GENERALIZED SHALIKA MODELS OF p-ADIC SO_{4n} AND FUNCTORIALITY

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Abstract. Let F denote a p-adic local field of characteristic zero. In this paper, we investigate the structures of irreducible admissible representations of $\mathrm{SO}_{4n}(F)$ having nonzero generalized Shalika models and find relations between the generalized Shalika models and the local Arthur parameters, which support our conjectures on the local Arthur parametrization and the local Langlands functoriality in terms of the dual group associated to the spherical variety, which is attached to the generalized Shalika models.

1. Introduction

This paper is a sequel of our previous work on the characterization of symplectic representations of p-adic group GL_{2n} in terms of the generalized Shalika models on the F-split special orthogonal group SO_{4n} ([JQ07], [JNQ08], [JNQ10-1], and [JNQ10-2]), where F is a p-adic local field of characteristic zero. For simplicity of notation, we use G for an algebraic group and also for its F-rational points of G. Our objective here is to study the structure of irreducible admissible representations of SO_{4n}, which have a nonzero generalized Shalika model.

Date: June, 2010; revised October, 2011.

2000 *Mathematics Subject Classification.* Primary 11F70, 22E50; Secondary 11F85, 22E55.

Key words and phrases. Generalized Shalika models, Representations of p-adic groups, Arthur parameters, Functoriality.

The work of the first named author is supported in part by NSF (USA) grant DMS-0653742 and DMS-1001672, and by The Chinese Academy of Sciences. The second name author is supported by NSC 99-2115-M-006-007- and NCTS Taiwan. The third named author is supported partly by the program PCSIRT (Program for Changjiang Scholars and Innovative Research Team) in East China Normal University. All three authors are supported in part by NSFC 10701034, P.R.China. The authors would like to thank The Morningside Center of Mathematics and the Institute of Mathematics, CAS, for strong support through the summer program over the years.

Let $\nu_1 = 1$ and inductively define

$$\nu_n = \begin{pmatrix} & 1 \\ \nu_{n-1} & \end{pmatrix}, \text{ for } n \geq 2, n \in \mathbb{N}. \tag{1.1}$$

We often use abbreviation ν for ν_n if there is no confusion for the rank n. Let SO_{4n} be the even special orthogonal group attached to the non-degenerate $4n$-dimensional quadratic vector space over F with respect to ν_{4n}. That is,

$$\mathrm{SO}_{4n} = \{g \in \mathrm{GL}_{4n} | \, {}^t g \cdot \nu_{4n} \cdot g = \nu_{4n}\}.$$

We recall from [JQ07] the definition of the *generalized Shalika models* on SO_{4n}. Let $P_{2n} = M_{2n}V_{2n}$ be the Siegel parabolic subgroup of SO_{4n} consisting of elements of the following form:

$$(g, X) = \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} \begin{pmatrix} \mathrm{I}_n & X \\ & \mathrm{I}_n \end{pmatrix}, \tag{1.2}$$

where $g \in \mathrm{GL}_{2n}$ and $g^* = \nu_{2n} \, {}^t g^{-1} \nu_{2n}$, and X satisfies ${}^t X = -\nu_{2n} X \nu_{2n}$.

The *generalized Shalika subgroup* $\mathcal{H}_{2n}$ of SO_{4n} is the subgroup of P_{2n} consisting of elements (g, X) with $g \in \mathrm{Sp}_{2n}$. Here the symplectic group is given by

$$\mathrm{Sp}_{2n} = \{g \in \mathrm{GL}_{2n} | \, {}^t g \cdot J_{2n} \cdot g = J_{2n}\},$$

where J_{2n} is given by

$$J_{2n} = \begin{pmatrix} & \nu_n \\ -\nu_n & \end{pmatrix}, \ n \in \mathbb{N}.$$

Define a character $\psi_{\mathcal{H}}$ of $\mathcal{H}_{2n}$ (We write $\mathcal{H} = \mathcal{H}_{2n}$ if there is no confusion) by letting

$$\psi_{\mathcal{H}}((g, X)) = \psi(\mathrm{tr}(J_{2n} X \nu_{2n})) \tag{1.3}$$

$$= \psi(\mathrm{tr}(\begin{pmatrix} -\mathrm{I}_n & \\ & \mathrm{I}_n \end{pmatrix} X)) \tag{1.4}$$

where ψ is a nontrivial character of F.

The *generalized Shalika functional* or *$\psi_{\mathcal{H}}$-functional* of an irreducible admissible representation (σ, V_σ) of SO_{4n} is a nonzero functional in the following space

$$\mathrm{Hom}_{\mathcal{H}}(V_\sigma, \psi_{\mathcal{H}}) \cong \mathrm{Hom}_{\mathrm{SO}_{4n}}(V_\sigma, \mathrm{Ind}_{\mathcal{H}}^{\mathrm{SO}_{4n}}(\psi_{\mathcal{H}})).$$

Here for any closed subgroup H of G, and for any admissible representation τ of p-adic H, $\mathrm{Ind}_H^G(\tau)$ denotes the *normalized smooth induction*, and $\mathrm{ind}_H^G(\tau)$ denotes the *normalized compact induction*.

The local uniqueness of the generalized Shalika models is proved by Nien in [N10], that is, for any irreducible admissible representation (σ, V_σ) of SO_{4n}, the dimension of the space

$$\mathrm{Hom}_{\mathcal{H}}(V_\sigma, \psi_{\mathcal{H}})$$

is at most one. Let $\ell_{\psi_{\mathcal{H}}}$ be a nonzero $\psi_{\mathcal{H}}$*-functional* of an irreducible admissible representation (σ, V_σ) of SO_{4n}. For $v \in V_\sigma$, and $g \in \mathrm{SO}_{4n}$, we define

$$H_{\psi_{\mathcal{H}}}(g, v) := \ell_{\psi_{\mathcal{H}}}(\sigma(g)(v)),$$

which is a $\psi_{\mathcal{H}}$-generalized Shalika function on SO_{4n} attached to v. The space consisting of all $\psi_{\mathcal{H}}$-generalized Shalika functions with $v \in V_\sigma$ is called the $\psi_{\mathcal{H}}$*-generalized Shalika model* of σ or just the $\psi_{\mathcal{H}}$*-model* of σ.

One of the key local results of [JQ07] is that the Shalika models (which is recalled below) and the generalized Shalika models are intrinsically related through the parabolic induction.

For an irreducible, unitary, supercuspidal representation (τ, V_τ) of GL_{2n}, we consider the following unitarily induced representation of SO_{4n}

$$\mathrm{I}^{\mathrm{SO}_{4n}}(s, \tau) = \mathrm{Ind}_{P_{2n}}^{\mathrm{SO}_{4n}}(|\det|^{\frac{s}{2}} \cdot \tau)$$

which consisting of all smooth V_τ-valued functions $\phi_{\tau,s}$ on SO_{4n}, such that

$$\phi_{\tau,s}(m(a)ng) = |\det a|^{\frac{s}{2}+\frac{2n-1}{2}} \tau(a)\phi_{\tau,s}(g),$$

where $m(a) \in M_{2n}$ with $a \in \mathrm{GL}_{2n}$, $n \in V_{2n}$.

Theorem 1.1 (Theorem 3.1, [JQ07])**.** *The unitarily induced representation* $\mathrm{I}^{\mathrm{SO}_{4n}}(s, \tau)$ *admits a nonzero generalized Shalika functionals only when* $s = 1$*. In that case,* $\mathrm{I}^{\mathrm{SO}_{4n}}(1, \tau)$ *admits a nonzero generalized Shalika functional if and only if the supercuspidal datum* τ *admits a nonzero Shalika functional. The generalized Shalika functionals of* $\mathrm{I}^{\mathrm{SO}_{4n}}(1, \tau)$ *is unique up to scalar, and if nonzero, they must factor through the unique Langlands quotient* $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$*.*

We recall from [JS90] the definition of the Shalika models for GL_{2n}. Take the maximal parabolic subgroup $\mathrm{P}_{n,n} = \mathrm{M}_{n,n}\mathrm{N}_{n,n}$ of GL_{2n} with

$$\mathrm{M}_{n,n} = \mathrm{GL}_n \times \mathrm{GL}_n,$$

and

$$\mathrm{N}_{n,n} = \{n(X) = \begin{pmatrix} \mathrm{I}_n & X \\ 0 & \mathrm{I}_n \end{pmatrix} \in \mathrm{GL}_{2n}\}.$$

Define a character

$$\psi_{\mathrm{N}_{n,n}}(n(X)) = \psi(\mathrm{tr}(X)).$$

The stabilizer of $\psi_{\mathrm{N}_{n,n}}$ in $\mathrm{M}_{n,n}$ is GL_n^{Δ}, the diagonal embedding of GL_n into $\mathrm{M}_{n,n}$. The Shalika subgroup $\mathcal{S}_n$ is defined to be

$$\mathcal{S}_n = \{s(a,X) = \mathrm{diag}(a,a)n(X) \mid a \in \mathrm{GL}_n, n(X) \in \mathrm{N}_{n,n}\}. \tag{1.5}$$

It is clear that $\mathcal{S}_n = \mathrm{GL}_n^{\Delta} \ltimes \mathrm{N}_{n,n}$, a semi-direct product with the radical $\mathrm{N}_{n,n}$ being a normal subgroup. Denote by $\psi_{\mathcal{S}_n}$ the extension of $\psi_{\mathrm{N}_{n,n}}$ from $\mathrm{N}_{n,n}$ to the Shalika subgroup $\mathcal{S}_n$, such that $\psi_{\mathcal{S}_n}$ is trivial on GL_n^{Δ}.

The Shalika functionals of an irreducible admissible representation (τ, V_τ) of GL_{2n} are nonzero elements of the following space

$$\mathrm{Hom}_{\mathcal{S}_n}(V_\tau, \psi_{\mathcal{S}_n}) \cong \mathrm{Hom}_{\mathrm{GL}_{2n}}(V_\tau, \mathrm{Ind}_{\mathcal{S}_n}^{\mathrm{GL}_{2n}}(\psi_{\mathcal{S}_n})).$$

Any nonzero Shalika functional ℓ_ψ in $\mathrm{Hom}_{\mathcal{S}_n}(V_\tau, \psi_{\mathcal{S}_n})$ gives rise to an embedding of V_τ into the full induction $\mathrm{Ind}_{\mathcal{S}_n}^{\mathrm{GL}_{2n}}(\psi_{\mathcal{S}_n})$, the image of which is called a local Shalika model of V_τ. The local uniqueness of Shalika models was first proved by Jacquet-Rallis in [JR96] and then by Nien in [N09-2] by using a different argument.

