h_\beta h_\beta^* + h_\alpha + h_\beta + h_{\alpha+\beta} \quad \text{(on left, on principal series)}$$

So we need to compute the action of h_α, h_β, and $h_{\alpha+\beta}$ on principal series. For a left N-invariant character χ on $P = P^{\min}$, the χ^{th} prinicpal series I_χ is the induced representation $\operatorname{Ind}_P^G \chi$, modelled by functions on G that are left P-equivariant by a character χ. We can write χ as $\chi = e^\mu$ for some $\mu \in \mathfrak{a}_{\mathbb{C}}^*$ (and extend by left N-invariance to a character on P). So take f in I_χ and compute the action of $\mathfrak{h}$.

$$\begin{aligned} h_\gamma f(g) \;&=\; \frac{d}{dt}\bigg|_{t=0} f(\exp(h_\gamma t)\cdot g) \\ &=\; \frac{d}{dt}\bigg|_{t=0} \chi(\exp(h_\gamma t))\cdot f(g) \\ &=\; \frac{d}{dt}\bigg|_{t=0} -\langle \mu, \gamma\rangle \cdot f(g) \end{aligned}$$

Similarly, $h_\gamma^* f(g) = -\langle \mu, \gamma^*\rangle \cdot f(g)$, and for general reasons,

$$\sum_\gamma \langle \mu, \gamma\rangle \langle \mu, \gamma^*\rangle = \langle \mu, \mu\rangle$$

So Casimir acts on principal series by the scalar

$$\Omega \;=\; \langle \mu,\mu\rangle - \langle \mu,\alpha\rangle - \langle \mu,\beta\rangle - \langle \mu,\alpha+\beta\rangle \;=\; \langle \mu,\mu\rangle - 2\langle \mu, \alpha+\beta\rangle$$

If we now renormalize by $\rho = \alpha+\beta$ (half the sum of the positive roots), $\mu = \rho + i\lambda$, the action of Ω is by

$$\begin{aligned} \Omega \;&=\; \langle \rho+i\lambda, \rho+i\lambda\rangle - 2\langle \rho+i\lambda, \rho\rangle \\ &=\; \langle \rho,\rho\rangle + 2\langle \rho, i\lambda\rangle - \langle \lambda,\lambda\rangle \;-\; 2\langle \rho,\rho\rangle - 2i\langle \lambda,\rho\rangle \\ &=\; -\langle \lambda,\lambda\rangle - \langle \rho,\rho\rangle \end{aligned}$$

Note. This argument works for $GL_n(\mathbb{R})$ as well.

2.2 Casimir on bi-K-invariant functions

It seems to work out nicer to choose the following basis for $\mathfrak{h}$:

$$h_1 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

Note. The reason things work out nicely is that h_i acts on bi-K-invariant f by $\frac{\partial}{\partial r_i}$, whereas using the basis that corresponds to the roots yields directional derivatives. However this might not be a good enough reason to choose this basis.

Then

$$h_1^* = h_1 \qquad h_2^* = h_2 \qquad h_3^* = h_3$$

Each $\sigma_\gamma^* = \frac{1}{2}\sigma_\gamma$ and each $\theta_\gamma^* = -\frac{1}{2}\theta_\gamma$, so Casimir is

$$\Omega = h_1^2 + h_2^2 + h_3^2 + \tfrac{1}{2}\sigma_\alpha^2 + \tfrac{1}{2}\sigma_\beta^2 + \tfrac{1}{2}\sigma_{\alpha+\beta}^2 - \tfrac{1}{2}\theta_\alpha^2 - \tfrac{1}{2}\theta_\beta^2 - \tfrac{1}{2}\theta_{\alpha+\beta}^2$$

On bi-K-invariant functions, each θ_γ acts by zero (on the left and the right.) We rewrite each σ_γ in terms of θ_γ and $\mathrm{Ad}(r)\theta_\gamma$ (as previously, when we wrote $\mathrm{Ad}(a_r)\theta$ in terms of σ and θ.)

$$\begin{aligned}
\sigma_\alpha &= \frac{1}{\sinh(r_1-r_2)}\cdot \mathrm{Ad}(a_r)\theta_\alpha - \coth(r_1-r_2)\cdot\theta_\alpha \\
\sigma_\beta &= \frac{1}{\sinh(r_2-r_3)}\cdot \mathrm{Ad}(a_r)\theta_\beta - \coth(r_2-r_3)\cdot\theta_\beta \\
\sigma_{\alpha+\beta} &= \frac{1}{\sinh(r_1-r_3)}\cdot \mathrm{Ad}(a_r)\theta_{\alpha+\beta} - \coth(r_1-r_3)\cdot\theta_{\alpha+\beta}
\end{aligned}$$

Note that $r_1-r_2=\alpha(\log a_r)$, $r_2-r_3=\beta(\log a_r)$, and $r_1-r_3=(\alpha+\beta)(\log a_r)$, so we could summarize the above by writing

$$\sigma_\gamma = \frac{1}{\sinh(\gamma(\log a_r))}\cdot\theta_{\gamma,r} - \coth(\gamma(\log a_r))\cdot\theta_\gamma$$

Dropping the subscript γ for the moment, the σ^2's that occur in Casimir are

$$\sigma^2 = \frac{1}{\mathrm{sh}^2}\cdot\theta_r^2 - \frac{\mathrm{cth}}{\mathrm{sh}}\cdot\theta_r\,\theta - \frac{\mathrm{cth}}{\mathrm{sh}}\cdot\theta\,\theta_r + \mathrm{cth}^2\cdot\theta^2$$