The first result of this paper is the following theorem:

Theorem 1.2. *Let (σ, V_σ) be an irreducible admissible representation of SO_{4n}. Assume that V_σ has a nonzero generalized Shalika model (or $\psi_{\mathcal{H}}$-model). Then there exists an irreducible admissible representation τ of GL_{2n} such that V_σ is a quotient of the following induced representation*

$$\mathrm{Ind}_{P_{2n}}^{\mathrm{SO}_{4n}}(\tau|\det|^{\frac{1}{2}}).$$

This theorem will be proved in Section 2. According to Theorem 1.2, we have

$$\mathrm{Ind}_{P_{2n}}^{\mathrm{SO}_{4n}}(\tau|\det|^{\frac{1}{2}}) \to V_\sigma \to 0. \tag{1.6}$$

The generalized Shalika functional on V_σ produces a generalized Shalika functional on the induced representation $\mathrm{Ind}_{P_{2n}}^{\mathrm{SO}_{4n}}(\tau|\det|^{\frac{1}{2}})$. By comparing Theorem 1.2 to Theorem 1.1, we expect to produce a certain model for the irreducible admissible representation τ of GL_{2n} if V_σ has a nonzero generalized Shalika model. To this end, we introduce the following new family of models.

For an integer $0 \le r \le n$, in the standard maximal parabolic subgroup

$$\mathrm{P}_{2r,2n-2r} = \mathrm{M}_{2r,2n-2r}\mathrm{N}_{2r,2n-2r} = (\mathrm{GL}_{2r} \times \mathrm{GL}_{2n-2r})\mathrm{N}_{2r,2n-2r}$$

of GL_{2n}, we define a subgroup $\mathcal{N}_r$ of GL_{2n} by

$$\mathcal{N}_r := (\mathcal{S}_r \times \mathrm{Sp}_{2n-2r})\mathrm{N}_{2r,2n-2r}, \tag{1.7}$$

where $\mathcal{S}_r$ is the Shalika subgroup of GL_{2r} as defined in (1.5), and the symplectic group Sp_{2n-2r} is embedded into GL_{2n-2r} naturally. We denote by $n(s,h,x)$ the elements of $\mathcal{N}_r$ with $s = s(a,X) \in \mathcal{S}_r$, $h \in \mathrm{Sp}_{2n-2r}$ and $x \in \mathrm{N}_{2r,2n-2r}$, and define a 1-dimensional representation $\theta_{\mathcal{N}_r}$ of $\mathcal{N}_r$ by

$$\theta_{\mathcal{N}_r}(n(s,h,x)) := \psi_{\mathcal{S}_r}(s)|\det a|^{n-r}. \tag{1.8}$$

It is easy to see from the definition of $\mathcal{N}_r$ that $\theta_{\mathcal{N}_r}$ is well defined.

For any irreducible admissible representation (τ, V_τ) of GL_{2n}, if the following space

$$\mathrm{Hom}_{\mathcal{N}_r}(V_\tau, \theta_{\mathcal{N}_r}) \neq 0, \tag{1.9}$$

we say that the representation V_τ has a nonzero *$\theta_{\mathcal{N}_r}$-functional* or a nonzero *$\theta_{\mathcal{N}_r}$-model*.

It is clear that when $r = 0$, $\mathcal{N}_0 = \mathrm{Sp}_{2n}$. Hence $\theta_{\mathcal{N}_0}$-model is the symplectic model of GL_{2n}, which was first studied by Klyachko in 1984 over a finite field and by Heumos and Rallis in 1990 over a p-adic local field ([HR90]). On the other hand, when $r = n$, $\mathcal{N}_n = \mathcal{S}_n$. Hence $\theta_{\mathcal{N}_n}$-model recovers the Shalika model of GL_{2n}, which was first used by Jacquet and Shalika in 1990 ([JS90]). In this sense, we may view this new family of models as an interpolation between the symplectic model and the Shalika model for GL_{2n}. It should be very interesting to study further properties of this new family of models. We will consider this in other occasion.

The second result of this paper is

Theorem 1.3. *Let τ be an irreducible admissible representation of GL_{2n}. If the induced representation*

$$\mathrm{Ind}_{P_{2n}}^{\mathrm{SO}_{4n}}(\tau|\det|^{\frac{1}{2}})$$

has a nonzero generalized Shalika model, then τ has a nonzero $\theta_{\mathcal{N}_r}$-model for some integer $0 \leq r \leq n$.

We note that Theorem 1.3 is to extend Theorem 1.1 (Theorem 3.1 in [JQ07]) to great generality. However, the statement in Theorem 1.3 is not as complete as that in Theorem 1.1. The main reason is that for the moment, we only have very limited knowledge about the family of $\theta_{\mathcal{N}_r}$-models with $0 \leq r \leq n$. We will come back to this issue at the end of §3 after we finish the proof of Theorem 1.3, since the geometric structures occurring in the proof yield more information about these models. We also remark that the proof of Theorem 1.3 looks close to that of Theorem 1.1, but it is much more technical since it needs more complete geometric structures of the $\mathcal{H}$-orbits on the generalized flag variety $P_{2n}\backslash \mathrm{SO}_{4n}$ over F. See §3.1 for the details. We also discuss the existence and the local uniqueness issues in §3.2. This discussion may lead the issues to the disjointness of the $\theta_{\mathcal{N}_r}$-models and the Klyachko models ([HR90], [O06], [OS07], [OS08-1], [OS08-2], and [N09-1]).

In §4, we will explain our main results in terms of the local Arthur parametrization of irreducible admissible representations of SO_{4n}, and state our conjecture (Conjecture 4.1) on the local Langlands functoriality in terms of the dual group associated to the spherical variety, which gives the generalized Shalika models. Based on this, we prove Theorem 4.2, which yields the relation between the local generalized Shalika models on SO_{4n} and the local Arthur parametrization, and state Conjecture 4.3, which characterizes the symplectic type of all irreducible admissible representations of GL_{2n} in terms of the $\theta_{\mathcal{N}_r}$-models with $r = 0, 1, 2, \cdots, n$. The special cases of this conjecture were established in [JNQ08] and in [OS07] for different types of irreducible admissible representations of GL_{2n}. We remark that the discussion in the section was motivated by Y. Sakellaridis' wonderful lecture in the workshop on Relative Trace Formula and Periods of Automorphic Forms at The American Institute of Mathematics, 2009, announcing his joint work with A.Venkatesh on periods and harmonic analysis on spherical varieties. In a sense, the conjectures and the results discussed in Section 4 support their more general conjectures on Plancherel formula on spherical varieties.

Finally, we would like to thank the referee for helpful comments, especially those on the presentation of the proof of Proposition 2.2.

2. PROOF OF THEOREM 1.2

The idea to prove Theorem 1.2 goes as follows. It is well-known (a version of the Jacquet submodule theorem) that any irreducible admissible representation (σ, V_σ) of SO_{4n} can be realized as a quotient of unitarily induced representation $\mathrm{Ind}_Q^{\mathrm{SO}_{4n}}(\pi)$. This means that there is a surjective SO_{4n}-equivariant mapping

$$\mathrm{Ind}_P^{\mathrm{SO}_{4n}}(\pi) \to V_\sigma \to 0, \tag{2.1}$$

where P is a standard parabolic subgroup of SO_{4n} with its Levi subgroup isomorphic to

$$\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r} \times \mathrm{SO}_{2m}$$

and the representation π can be expressed as

$$\pi = \tau_1 |\det|^{s_1} \otimes \cdots \otimes \tau_r |\det|^{s_r} \otimes \rho$$

with τ_i being supercuspidal representation of $\mathrm{GL}_{n_i}, i = 1, \cdots, r$ and ρ being supercuspidal representation of SO_{2m}.

We first consider the case when $m \leq 1$.

Note that SO_{4n} is F-split. When $m = 1$, we have

$$\mathrm{SO}_2(F) = \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in F^\times\},$$

and the supercuspidal representation ρ of the F-split SO_2 is just a character χ of $F^\times$. Note in this case that the parabolic subgroup P is the same as the standard parabolic subgroup $P_{n_1,\cdots,n_r,1}$ which has the Levi subgroup

$$\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r} \times \mathrm{GL}_1,$$

where $2n = n_1 + \cdots + n_r + 1$. Therefore, we may rewrite the induced representation $\mathrm{Ind}_P^{\mathrm{SO}_{4n}}(\pi)$ via the induction by stages as follows:

$$\begin{aligned} \mathrm{Ind}_P^{\mathrm{SO}_{4n}}(\pi) &= \mathrm{Ind}_{P_{n_1,\cdots,n_r,1}}^{\mathrm{SO}_{4n}}(\tau_1|\det|^{s_1} \otimes \cdots \otimes \tau_r|\det|^{s_r} \otimes \chi) \\ &= \mathrm{Ind}_{P_{2n}}^{\mathrm{SO}_{4n}}(\mathrm{Ind}_{Q_{n_1,\cdots,n_r,1}}^{\mathrm{GL}_{2n}}(\tau_1|\det|^{s_1} \otimes \cdots \otimes \tau_r|\det|^{s_r} \otimes \chi)), \end{aligned}$$

where $Q_{n_1,\cdots,n_r,1} = \mathrm{GL}_{2n} \cap P_{n_1,\cdots,n_r,1}$ is the standard maximal parabolic subgroup of GL_{2n} with Levi subgroup

$$\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r} \times \mathrm{GL}_1.$$

If we denote the induced representation

$$\mathrm{Ind}_{Q_{n_1,\cdots,n_r,1}}^{\mathrm{GL}_{2n}}(\tau_1|\det|^{s_1} \otimes \cdots \otimes \tau_r|\det|^{s_r} \otimes \chi)$$

of GL_{2n} by $\tau|\det|^{\frac{1}{2}}$, this proves Theorem 1.2 in the case of $m = 1$.

Note here that the representation τ constructed above may not be irreducible. However, since V_σ is an irreducible quotient of the induced representation $\mathrm{Ind}_{P_{2n}}^{\mathrm{SO}_{4n}}(\tau|\det|^{\frac{1}{2}})$ (see (2.1)), one may always replace τ by an irreducible quotient of τ.

It is clear that the above argument also proves Theorem 1.2 for the case of $m = 0$.

It remains to prove Theorem 1.2 for the case of $m > 1$. In this case, if V_σ has a nonzero generalized Shalika functional, then the induced representation $\mathrm{Ind}_P^{\mathrm{SO}_{4n}}(\pi)$ has a nonzero generalized Shalika functional, following (2.1). It is enough to show that when $m > 1$, the induced representation

$$\mathrm{Ind}_P^{\mathrm{SO}_{4n}}(\pi) = \mathrm{Ind}_P^{\mathrm{SO}_{4n}}(\tau_1|\det|^{s_1} \otimes \cdots \otimes \tau_r|\det|^{s_r} \otimes \rho)$$

has no nonzero generalized Shalika functionals at all.

A special case when $m = 2n$ is given by the following theorem.

Theorem 2.1 (Local Version of Lemma 5.1, [G95]). *Let ρ be an irreducible supercuspidal representation of* SO_{4n}. *Then ρ does not have a nontrivial generalized Shalika model.*

Ginzburg proved in [G95] (Lemma 5.1) that any cuspidal automorphic form on GSO_{4n} has no nonzero generalized Shalika period (although this terminology was not used in [G95]). From the proof of Lemma 5.1 of [G95], it is not hard to see that the proof works also for cuspidal automorphic forms on SO_{4n}. Furthermore, the local analogy of the proof shows that any irreducible supercuspidal representation σ of SO_{4n} has no nonzero generalized Shalika module. This can also be proved by using the globalization argument of Prasad and Schulze-Pillot (Theorem 4.1, [PSP08]). We omit the details without repeating the same argument here.

This leaves us the case $1 < m < 2n$ of Theorem 1.2 to prove. To this end, we formulate and prove a more general result as follows.

Write $r + m = 2n$ with $1 < m < 2n$, denote by

$$P_{r,2m} = (\mathrm{GL}_r \times \mathrm{SO}_{2m})N_{r,2m} \tag{2.2}$$

the standard maximal parabolic subgroup of SO_{4n}.

Proposition 2.2. *Let $P_{r,2m}$ with $r + m = 2n$ with $1 < m < 2n$ be the standard maximal parabolic subgroups of* SO_{4n}*, τ be a smooth representation of* GL_r*, which is not necessarily of finite length, and ρ be a supercuspidal representation of* SO_{2m}*. Then the induced representation*

$$\pi = \mathrm{Ind}_{P_{r,2m}}^{\mathrm{SO}_{4n}}(\tau \otimes \rho)$$

does not admit any generalized Shalika model of SO_{4n}.

The proof of Proposition 2.2 is given in Subsection 2.2. To do so, we have to define in Section 2.1 the notion of admissibility of double cosets involved in the proof.

2.1. **Admissible double cosets.**

In order to prove Proposition 2.2, we have to introduce the notion of admissible double cosets.

Let $\pm e_i \pm e_j$ denote the roots of SO_{4n} with respect to the maximal torus, which consists of all diagonal matrices, as convention. We also denote by U_α the one parameter subgroup of SO_{4n} corresponding to the root α.

Proposition 2.3. *Let W_1 be the set consisting of elements w in the Weyl group $W(\mathrm{SO}_{4n})$ which satisfy the following conditions*

$$\begin{cases} w(e_k) &= & e_k, & \text{if } k \le t_w; \\ w(e_k) &= & -e_k, & \text{if } t_w < k \le r; \\ w(e_k) &= & -e_k, & \text{if } r < k \le t'_w; \\ w(e_k) &= & e_k, & \text{if } t'_w < k \le 2n, \end{cases} \tag{2.3}$$

for some $0 \le t_w \le r \le t'_w \le 2n$ and $t'_w - t_w$ is even. Let S_1 be a complete set of representatives for $[\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}]^{\triangle}$ Then

$$P_{r,2m}\backslash \mathrm{SO}_{4n}/\mathcal{H}_{2n} = \cup_{w\in W_1, g\in S_1} P_{r,2m} w g \mathcal{H}_{2n}.$$

Proof. The result follows that W_1 is a complete set of representatives for

$$P_{r,2m}\backslash \mathrm{SO}_{4n}/P_{2n}$$

and

$$P_{2n} = [\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}]^{\triangle} \cdot \mathcal{H}_{2n}.$$

□

In order to prove Proposition 2.2, it is enough to show the following:

$$\mathrm{Hom}_{\mathrm{SO}_{4n}}(\mathrm{Ind}_{P_{r,2m}}^{\mathrm{SO}_{4n}}(\tau\otimes\rho), \mathrm{Ind}_{\mathcal{H}}^{\mathrm{SO}_{4n}}(\psi_{\mathcal{H}})) = \{0\}.$$

Here $\mathcal{H} = \mathcal{H}_{2n}$ and $\tau\otimes\rho$ also denotes its extension to $P_{r,2m}$, which is trivial on the unipotent radical of $P_{r,2m}$. We also use the same identification for other inducing data in parabolic induction. By reciprocity law,

$$\mathrm{Hom}_{\mathrm{SO}_{4n}}(\mathrm{Ind}_{P_{r,2m}}^{\mathrm{SO}_{4n}}\tau\otimes\rho, \mathrm{Ind}_{\mathcal{H}}^{\mathrm{SO}_{4n}}\psi_{\mathcal{H}}) \cong \mathrm{Hom}_{\mathcal{H}}(\mathrm{Ind}_{P_{r,2m}}^{\mathrm{SO}_{4n}}(\tau\otimes\rho)|_{\mathcal{H}}, \psi_{\mathcal{H}}).$$

Note that $\mathcal{H}$ is unimodular. Let S denote a complete set of representatives of $P_{r,2m}\backslash \mathrm{SO}_{4n}/\mathcal{H}$. Let δ_G be the modular function of a group G. Then up to semisimplification, we have as vector spaces:

$$\mathrm{Ind}_{P_{r,2m}}^{\mathrm{SO}_{4n}}(\tau\otimes\rho)|_{\mathcal{H}} \cong \oplus_{g\in S}\mathrm{ind}_{\mathcal{H}^g}^{\mathcal{H}}(\tau\otimes\rho\otimes\delta_{P_{r,2m}}^{\frac{1}{2}})^g$$

where $\mathcal{H}^g = g^{-1}P_{r,2m}g \cap \mathcal{H}$, and the representation $(\tau\otimes\rho\otimes\delta_{P_{r,2m}}^{\frac{1}{2}})^g$ acts on $V_{\tau\otimes\rho}$ by

$$(\tau\otimes\rho\otimes\delta_{P_{r,2m}}^{\frac{1}{2}})^g(h) = (\tau\otimes\rho\otimes\delta_{P_{r,2m}}^{\frac{1}{2}})(ghg^{-1}), \text{ for } h\in\mathcal{H}^g.$$

Definition 2.4. *Let τ be a representation of GL_r and ρ be a representation of SO_{2m}. We say that a double coset $P_{r,2m}g\mathcal{H}$, $g\in\mathrm{GL}_{2n}$, is* **admissible** *for representation $\pi = \mathrm{Ind}_{P_{r,2m}}^{\mathrm{SO}_{4n}}(\tau\otimes\rho)$ if the following inequality holds*

$$\mathrm{Hom}_{\mathcal{H}^g}((\tau\otimes\rho\otimes\delta_{P_{r,2m}}^{\frac{1}{2}})^g, \psi_{\mathcal{H}}\delta_{\mathcal{H}^g}^{-1}) \ne \{0\},$$

where $\psi_{\mathcal{H}}$ *is identified with its restriction to* $\mathcal{H}^g$*.*

More generally, let G *be a group and* H_i *its subgroups,* $i = 1, 2$*. Assume that* H_2 *is unimodular. Let* σ_i *be a representation of* H_i*. We say that a double coset* $H_1 g H_2$*,* $g \in G$*, is* **admissible between** (σ_1, H_1) **and** (σ_2, H_2) *if the following inequality holds*

$$\mathrm{Hom}_{H^g}((\sigma_1 \otimes \delta_{H_1}^{\frac{1}{2}})^g, \sigma_2 \otimes \delta_{H^g}^{-1}) \neq \{0\},$$

where $H^g = g^{-1} H_1 g \cap H_2$*, and the representation* $(\sigma_1 \otimes \delta_{H_1}^{\frac{1}{2}})^g$ *acts on* V_{σ_1} *by*

$$(\sigma_1 \otimes \delta_{H_1}^{\frac{1}{2}})^g(h) = (\sigma_1 \otimes \delta_{H_1}^{\frac{1}{2}})(ghg^{-1}), \text{ for } h \in H^g.$$

To show Proposition 2.2, now it reduces to show that

$$\dim \mathrm{Hom}_{\mathcal{H}^g}((\tau \otimes \rho \otimes \delta_{P_{r,2m}}^{\frac{1}{2}})^g, \psi_{\mathcal{H}} \delta_{\mathcal{H}^g}^{-1}) = 0.$$

for all representatives $g \in S$. That is to show that there is no admissible cosets $P_{r,2m} g \mathcal{H}$ for all representatives g. From Proposition 2.3, we need to consider the admissibility of $P_{r,2m} w g \mathcal{H}$, with $w \in W_1$, and $g \in S_1$.

To simplify the calculation of admissibility of double cosets, we extend the notion of generalized Shalika groups as follows. Let A be a nonsingular skew symmetric matrix of degree $2n$. Define

$$\mathcal{H}_A = \mathrm{Sp}_{2n}(A) V_{2n},$$

where elements in $\mathcal{H}_A$ are $(h, X) \in P_{2n}$ (notations as in Eq. (1.2)) with $h \in \mathrm{Sp}_{2n}(A)$, and

$$\mathrm{Sp}_{2n}(A) = \{x \in \mathrm{GL}_{2n} \mid {}^t x A x = A\}.$$

Let $g \in \mathrm{GL}_{2n}$ satisfy ${}^t g A g = J_{2n}$. Then $\mathrm{Sp}_{2n}(A) = g \mathrm{Sp}_{2n}(J_{2n}) g^{-1}$, and $g \mathcal{H} g^{-1} = \mathcal{H}_A$. Define a character $\psi_{\mathcal{H}_A}$ on $\mathcal{H}_A$ by

$$\psi_{\mathcal{H}_A}(h, X) = \psi(\mathrm{tr}(A X \nu_{2n})), \quad (h, X) \in \mathcal{H}_A. \tag{2.4}$$

It is well defined. (Refer to [JNQ10-1].) When $A = J_{2n}$, $\mathcal{H}_A = \mathcal{H}$ is the generalized Shalika group as defined before, and $\psi_{\mathcal{H}_A} = \psi_{\mathcal{H}}$.

Definition 2.5. *For any skew symmetric matrix* A *and* $w \in W_1$*, we say that a double coset* $P_{r,2m} w \mathcal{H}_A$ *is* **admissible** *for representation* $\pi = \mathrm{Ind}_{P_{r,2m}}^{\mathrm{SO}_{4n}}(\tau \otimes \rho)$ *if the following inequality holds*

$$\mathrm{Hom}_{\mathcal{H}^{w,A}}((\tau \otimes \rho \otimes \delta_{P_{r,2m}}^{\frac{1}{2}})^w, \psi_{\mathcal{H}_A} \delta_{\mathcal{H}^{w,A}}^{-1}) \neq \{0\},$$

where $\mathcal{H}^{w,A} = w^{-1} P_{r,2m} w \cap \mathcal{H}_A$*.*

It is easy to check from the definition that the following lemma holds.

Lemma 2.6. *The double coset* $P_{r,2m} w g \mathcal{H}_A$ *with* $w \in W_1, g \in \mathrm{GL}_{2n}$ *is admissible if and only if* $P_{r,2m} w \mathcal{H}_B$ *is admissible where* $B = {}^t g^{-1} A g^{-1}$*.*

Let U_k denote the upper triangular maximal unipotent subgroup of GL_k and $W(\mathrm{GL}_k)$ the Weyl group of GL_k, identified with the group of permutation matrices in GL_k. The following lemma gives a nice characterization of nonsingular skew symmetric matrices.