Use the commutator relations

$$[\theta_\gamma,\theta_r] = -2\sinh(\gamma(\log a_r))\cdot h_\gamma$$

to write

$$\theta\,\theta_r = -2\sinh(\gamma(\log a_r))\cdot h_\gamma + \theta_r\,\theta$$

So, on bi-K-invariant functions

$$\sigma_\gamma^2 = 2\coth(\gamma(\log a_r))\cdot h_\gamma = 2\coth(r_i-r_j)\cdot(h_i-h_j)$$

and Casimir is

$$\begin{aligned}
\Omega &= h_1^2 + \big(\coth(r_1-r_2)+\coth(r_1-r_3)\big)h_1 \\
&\quad + h_2^2 + \big(\coth(r_2-r_3)-\coth(r_1-r_2)\big)h_2 \\
&\quad + h_3^2 + \big(-\coth(r_2-r_3)-\coth(r_1-r_3)\big)h_3
\end{aligned}$$

For $GL_3(\mathbb{C})$,

$$\begin{aligned}
\Omega &= h_1^2 + 2\big(\coth(r_1-r_2)+\coth(r_1-r_3)\big)h_1 \\
&\quad + h_2^2 + 2\big(\coth(r_2-r_3)-\coth(r_1-r_2)\big)h_2 \\
&\quad + h_3^2 + 2\big(-\coth(r_2-r_3)-\coth(r_1-r_3)\big)h_3
\end{aligned}$$

Note. Give at least a short explanation of how to get from $GL_3(\mathbb{R})$ to $GL_3(\mathbb{C})$.

The action of h_i on a bi-K-invariant function is just $\frac{\partial}{\partial r_i}$. So if we write a bi-$K$-invariant function f as

$$f=\frac{\varphi}{g}\qquad\text{with } g=\sinh(r_1-r_2)\sinh(r_2-r_3)\sinh(r_1-r_3)$$

Then taking derivatives we see

$$\left(\frac{\varphi}{g}\right)' = \frac{\varphi'}{g} - \frac{g'}{g}\cdot\frac{\varphi}{g}$$

and

$$\left(\frac{\varphi}{g}\right)'' = \frac{\varphi''}{g} - 2\,\frac{g'}{g}\cdot\frac{\varphi'}{g} + \left(2\,\frac{g'}{g}\cdot\frac{g'}{g} - \frac{g''}{g}\right)\cdot\frac{\varphi}{g}$$

For the moment $\rho_1=r_1-r_2$, $\rho_2=r_2-r_3$, and $\rho_3=r_1-r_3$. (These ρ's have no connection to ρ =half the sum of positive roots.) Taking derivatives with respect to r_1,

$$\frac{g'}{g} = \frac{\mathrm{ch}(\rho_1)}{\mathrm{sh}(\rho_1)} + \frac{\mathrm{ch}(\rho_3)}{\mathrm{sh}(\rho_3)}$$

$$\frac{g''}{g} = 2\left(1 + \frac{\mathrm{ch}(\rho_1)}{\mathrm{sh}(\rho_1)} \cdot \frac{\mathrm{ch}(\rho_3)}{\mathrm{sh}(\rho_3)}\right)$$

Taking derivatives with respect to r_2,

$$\frac{g'}{g} = -\frac{\mathrm{ch}(\rho_1)}{\mathrm{sh}(\rho_1)} + \frac{\mathrm{ch}(\rho_2)}{\mathrm{sh}(\rho_2)}$$

$$\frac{g''}{g} = 2\left(1 - \frac{\mathrm{ch}(\rho_1)}{\mathrm{sh}(\rho_1)} \cdot \frac{\mathrm{ch}(\rho_2)}{\mathrm{sh}(\rho_2)}\right)$$

Taking derivatives with respect to r_3,

$$\frac{g'}{g} = -\frac{\mathrm{ch}(\rho_2)}{\mathrm{sh}(\rho_2)} - \frac{\mathrm{ch}(\rho_3)}{\mathrm{sh}(\rho_3)}$$

$$\frac{g''}{g} = 2\left(1 + \frac{\mathrm{ch}(\rho_2)}{\mathrm{sh}(\rho_2)} \cdot \frac{\mathrm{ch}(\rho_3)}{\mathrm{sh}(\rho_3)}\right)$$

Notice that the g'/g exactly matches the $\pm\coth\pm\coth$ factors that occured in the differential operator. So looking at the action of Casimir one component at a time,

$$\left(\frac{\varphi}{g}\right)'' + 2\,\frac{g'}{g} \cdot \left(\frac{\varphi}{g}\right)' = \ldots = \frac{\varphi''}{g} - \frac{g''}{g} \cdot \frac{\varphi}{g}$$

Putting them all together,

$$\Omega f = \frac{\Delta\varphi}{g} - \frac{\Delta g}{g}\,\frac{\varphi}{g}$$

where $\Delta = \sum \frac{\partial^2}{\partial r_i^2}$. Careful computation shows that $\frac{\Delta g}{g} = 8$ So we have,

$$\Omega\,\frac{\varphi}{g} = \frac{\Delta\varphi}{g} - 8\frac{\varphi}{g}$$

In particular, this means that the equation

$$\Omega f = \lambda f$$

becomes

$$\Delta\varphi - 8\varphi = \lambda\varphi$$

which is a *constant coefficient* partial differential equation. (Note that the the r_1, r_2, r_3 are not independent of each other; they sum to zero.)

However, this PDE is not sufficient to determine the spherical functions uniquely, because we need the *whole* center of the universal enveloping algebra:

$$z\,f = \lambda(z) \cdot f \qquad \text{all } z \in \mathfrak{z}$$

In particular, in the case of GL_3, we need another operator, which is of order 4, together with Casimir, to generate the center of the enveloping algebra.