Lemma 2.7 ([JR92], Lemma 2)**.** *Every nonsingular skew symmetric matrix of degree $2n$ can be written in the form*

$$s = u\sigma\lambda u^t$$

with $u \in \mathrm{U}_{2n}$, λ is a diagonal matrix in GL_{2n}, and $\sigma \in W(\mathrm{GL}_{2n})$ such that

$$\sigma^2 = 1,\ \sigma\lambda\sigma^{-1} = -\lambda. \tag{2.5}$$

Let $\mathcal{A}$ be a subset of GL_{2n} defined by

$$\mathcal{A} = \{\sigma\lambda \mid \lambda \text{ is diagonal, } \sigma \in W(\mathrm{GL}_{2n}), \sigma^2 = 1, \sigma\lambda\sigma^{-1} = -\lambda\}.$$

Then $\mathcal{A}$ consists of nonsingular skew symmetric matrices with one and only one nonzero element at each row and column, which is called monomial. Therefore, by Lemma 2.6 and 2.7, to prove Proposition 2.2, it suffices to show that the coset $P_{r,2m}wu\mathcal{H}_A$ is not admissible for any $w \in W_1$, $A \in \mathcal{A}$, and $u \in \mathrm{U}_{2n}$ (identified with its diagonal embedding in SO_{4n}).

2.2. **Proof of Proposition 2.2.** In Proposition 2.2, we assume that $1 < m < 2n$. In order to prove Proposition 2.2, it is enough to show that $P_{r,2m}wu\mathcal{H}_A$ is not admissible for any fixed $w \in W_1$, $u \in \mathrm{U}_{2n}$, and $A \in \mathcal{A}$. We will proceed by induction on n and equivalently on r.

We assume first that for all $r' + m = 2q$ with $q < n$, the double coset

$$P_{r',2m}wu\mathcal{H}_A \subset \mathrm{SO}_{4q}$$

is not admissible with respect to any

$$\pi' = \mathrm{Ind}_{P_{r',2m}}^{\mathrm{SO}_{4q}}(\tau' \otimes \rho'),$$

where τ' is any smooth representation (not necessarily of finite length) of $\mathrm{GL}_{r'}$ and ρ' is a supercuspidal representation of SO_{2m}.

Write $A = (a_{i,j})$ and define a permutation $\iota = \iota_A$ on $[2n]$ with respect to A such that $a_{s,\iota(s)} \neq 0$. It is well-defined since A is monomial and $\iota^2 = \mathrm{id}$ since A is skew symmetric. For any skew symmetric matrix T (not necessary in $\mathcal{A}$), write $T = (t_{i,j}), 1 \leq i, j \leq 2n$. If $t_{k,s} \neq 0$, then $\psi_{\mathcal{H}_T}|_{U_\alpha}$ is not trivial, where α is the root corresponding to $(k, 4n+1-s)$-entry in SO_{4n}. Especially, for $A \in \mathcal{A}$,

$$\psi_{\mathcal{H}_A}|_{U_\beta} \text{ is not trivial,}$$

where β is the root corresponding to $(s, 4n + 1 - \iota_A(s))$-entry in SO_{4n}.

Given any $u \in \mathrm{U}_{2n}$ and $A \in \mathcal{A}$, to show that $P_{r,2m}wu\mathcal{H}_A$ is not admissible is equivalent to show that $P_{r,2m}w\mathcal{H}_B$ is not admissible, where

$$B = {}^t u^{-1} A u^{-1}.$$

Write

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \text{ and } B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

where $A_1, B_1 \in \mathrm{Mat}_r$ and $A_4, B_4 \in \mathrm{Mat}_m$. For all skew symmetric matrix $T = (t_{i,j})$, denote

$$C_T = \{k \leq r \mid t_{k,s} \neq 0 \text{ for some } s \geq r+1\}. \tag{2.6}$$

This means that C_T is the set of row indices corresponding to nonzero rows of T_2, when

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

with $T_1 \in \mathrm{Mat}_r$ and $T_4 \in \mathrm{Mat}_m$.

Lemma 2.8. *Assume that C_B is nonempty, i.e. $B_2 \neq 0$. If $P_{r,2m}w\mathcal{H}_B$ is admissible, then $w(e_k) = -e_k$ for $k \in C_B$.*

Proof. Assume on the contrary that $w(e_k) = e_k$ and $b_{k,s} \neq 0$ for some $s > r$ and $k \in C_B$. For $\lambda \in F$, take

$$X(\lambda) = \lambda(e_s + e_k) \in U_{e_s+e_k}.$$

Since $w(e_s + e_k) = e_k \pm e_s$, the element $w^{-1}(X(\lambda))w$ belongs to the unipotent radical of $P_{r,2m}$. Hence we have

$$\begin{aligned} \psi_{\mathcal{H}_B}(X(\lambda)) &= \psi(\pm b_{k,s}\lambda), \\ (\tau \otimes \rho)(w^{-1}(X(\lambda))w) &= 1, \end{aligned}$$

where the last equality means that the representation $\tau \otimes \rho$ is trivial on elements of the form $w^{-1}(X(\lambda))w$ and similar expression will also be used in later paragraphs. Therefore $P_{r,2m}w\mathcal{H}_B$ is not an admissible coset. □

Since $B = {}^t u^{-1} A u^{-1}$, if the k-th row of A_2 is nonzero, then so is the k-th row of B_2. By Lemma 2.8, if $P_r wu\mathcal{H}_A$ is admissible (which is equivalent to $P_r w\mathcal{H}_B$ is admissible) and C_A is nonempty, then $w(e_k) = -e_k$ for all $k \in C_A$.

Write $u \in \mathrm{U}_{2n}$ as

$$u = \begin{pmatrix} u_1 & u_2 & u_3 \\ & u_4 & u_5 \\ & & u_6 \end{pmatrix},$$

where $u_1 \in \mathrm{U}_{t_w}$, $u_4 \in \mathrm{U}_{r-t_w}$, $u_6 \in \mathrm{U}_m$, and t_w is defined in Equation (2.3). For $g \in \mathrm{GL}_{2n}$, denote the embedding of elements of GL_{2n} into SO_{4n} by

$$m(g) = \mathrm{diag}(g, g^*) \in \mathrm{SO}_{4n}.$$

Since

$$m(\begin{pmatrix} u_1 & u_2 & u_3 \\ & u_4 & 0 \\ & & u_6 \end{pmatrix}) \in w^{-1}P_{r,2m}w,$$

we can even take

$$u = \begin{pmatrix} \mathrm{I}_{t_w} & 0 & 0 \\ & \mathrm{I}_{r-t_w} & u_5 \\ & & \mathrm{I}_m \end{pmatrix} \tag{2.7}$$

as the representative in the coset $P_{r,2m}wu\mathcal{H}_A$.

Lemma 2.9. *Assume that $P_{r,2m}wu\mathcal{H}_A$ is admissible, with u of the form in (2.7). If $s \le t_w$, then $t_w < \iota(s) \le r$.*

Proof. By Lemma 2.8, $\iota(s) \le r$. Assume on the contrary that $\iota(s) \le t_w$. Take $X(\lambda) = \lambda(e_s + e_{\iota(s)})$ with $\lambda \in F^*$. Then $wuX(\lambda)u^{-1}w^{-1} = X(\lambda)$ belongs to the unipotent radical of P_r and

$$\psi_{\mathcal{H}_A}(X(\lambda)) = \psi(\pm a_{s,\iota(s)}\lambda),$$

which is not trivial for a suitable chosen λ. It contradicts to the admissibility of $P_{r,2m}wu\mathcal{H}_A$ and hence $t_w < \iota(s) \le r$. ☐

Lemma 2.10. *If $t_w \ne 0, C_B \ne \emptyset$ and $w(e_k) = -e_k$, for all $k \in C_B$, where $B = {}^tu^{-1}Au^{-1}$ with u as in (2.7), then $P_{r,2m}w\mathcal{H}_B$ is not admissible.*

Proof. Assume that $P_{r,2m}w\mathcal{H}_B$ is an admissible coset. Let $A_1' \in \mathrm{Mat}_{2t_w}$ be the left-upper block of A. By Lemma 2.9, we can write A_1' in the following form

$$A_1' = \begin{pmatrix} 0_{t_w} & C \\ -C^t & D \end{pmatrix}.$$

Moreover, with the help of some diagonal matrix and permutation matrix on $\{e_i \mid t_w \le i \le r\}$, which are in $w^{-1}P_{r,2m}w$ and commute with the set of unipotent matrices of the form (2.7), we may assume that

$$A_1' = \begin{pmatrix} & \nu_{t_w} \\ -\nu_{t_w} & \end{pmatrix}.$$

Since $\iota(1) = 2t_w$, by Lemma 2.9 and Eq. (2.3), $r \geq 2t_w$. Let

$$u = \begin{pmatrix} \mathrm{I}_{t_w} & 0 & 0 & 0 \\ & \mathrm{I}_{t_w} & 0 & z_1 \\ & & \mathrm{I}_{r-2t_w} & z_2 \\ & & & \mathrm{I}_m \end{pmatrix}.$$

Then

$$B = {}^t u^{-1} A u^{-1} = \begin{pmatrix} 0 & \nu_{t_w} & 0 & -\nu_{t_w} z_1 \\ -\nu_{t_w} & 0 & 0 & 0 \\ 0 & 0 & * & * \\ (\nu_{t_w} z_1)^t & 0 & * & * \end{pmatrix}.$$

By Lemma 2.8, $-\nu_{t_w} z_1 = 0$ and B is of the form

$$B = \mathrm{diag}(B'_1, B'_4), \text{ where } B'_1 = \begin{pmatrix} 0 & \nu_{t_w} \\ -\nu_{t_w} & 0 \end{pmatrix} \tag{2.8}$$

and $B'_4 \in \mathrm{Mat}_{2n-2t_w}$ is skew symmetric of even rank. Since B is the form of Eq. (2.8) and $C_B \neq \emptyset$, we must have

$$r' := r - 2t_w > 0.$$

For $t_w \neq 0$ (i.e. B'_1 is nontrivial), the admissibility (as defined in Definition 2.4) of $P_{r,2m} w \mathcal{H}_B$ with respect to π gives

$$\mathrm{Hom}_{\mathcal{H}^{w,B}}((\tau \otimes \rho \otimes \delta_{P_{r,2m}}^{\frac{1}{2}})^w, \psi_{\mathcal{H}_B} \delta_{\mathcal{H}^{w,B}}^{-1}) \neq \{0\},$$

where $\mathcal{H}^{w,B} = w^{-1} P_{r,2m} w \cap \mathcal{H}_B$, which is equivalent to

$$\mathrm{Hom}_{\mathcal{H}^{w,B}}((\tau \otimes \rho \otimes \delta_{P_{r,2m}}^{\frac{1}{2}} \delta_{\mathcal{H}_{w,B}})^w, \psi_{\mathcal{H}_B}) \neq \{0\},$$

where $\mathcal{H}_{w,B} = P_{r,2m} \cap w \mathcal{H}_B w^{-1}$. This implies that there exist a nontrivial functional

$$T \ : \ V_{\tau \otimes \rho} \mapsto \mathbb{C},$$

such that

$$T(\tau \otimes \rho \otimes \delta_{P_{r,2m}}^{\frac{1}{2}} \delta_{\mathcal{H}_{w,B}}(x) \cdot v) = \psi_{\mathcal{H}_B}(w^{-1} x w) T(v), \tag{2.9}$$

for $x \in \mathcal{H}_{w,B}, v \in V_{\tau \otimes \rho}$.

Let $w' = w|_{\mathrm{SO}_{4n-4t_w}}$, where w' and SO_{4n-4t_w} are identified with their embedding in the middle part of SO_{4n}. More precisely, we have

$$w = \begin{pmatrix} \mathrm{I}_{t_w} & & & & \\ & & & \nu_{t_w} & \\ & & w' & & \\ & \nu_{t_w} & & & \\ & & & & \mathrm{I}_{t_w} \end{pmatrix},$$

and $P_{r',2m}$ can be identified with

$$\begin{pmatrix} \mathrm{I}_{2t_w} & & \\ & P_{r',2m} & \\ & & \mathrm{I}_{2t_w} \end{pmatrix} \subset P_{r,2m}.$$

Consider Eq. (2.9) for $y \in \mathcal{H}^{w',B'_4} = w'^{-1}P_{r',2m}w' \cap \mathcal{H}_{B'_4}$. Then

$$\mathrm{Hom}_{\mathcal{H}^{w',B'_4}}((\tau \otimes \rho \otimes \delta_{P_{r,2m}}^{\frac{1}{2}} \delta_{\mathcal{H}_{w,B}})^{w'}, \psi_{\mathcal{H}_{B'_4}}) \neq \{0\},$$

where $\mathcal{H}_{w',B'_4} = P_{r',2m} \cap w'\mathcal{H}_{B'_4}w'^{-1}$. Hence when we restrict ourselves to the middle $\mathrm{SO}_{4(n-t_w)}$, we obtain

$$\mathrm{Hom}_{\mathcal{H}^{w',B'_4}}((\tau' \otimes \rho' \otimes \delta_{P_{r',2m}}^{\frac{1}{2}} \delta_{\mathcal{H}_{w',B'_4}})^{w'}, \psi_{\mathcal{H}_{B'_4}}) \neq \{0\},$$

where τ' is the restriction to $\mathrm{GL}_{r'}$ of the representation

$$\tau \otimes \delta_{P_{r,2m}}^{\frac{1}{2}} \delta_{\mathcal{H}_{w,B}} \delta_{P_{r',2m}}^{-\frac{1}{2}} \delta_{\mathcal{H}_{w',B'_4}}^{-1},$$

which is a smooth representation of $\mathrm{GL}_{r'}$, but may not be of finite length, and

$$\rho' = \rho \otimes (\delta_{P_{r,2m}}^{\frac{1}{2}} \delta_{\mathcal{H}_{w,B}} \delta_{P_{r',2m}}^{-\frac{1}{2}} \delta_{\mathcal{H}_{w',B'_4}}^{-1})|_{\mathrm{SO}_{2m}}$$

is still a supercuspidal representation of SO_{2m}. In fact, it is clear that $\rho' = \rho$.

This means that the double coset $P_{r',2m}w'\mathcal{H}_{B'_4}$ is admissible with respect to

$$\pi' = \mathrm{Ind}_{P_{r',2m}}^{\mathrm{SO}_{4(n-t_w)}}(\tau' \otimes \rho').$$

By Lemma 2.7, there exists $u' \in \mathrm{U}_{4n-4t_w}$ and $A' \in \mathcal{A}(\mathrm{GL}_{2n-2t_w})$ such that $B_4 = {}^t u'^{-1}A'u'^{-1}$. By Lemma 2.6, the admissibility with respect to π' of $P_{r',2m}w'\mathcal{H}_{B'_4}$ is the same as the admissibility of $P_{r',2m}w'u'\mathcal{H}_{A'}$, which contradicts to the induction assumption at the beginning of Subsection 2.2, with $4n - 4t_w < 4n$. Therefore, we finish the proof. □

We remark that in the proof of Lemma 2.10, we deduce that $r' = r - 2t_w$ must be positive. Hence in the case when $r = 1$ and $m = 2n-1$, the double cosets of the type in Lemma 2.10 will not occur. Also, in the inductive argument, m remains unchanged.

Lemma 2.11. *Assume that $t_w = 0$, then $P_{r,2m}wu\mathcal{H}_A$ is not admissible for all $A \in \mathcal{A}$ and $u \in \mathrm{U}_{2n}$.*

Proof. Let $\bar{\mathrm{U}}_{2n}$ denote the opposite unipotent radical of U_{2n}. When $t_w = 0$, the admissibility of the double coset $P_{r,2m}wu\mathcal{H}_A$ is equivalent to the admissibility of $P_{r,2m}w\bar{u}\mathcal{H}_{A'}$ for some $A' \in \mathcal{A}$ and $\bar{u} \in \bar{\mathrm{U}}_{2n}$. Since

$\bar{\mathrm{U}}_{2n} \subset w^{-1}P_{r,2m}w$, $P_{r,2m}w\bar{u}\mathcal{H}_{A'} = P_{r,2m}w\mathcal{H}_{A'}$. Now it leaves us to show the non-admissibility of $P_{r,2m}w\mathcal{H}_{A'}$, which will be established in the following Lemmas 2.12 and 2.13. $\square$

Lemma 2.12. *If $C_A \neq \emptyset$ and $w(e_k) = -e_k$, for all $k \in C_A$. Then $P_{r,2m}w\mathcal{H}_A$ is not admissible.*

Proof. Let

$$D_A = \{t \geq r+1 \mid \iota(t) \geq r+1\},$$

and q be the cardinality of D_A (q may equal to zero). Then the cardinality of $C_A = m - q$. For $k \in C_A$, $t \in D_A$, for all $\alpha \in F^*$, choose $\beta(\alpha) \in F^*$ such that

$$g_{k,t}(\alpha) = \mathrm{I}_{2n} + \alpha E_{\iota(k),t} + \beta(\alpha) E_{\iota(t),k} \in \mathrm{Sp}_A,$$

where $E_{i,j} = (e_{k,l})$ denotes the elementary matrix with its (k,l)-entry $e_{k,l} = \delta_{k,i}\delta_{l,j}$. Define

$$X_{k,t}(\lambda) := \lambda(E_{\iota(k),2n+1-t} - E_{t,2n+1-\iota(k)}), \text{ for } \lambda \in F,$$

and

$$Y_{k,k'}(\eta) := \eta(E_{\iota(k),2n+1-k'} - E_{k',2n+1-\iota(k)}),$$

where $\eta \in F, k \neq k' \in C_A$. Then we have

$$(g_{k,t}(\alpha), X_{k,t})(\mathrm{I}_{2n}, Y_{k,k'}(\lambda)) \in \mathcal{H}_A \cap w^{-1}P_{r,2m}w$$

and

$$\psi_{\mathcal{H}_A}((g_{k,t}(\alpha), X_{k,t}(\lambda))(\mathrm{I}_{2n}, Y_{k,k'}(\lambda))) = 1.$$

If $w' = w|_{\mathrm{SO}_{2m}}$ is an even permutation, then according to admissibility, $\rho|_{N'} = 1$, where N' is isomorphic to the unipotent radical $N_{m-q,2q}$ of $P_{m-q,2q} \subset \mathrm{SO}_{2m}$. If $w' = w|_{\mathrm{SO}_{2m}}$ is an odd permutation, then $\tilde{\rho} \cong \rho'$ (Refer to Lemma 4.2, [N10]), where the representation space of ρ' is the same as ρ such that

$$\rho'(g) = \rho(w' g w'^{-1}), \text{ for } g \in \mathrm{SO}_{2m}.$$

Note that if ρ is supercuspidal then so is its contragradient $\tilde{\rho}$. Hence the admissibility implies $\rho'|_{N'} = 1$, where N' is isomorphic to the unipotent radical $N_{m-q,2q}$ of $P_{m-q,2q} \subset \mathrm{SO}_{2m}$. Therefore, both of the cases contradict to the supercuspidality of ρ, and $P_{r,2m}w\mathcal{H}_A$ is not admissible. $\square$

We remark that in the proof of Lemma 2.12, we use the supercuspidality of ρ of SO_{2m} and hence we use the assumption that $m > 1$.

Lemma 2.13. *Let $B = \begin{pmatrix} B_1 & \\ & B_4 \end{pmatrix}$, for some skew symmetric matrices $B_1 \in \mathrm{Mat}_r$ and $B_4 \in \mathrm{Mat}_m$, where m and r are both even and nonzero. Then $P_{r,2m}w\mathcal{H}_B$ is not admissible.*

Proof. Assume that $P_{r,2m}w\mathcal{H}_B$ is admissible. Note that the SO_{2m} part is invariant under the conjugate action of $w \in W_1$ and only regards to the skew symmetric form B_4. According to whether $w|_{\mathrm{SO}_{2m}}$ is even or odd, the admissibility of the coset implies that either ρ or $\tilde{\rho}$ has a generalized Shalika model of SO_{2m}. If ρ is supercuspidal, then so is its contragradient $\tilde{\rho}$. (Refer to [N10].) Either of the cases contradicts to Theorem 2.1. □

We remark again that in the proof of Lemma 2.13, we use the supercuspidality of ρ of SO_{2m} and hence we use the assumption that $m > 1$. Note that in Lemma 2.13, r must be even.

Finally, we are able to complete the proof of Proposition 2.2 based on Lemmas 2.8 through 2.13 and hence that of Theorem 1.2.

In fact, all double cosets are not admissible except those considered in Lemma 2.10. For the double cosets considered in Lemma 2.10, we reduce the problem to the group $\mathrm{SO}_{4(n-t_w)}$ and the parabolic subgroup with Levi part isomorphic to $\mathrm{GL}_{r'} \times \mathrm{SO}_{2m}$ and the supercuspidal ρ.

By repeating the same argument, if $m = 2l$ is even, then r must be even and hence we reduce the problem to SO_{4l} with supercuspidal ρ. In this case, Proposition 2.2 reduces to Theorem 2.1.

Now if m is odd and greater than 1, then r must be odd. We eventually reduce the problem to the case with the group SO_{4n} and the parabolic subgroup with Levi part isomorphic to $\mathrm{GL}_1 \times \mathrm{SO}_{2(2n-1)}$ (i.e. $r = 1$ and $m = 2n - 1 > 1$) and the supercuspidal ρ of $\mathrm{SO}_{2(2n-1)}$. As we remarked after finishing the proof of Lemma 2.10, when $r = 1$, the double cosets of the type in Lemma 2.10 will not occur. Also they are not of the types occurring in Lemma 2.13, where r must be even. Hence in this case (i.e. $r = 1$ and $1 < m = 2n - 1$), the double cosets are of the types considered in Lemmas 2.11 and 2.12, and hence all of them are not admissible. This completes the inductive argument, which proves Proposition 2.2.

3. PROOF OF THEOREM 1.3

In this section, we fix any $n \in \mathbb{N}$, and write $P = P_{2n}$ the Siegel parabolic subgroup of SO_{4n}. We first prove Theorem 1.3 and then discuss the conditions for the existence and uniqueness of the generalized Shalika functionals on such an induced representation of SO_{4n}.

3.1. **Proof of Theorem 1.3.**

We want to show that if the induced representation

$$\pi := \mathrm{Ind}_P^{\mathrm{SO}_{4n}}(\tau|\det|^{\frac{1}{2}})$$

for some irreducible admissible representation τ of GL_{2n}, has a nonzero generalized Shalika model, then τ admits a nonzero $\theta_{\mathcal{N}_r}$-model of GL_{2n} for some $0 \leq r \leq n$.

To compare with Theorem 1.1, we consider first the general case when

$$\Sigma := \mathrm{Ind}_P^{\mathrm{SO}_{4n}}(\tau|\det|^{\frac{s}{2}}).$$

Following the standard argument, we have to consider the $\mathcal{H}$-orbit decomposition of the generalized flag variety $P\backslash\mathrm{SO}_{4n}$ and consider the admissibility of the $\mathcal{H}$-orbits, from which we deduce the property of τ as stated in Theorem 1.3.

Define certain representatives of Weyl group elements of SO_{4n}:

$$w_i := \begin{pmatrix} & & & \mathrm{I}_{2i} \\ & \mathrm{I}_{2n-2i} & 0 & \\ & 0 & \mathrm{I}_{2n-2i} & \\ \mathrm{I}_{2i} & & & \end{pmatrix}, \quad 0 \leq i \leq n.$$

Then the set $\{w_i \mid 0 \leq i \leq n\}$ is a complete set of representatives for the generalized Bruhat decomposition

$$P\backslash\mathrm{SO}_{4n}/P$$

of SO_{4n} with respect to the parabolic subgroup P. Hence a complete set of the representatives of double coset decomposition

$$P\backslash\mathrm{SO}_{4n}/\mathcal{H}$$

consists of elements of type $w_i h$ with $0 \leq i \leq n$ and certain $h \in \mathrm{GL}_{2n}$.

Now we consider all double cosets in the form of $Pw_ih\mathcal{H}$, where $h \in \mathrm{GL}_{2n}, 0 \leq i \leq n$. By the same argument as in the previous section, it suffices to consider all double cosets in the form of

$$Pw_i\bar{u}\mathcal{H}_A,$$

with $\bar{u} \in \bar{\mathrm{U}}_{2n}, A \in \mathcal{A}$, where we use $\bar{u} \in \bar{\mathrm{U}}_{2n}$ (the radical of the opposite of P) to simplify the computation.

To figure out the admissibility of the cosets $Pw_i\bar{u}\mathcal{H}_A$ with $A \in \mathcal{A}$, we have the following cases: $i = 0$ and $1 \leq i \leq n$.

When $i = 0$, we have the coset $Pw_0\bar{u}\mathcal{H}_A = P\mathcal{H}_A,$ for all $\bar{u} \in \bar{\mathrm{U}}_{2n}, A \in \mathcal{A}$. For any $(\mathrm{I}_{2n}, X) \in V_{2n}$, then $(\mathrm{I}_{2n}, X) \in P \cap \mathcal{H}_A$. On one hand we have

$$(\tau \cdot |\det|^{\frac{s}{2}})((\mathrm{I}_{2n}, X)) = 1$$

and on the other hand the character $\psi_{\mathcal{H}_A}((\mathrm{I}_{2n}, X)) = \psi(\mathrm{tr}AX\nu_{2n})$ is non-trivial. Hence $Pw_0\bar{u}\mathcal{H}_A$ is not an admissible coset.

When $i = k$ with $1 \leq k \leq n$, we consider the double coset $Pw_i\bar{u}\mathcal{H}_A$. Define

$$p = \begin{pmatrix} G & B & X & Y \\ C & D & Z & W \\ & & D' & B' \\ & & C' & G' \end{pmatrix} \in P,$$

where $G, G' \in \mathrm{Mat}_{2k}$ and $D, D' \in \mathrm{Mat}_{2n-2k}$. Then

$$w_k p w_k^{-1} = \begin{pmatrix} G' & & C' & \\ W & D & Z & C \\ B' & & D' & \\ Y & B & X & G \end{pmatrix}.$$

Since $w_k^{-1}Pw_k = (\bar{u}w_k)^{-1}Pw_k\bar{u}$ for all $\bar{u} \in \bar{\mathrm{U}}_{2n}$, it suffices to consider the admissibility of $Pw_k\mathcal{H}_A$ for $A \in \mathcal{A}$.

Write

$$A := \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

with $A_1 \in \mathrm{Mat}_{2k}$ and $A_4 \in \mathrm{Mat}_{2n-2k}$. Define $t := (\mathrm{I}_{2n}, T) \in V_{2n}$(Refer to Eq. (1.2) for the notation.) with

$$T = \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix}, \quad Z \in \mathrm{Mat}_{2n-2k}.$$

Then $t \in w_k^{-1}Pw_k \cap \mathcal{H}$. Now on the left hand side, we have

$$(\tau \cdot |\det|^{\frac{s}{2}})(t) = 1,$$

and on the right hand side we have

$$\psi_{\mathcal{H}_A}(t) = \psi(\mathrm{tr} A_4 Z \nu_{2n-2k}),$$

which is the key condition for us to determine the admissibility of the cosets $Pw_k\mathcal{H}_A$ for $A \in \mathcal{A}$ and $k = 1, 2, \cdots, n$.

For $2k < n$, since A is invertible, $A_4 \neq 0$. Hence the cosets $Pw_k\mathcal{H}_A$ are not admissible, and when $2k \geq n$ and $A_4 \neq 0$, the cosets $Pw_k\mathcal{H}_A$ are also not admissible. It remains to consider the cosets $Pw_k\mathcal{H}_A$ with the conditions that $2k \geq n$ and $A_4 = 0$.

In this case, we may re-write the $2n \times 2n$-matrix A as follows:

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & 0 \end{pmatrix}$$

with $a_1 \in \mathrm{Mat}_{2n-2k}$ and $a_5 \in \mathrm{Mat}_{4k-2n}$. Since A is monomial, there exists a suitable permutation matrix

$$B = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix}, \ B_1 \in \mathrm{Mat}_{2k},\ B_2 \in \mathrm{Mat}_{2n-2k}$$

and a diagonal matrix $d = \mathrm{diag}(d_1, \cdots, d_n, d_n, \cdots, d_1)$ such that

$$(dB)A(dB)^t = J.$$

Notice

$$(w_k dB)^{-1} P w_k dB = (w_k)^{-1} P w_k.$$

Hence $P w_k \mathcal{H}_A = P w_k \mathcal{H}$.

In order to consider the admissibility of the cosets $P w_k \mathcal{H}$ with the condition that $2k \geq n$, we consider the stabilizer of the coset in P, which is $P \cap w_k \mathcal{H} w_k^{-1}$. The elements of the stabilizer can be written as

$$\begin{pmatrix} g & X \\ & g^* \end{pmatrix} \in P \cap w_k \mathcal{H} w_k^{-1},$$

with

$$g = \begin{pmatrix} G_1 & & \\ G_3 & D & \\ R_1 & R_2 & D \end{pmatrix}, \quad X = \begin{pmatrix} W & & \\ Y & & \\ Z & Y^* & W^* \end{pmatrix}, \tag{3.1}$$

where the matrices satisfy the conditions

- $G_1 \in \mathrm{Sp}_{4k-2n}$, and G_3, R_1, and R_2 are arbitrary,
- $D \in \mathrm{GL}_{2n-2k}$ and $D^* = \nu\ {}^tD^{-1}\nu$,
- $W \in \mathrm{Mat}_{4k-2n,2n-2k}$ and $W^* = \nu_{2n-2k} W^t \nu_{4k-2n}$,
- $Y,\ Z \in \mathrm{Mat}_{2n-2k}$ such that $Z^t = \nu_{2n-2k} Z \nu_{2n-2k}$ and $Y^* = \nu_{2n-2k} Y^t \nu_{2n-2k}$,

such that

$$h = \begin{pmatrix} D^* & & \\ W & G_1 & \\ Y & G_3 & D \end{pmatrix} \in \mathrm{Sp}_{2n}. \tag{3.2}$$

Clearly, the subgroup of GL_{2n} consisting of all elements g of the form in Eq. (3.1) is isomorphic to the group $\mathcal{N}_{2n-2k}$ (with $2k \geq n$) as defined in (1.7), which defines the $\theta_{\mathcal{N}_{2n-2k}}$-model for GL_{2n} as in (1.9).

More explicitly, we write

$$\begin{pmatrix} g & X \\ & g^* \end{pmatrix} = \begin{pmatrix} G_1 & & & W & & \\ G_3 & D & & Y & & \\ R_1 & R_2 & D & Z & Y^* & W^* \\ & & & D^* & & \\ & & & R_2^* & D^* & \\ & & & R_1^* & G_3^* & G_1^* \end{pmatrix}.$$

Then after conjugating by w_k we have

$$w_k^{-1} \begin{pmatrix} g & X \\ & g^* \end{pmatrix} w_k = \begin{pmatrix} D^* & & & R_2^* & & \\ G_3^* & G_1^* & & R_1^* & & \\ Y^* & W^* & D & Z & R_1 & R_2 \\ & & & D^* & & \\ & & & W & G_1 & \\ & & & Y & G_3 & D \end{pmatrix},$$

which is an element in $\mathcal{H}^{w_k} := w_k^{-1} P w_k \cap \mathcal{H}$. Denote by $h^* = \nu({}^t h^{-1})\nu$, where h is as in (3.2). Then $\mathrm{diag}(h, h^*) \in \mathcal{H}$.

Now the admissibility implies that

$$|\det g|^{\frac{s}{2}} \delta_P^{\frac{1}{2}}(g) \tau(g) = \psi(\mathrm{tr} R_2) \delta_{\mathcal{H}^{w_k}}^{-1}(h). \tag{3.3}$$

Denote by $f = \mathrm{diag}(D^*, G_1^*, D)$ and $f^* = \mathrm{diag}(D^*, G_1, D)$ such that $\mathrm{diag}(f, f^*)$ is an element in the Levi part of $\mathcal{H}^{w_k}$. Denote by

$$R = \begin{pmatrix} \mathrm{I}_{2n-2k} & & & R_2^* & & \\ G_3^* & \mathrm{I}_{4k-2n} & & R_1^* & & \\ Y^* & W^* & \mathrm{I}_{2n-2k} & Z & R_1 & R_2 \\ & & & I_{2n-2k} & & \\ & & & W & \mathrm{I}_{4k-2n} & \\ & & & Y & G_3 & \mathrm{I}_{2n-2k} \end{pmatrix}$$

an element in the unipotent radical of $\mathcal{H}^{w_k}$. Then we have

$$fRf^{-1} = \begin{pmatrix} \mathrm{I} & & & * & & \\ * & \mathrm{I} & & * & & \\ * & * & \mathrm{I} & DZD\nu^t\nu & DR_1\nu G_1^{-t}\nu & DR_2\nu D^{-t}\nu \\ & & & \mathrm{I} & & \\ & & & G_1 W \nu D^t \nu & \mathrm{I} & \\ & & & DY\nu D^t\nu & DG_3G_1^{-1} & \mathrm{I} \end{pmatrix}.$$

By Eq. (3.2), we obtain the following conditions:

$$\nu Y + W^t J W - Y^t \nu = 0, \ \textit{and} \ \ G_3 = -\nu W^t J.$$

It follows that the dimension of Y is $\frac{(2n-2k)(2n-2k+1)}{2}$, the dimension of Z is $\frac{(2n-2k)(2n-2k-1)}{2}$, and the dimensions of R_1 and W are $(4k-2n) \times (2n-2k)$. Now it is an easy calculation to show that

$$\delta_{\mathcal{H}^{w_k}}^{-1}(h) = |\det D|^{2(2n-2k)+2(4k-2n)} = |\det g|^{2k}$$

and

$$|\det g|^{\frac{s}{2}}\delta_P^{\frac{1}{2}}(g) = |\det g|^{\frac{s}{2}}|\det g|^{\frac{2n-1}{2}} = |\det g|^{n+\frac{s-1}{2}}.$$

Eq. (3.3) becomes

$$|\det g|^{n+\frac{s-1}{2}}\tau(g) = \psi(\mathrm{tr}R_2)|\det g|^{2k}.$$

Hence in order for the double coset $Pw_k\mathcal{H}$ with the condition that $2k \geq n$ to be admissible, the irreducible representation τ must has a nonzero $\theta_{\mathcal{N}_{2n-2k}}$-model when $s = 1$. This proves Theorem 1.3.

3.2. **On the existence and the uniqueness.** From the proof of Theorem 1.3 given above, we have the following existence result.

Corollary 3.1. *Let τ be an irreducible, unitary, admissible representation of GL_{2n}. Assume that τ has a nonzero $\theta_{\mathcal{N}_{2n-2k}}$-functional for some k with $[\frac{n}{2}] \leq k \leq n$. Then the induced representation*

$$\mathrm{Ind}_{P_{2n}}^{\mathrm{SO}_{4n}}(\tau \otimes |\det|^{\frac{s}{2}})$$

has nonzero generalized Shalika functionals only when $s = 1$.

By comparing with Theorem 1.1, we may ask if such generalized Shalika functionals exist on the induced representation

$$\mathrm{Ind}_{P_{2n}}^{\mathrm{SO}_{4n}}(\tau \otimes |\det|^{\frac{1}{2}})$$

and if they are also unique.

For the existence, we may assume that the representation τ has a nonzero $\theta_{\mathcal{N}_{2n-2k}}$-functional with the smallest possible k satisfying $[\frac{n}{2}] \leq k \leq n$. Then the corresponding admissible double coset produces a nonzero quasi-invariant functional with $s = 1$. It is technical to show that such a quasi-invariant functional is in fact supported on a closed $P \times \mathcal{H}$-stable subset of SO_{4n}, which implies that it extends to a nonzero generalized Shalika functional on the induced representation with $s = 1$. Hence we will not pursue that matter in this paper. We leave further discussion towards the end of Section 4 related to Theorem 4.2.

Recall that the local uniqueness of the generalized Shalika functionals for irreducible admissible representations of SO_{4n} was proved in [N10]. However, the representation $\mathrm{Ind}_{P_{2n}}^{\mathrm{SO}_{4n}}(\tau \otimes |\det|^{\frac{1}{2}})$ may be reducible. From the proof of Theorem 1.3, it is easy to see that such a local uniqueness problem reduces to the following problem on intersection of different models in the family of $\theta_{\mathcal{N}_r}$-models for irreducible admissible representations of GL_{2n}. It is not clear to us in general if the models in this family are disjoint in the sense that there exists no irreducible admissible representation τ having more than one models in this family, except that when τ is supercuspidal. In this case, τ can not have $\theta_{\mathcal{N}_r}$-models for $0 < r < n$ because of the supercuspidality of τ. Since an irreducible supercuspidal representation GL_{2n} is generic, i.e. has a nonzero Whittaker model. By [HR90], the symplectic model, which is the $\theta_{\mathcal{N}_0}$-model here, is disjoint with the Whittaker model for GL_{2n}. Hence an irreducible supercuspidal representation τ of GL_{2n} has the only possibility to have the $\theta_{\mathcal{N}_n}$-model, which is the Shalika model. This leads to the uniqueness assertion in Theorem 1.1, since uniqueness of Shalika model is known.

In order to extend this uniqueness to the case with more general representation τ, we prove a more general disjointedness result (Theorem 3.2) below. To state the result, we denote the standard maximal parabolic subgroup of GL_m by

$$\mathrm{P}_{r,m-r} = \mathrm{M}_{r,m-r}\mathrm{N}_{r,m-r}$$

for any integer $0 \leq r \leq m$ with $m \in \mathbb{N}$. Also, the Whittaker character ψ_{U_m} of U_m is given by

$$\psi_{\mathrm{U}_m}(u) = \psi(\sum_{i=1}^{m-1} u_{i,i+1}), \text{ for } u = (u_{i,j}) \in \mathrm{U}_m.$$

For $2k \leq m$, and given any representation (not necessarily irreducible) τ_{m-2k} of GL_{m-2k}, define

$$\pi(\tau_{m-2k}) = \mathrm{Ind}_{A_{m,2k}}^{\mathrm{GL}_m}(\tau_{m-2k} \otimes 1_{\mathrm{Sp}_{2k}} \otimes 1_{\mathrm{N}_{m-2k,2k}}),$$

where

$$A_{m,2k} = (\mathrm{GL}_{m-2k} \otimes \mathrm{Sp}_{2k})\mathrm{N}_{m-2k,2k}$$

whose elements are expressed in matrix form as

$$\begin{pmatrix} \mathrm{GL}_{m-2k} & * \\ 0 & \mathrm{Sp}_{2k} \end{pmatrix}.$$

Theorem 3.2. *For $0 < 2k \leq m$ and any representation τ_{m-2k} of GL_{m-2k}, the Whittaker model of GL_m and the model $\pi(\tau_{m-2k})$ of GL_m are disjoint in the sense that there is no GL_m-intertwining mapping between the Whittaker model and the model $\pi(\tau_{m-2k})$ of GL_m.*

Proof. When $m = 2k$, this theorem reduces to the disjointedness of Whittaker model and Symplectic model, which is known by the result of Klyachko for finite field case and by Heumos and Rallis for p-adic field case ([HR90]).

Now we consider the disjointedness between Whittaker model and $\pi(\tau_{m-2k})$ for $0 < 2k < m$. Let

$$\mathrm{P} = \mathrm{P}_{m-2k,2k} = \mathrm{M}_{m-2k,2k}\mathrm{N}_{m-2k,2k}.$$

Let S' be a complete set of representatives for $W(\mathrm{P})\backslash W(\mathrm{GL}_m)$, where $W(G)$ denotes the Weyl group of a group G. By Bruhat decomposition,

$$\mathrm{P}\backslash \mathrm{GL}_m/\mathrm{U}_m = \cup_{w\in S'}\mathrm{P}w\mathrm{U}_m,$$

where U_m denotes the upper triangular unipotent subgroup of GL_m. Let

$$\Lambda = \{\mathrm{diag}(\mathrm{I}_{m-2k}, B) \mid B \in B'\},$$

where B' is a complete set of representatives for $\mathrm{Sp}_{2k}\backslash \mathrm{GL}_{2k}$. Then

$$A_{m,2k}\backslash \mathrm{GL}_m/\mathrm{U}_m = \cup_{w\in S',\lambda\in\Lambda}A_{m,2k}w\lambda\mathrm{U}_m.$$

For any $w \in S'$ and $\mathrm{diag}(\mathrm{I}_{m-2k}, B) \in \Lambda$, we claim that

$$A_{m,2k}\mathrm{diag}(\mathrm{I}_{m-2k}, B)w\mathrm{U}_m$$

is not an admissible coset. Let

$$D' = \begin{pmatrix} Q' & R' \\ & T' \end{pmatrix} \in A_{m,2k},$$

where $Q' \in \mathrm{GL}_{m-2k}$ and $T' \in \mathrm{Sp}_{2k}$. Write

$$D = \begin{pmatrix} Q & R \\ & T \end{pmatrix} = \mathrm{diag}(\mathrm{I}_{m-2k}, B)^{-1}D'\mathrm{diag}(\mathrm{I}_{m-2k}, B).$$

If the conjugate action of w^{-1} on D (which maps D to $w^{-1}Dw$) separates T into more than two blocks, then some element n in the unipotent radical of P will make $\psi_{\mathrm{U}_m}(n) \neq 1$, and the coset is not admissible.

We use the following example to explain the idea and the method. Let

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix},\ T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \text{ and } R = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}.$$

Assume that $w = (1, 4)$, i.e. the permutation which interchanges the first and the fourth blocks, then

$$w^{-1}Dw = \begin{pmatrix} T_4 & 0 & T_3 & 0 \\ R_4 & Q_4 & R_3 & Q_3 \\ T_2 & 0 & T_1 & 0 \\ R_2 & Q_2 & R_1 & Q_1 \end{pmatrix}.$$

For some element D with nontrivial R_3 part in $w^{-1}Dw$, we have

$$\psi_{\mathrm{U}_m}(w^{-1}Dw) \neq 1$$

and the coset is not admissible.

Next we assume that the conjugate action of w^{-1} keeps the symplectic part T-block a whole piece (may perform a permutation on the T-block's interior). Let $w' = w|_{\mathrm{Sp}_{2k}}$ be the restriction of w to Sp_{2k}-part (as its embedding in GL_m) and consider the restriction of this coset to its corresponding GL_{2k} part (i.e. the coset $\mathrm{Sp}_{2k}Bw'\mathrm{U}_{2k}$ in GL_{2k}). Then we can see this coset can not be admissible by the disjointedness of Whittaker model and Symplectic model.

This completes the proof. $\square$

Now we apply Theorem 3.2 to the case of the mixed model $\theta_{\mathcal{N}_r}$-model in GL_{2n}. Let $m = 2n$, and for $0 \leq 2r < 2n$, take

$$\tau = \tau_{2r} = \mathrm{Ind}_{\mathcal{S}_r}^{\mathrm{GL}_{2r}}(\psi_{\mathcal{S}_r}).$$

It is clear from induction by stages that

$$\mathrm{Ind}_{\mathcal{N}_r}^{\mathrm{GL}_{2n}}(\theta_{\mathcal{N}_r}) = \mathrm{Ind}_{(\mathrm{GL}_{2r}\otimes\mathrm{Sp}_{2n-2r})N_{2r,2n-2r}}^{\mathrm{GL}_{2n}}(\tau_{2r} \otimes 1_{\mathrm{Sp}_{2n-2r}} \otimes 1_{N_{2r,2n-2r}}).$$

As a consequence of Theorem 3.2, we have the following Proposition.

Proposition 3.3. *For $0 \leq r < n$, the $\theta_{\mathcal{N}_r}$-model and Whittaker model are disjoint as representations of* GL_{2n}.

Finally the uniqueness for the induced representation in this case follows.

Corollary 3.4. *Let τ be an irreducible, unitary, generic representation of* GL_{2n}. *Then the induced representation*

$$\mathrm{Ind}_{P_{2n}}^{\mathrm{SO}_{4n}}(\tau \otimes |\det|^{\frac{s}{2}})$$

has at most one nonzero generalized Shalika functional up to scalar. If it exists, then $s = 1$ and τ has a nonzero Shalika functional.

When τ is a non-generic representation of GL_{2n}, such a uniqueness will be a more technical issue. We omit the discussion here.

We remark that it is also very interesting to study further properties of this new family of models, $\theta_{\mathcal{N}_r}$-models, and find applications to representation theory and automorphic forms, following the lines of ideas in [OS07], [OS08-1], and [OS08-2].

4. Conjectures Related to the Generalized Shalika Models

We are going to discuss relations of our previous work on the generalized Shalika models to the local Langlands functoriality and state conjectures on these issues.

The study of the generalized Shalika models for irreducible admissible representations of SO_{4n} over a p-adic local field F can be viewed as a special case of establishing the general theory of representations and harmonic analysis related to spherical varieties. In this case, the generalized Shalika subgroup $\mathcal{H}_{2n}$ is a spherical subgroup of SO_{4n}. From the general theory of spherical varieties ([K96]), from the context of the geometric Langlands program ([GN10] and [GN09]), and from the harmonic analysis on spherical functions ([S08]), for each spherical variety X attached to a reductive algebraic group G over F, there exists a dual group $G_X^\vee$ which is a subgroup of the Langlands dual group ${}^LG$ of G, such that the irreducible admissible representations of G attached to the spherical variety X are conjecturally parametrized in terms of the dual group $G_X^\vee$ through the local Langlands functoriality conjecture.

In the summer of 2009, in the workshop on Relative Trace Formula and Periods of Automorphic Forms at The American Institute of Mathematics, Y. Sakellaridis gave a wonderful lecture announcing his joint work with A. Venkatesh on periods and harmonic analysis on spherical varieties, which motivated us to consider the relative functorial parametrization of distinguished representations of p-adic groups. In a sense, the conjectures and the results discussed here support their more general conjectures on Plancherel formula on spherical varieties. We are taking a close look at the results obtained in terms of the generalized Shalika models from this perspective.

Recall that the complex dual group of F-split SO_{4n} is $\mathrm{SO}_{4n}(\mathbb{C})$. Let $\mathcal{W}_F$ be the local Weil group of F. Recall from [A05] that a local Arthur parameter is a homomorphism

$$\psi \;:\; \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{4n}(\mathbb{C})$$

such that the restriction of ψ to $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ is algebraic and the restriction of ψ to $\mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C})$ is a tempered local Langlands parameter. Assume that the local Arthur conjecture holds, that is, there exists a finite set of irreducible admissible representations of SO_{4n} attached to the local Arthur parameter ψ, which is called **the local Arthur packet of** ψ and is denoted by $\Pi(\psi)$. For each local Arthur parameter ψ, one defines a local Langlands parameter

$$\phi_\psi \;:\; \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{4n}(\mathbb{C})$$

by

$$\phi_\psi(w,h) := \psi(w,h,\begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}).$$

For this local Langlands parameter ϕ_ψ, following the local Langlands conjecture, there exists a finite set of irreducible admissible representations of SO_{4n}, which is denoted by $\Pi(\phi_\psi)$ and is called **the local L-packet associated to ϕ_ψ**. Conjecturally, the local L-packet $\Pi(\phi_\psi)$ should be contained in the local Arthur packet $\Pi(\psi)$. See the recent work of Moeglin ([M09]) for more discussion on this issue.

In general, the local Arthur parameter ψ is a direct sum of stable local Arthur parameters

$$\psi = \psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r \tag{4.1}$$

where the stable local Arthur parameters ψ_i $(i = 1, 2, \cdots, r)$ are given by

$$\psi_i := (\rho_i, a_i, b_i)$$

where a_i and b_i are positive integers, which are the dimensions of the irreducible representations of the two copies of $\mathrm{SL}_2(\mathbb{C})$, respectively, and ρ_i is irreducible continuous homomorphism from the local Weil group $\mathcal{W}_F$ to $\mathrm{GL}_{d_{\rho_i}}(\mathbb{C})$.

We first consider the result from Theorem 1.1 obtained in [JQ07]. The Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ with τ an irreducible supercuspidal representation of GL_{2n} has a stable local Arthur parameter

$$\psi = (\rho_\tau, 1, 2)$$

where ρ_τ is the local Langlands parameter attached to τ by the local Langlands conjecture for GL_{2n} ([HT01] and [H00]). Of course the local Langlands parameter for the quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ is

$$\phi_\psi = \rho_\tau |\cdot|^{\frac{1}{2}} \oplus \rho_\tau^\vee |\cdot|^{-\frac{1}{2}}.$$

In this case, the spherical space $X := \mathcal{H}_{2n} \backslash \mathrm{SO}_{4n}$ has the dual group

$$G_X^\vee = \mathrm{Sp}_{2n}(\mathbb{C}).$$

Theorem 1.1 shows that if the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ has a nonzero generalized Shalika model, i.e. is $(\mathcal{H}_{2n}, \psi_{\mathcal{H}_{2n}})$-distinguished if and only if τ has a nonzero Shalika model. By Theorem 1.1, [JNQ10-1], the irreducible supercuspidal representation τ has a nonzero Shalika model if and only if τ is the image under the local Langlands functorial transfer from an irreducible generic supercuspidal representation of SO_{2n+1}. This means that the local Langlands parameter

$$\psi_{1,2} := (\rho_\tau, 1)$$

factors through the complex dual group $\mathrm{Sp}_{2n}(\mathbb{C})$ of SO_{2n+1}. Since $\mathrm{Sp}_{2n}(\mathbb{C})$ is also the dual group $G_X^\vee$ associated to the spherical variety $X = \mathcal{H}_{2n} \backslash \mathrm{SO}_{4n}$, the result in Theorem 1.1 proved in [JQ07] can be expressed by the following diagram:

$$\begin{tikzcd}
\mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \arrow[rr] \arrow[dr] & & \mathrm{SO}_{4n}(\mathbb{C}) \\
& G_X^\vee \times \mathrm{SL}_2(\mathbb{C}) \arrow[ur] &
\end{tikzcd} \tag{4.2}$$

where $\psi_{1,2}(\mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}))$ is included in $G_X^\vee$ and the restriction to the second copy of $\mathrm{SL}_2(\mathbb{C})$ of the stable local Arthur parameter $\psi = (\rho_\tau, 1, 2)$ which is denoted by ψ_3, is the identity homomorphism from $\mathrm{SL}_2(\mathbb{C})$ onto $\mathrm{SL}_2(\mathbb{C})$. This interpretation leads to the following conjecture.

Conjecture 4.1. *Let ψ be a local Arthur parameter of SO_{4n}. Let $\Pi(\phi_\psi)$ be the local L-packet associated to the local Langlands parameter ϕ_ψ, which is included in the local Arthur packet $\Pi(\psi)$ associated to the local Arthur parameter ψ. Then there exists a member in $\Pi(\phi_\psi)$ having a nonzero generalized Shalika model if and only if the local Arthur parameter ψ factors through $G_X^\vee \times \mathrm{SL}_2(\mathbb{C})$, which is a subgroup of $\mathrm{SO}_{4n}(\mathbb{C})$, i.e.*

$$\psi(\mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})) \subset G_X^\vee \times \mathrm{SL}_2(\mathbb{C}).$$

Furthermore, the projection to the $\mathrm{SL}_2(\mathbb{C})$ of $\psi(1 \times 1 \times \mathrm{SL}_2(\mathbb{C}))$ yields the identity homomorphism from the second copy of $\mathrm{SL}_2(\mathbb{C})$ in $\mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ onto $\mathrm{SL}_2(\mathbb{C})$.

Next, we interpret the new results obtained in this paper along the line of Conjecture 4.1. We start with Theorem 1.2. If an irreducible admissible representation σ of SO_{4n} has a nonzero generalized Shalika model, then it can be realized as a quotient of an induced representation of SO_{4n} of type

$$\mathrm{Ind}_{P_{2n}}^{\mathrm{SO}_{4n}}(\tau|\det|^{\frac{1}{2}}) \tag{4.3}$$

where τ is an irreducible admissible representation of GL_{2n}. Let ϕ_τ be the local Langlands parameter associated to τ by the local Langlands conjecture for GL_{2n} (the Local Langlands Reciprocity Theorem of Harris-Taylor ([HT01]) and of Henniart ([H00])). Then the expected local Langlands parameter ϕ_σ for σ should be

$$\phi_\sigma = \phi_\tau|\cdot|^{\frac{1}{2}} \oplus \phi_\tau^\vee|\cdot|^{-\frac{1}{2}}.$$

Assume that the irreducible admissible representation τ of GL_{2n} is an Arthur representation, i.e. τ has a local Arthur parameter ψ_τ. Then we write ψ_τ as a direct sum of stable ones

$$\psi_\tau = \oplus_{j=1}^r \psi_j = \oplus_{j=1}^r (\rho_j, a_j, l_j).$$

Then the local Arthur parameter of σ should be

$$\psi_\sigma = \oplus_{j=1}^{r}(\rho_j, a_j, 2l_j).$$

In other words, Theorem 1.2 shows, under the local Arthur parametrization conjecture, that if ψ is the local Arthur parameter of σ such that $\sigma \in \Pi(\phi_\psi)$ (the local L-packet included in the local Arthur packet $\Pi(\psi)$) and if σ has a nonzero generalized Shalika model, then in the expression as a formal sum of stable local Arthur parameters

$$\psi = \oplus_{j=1}^{r}(\rho_j, a_j, b_j),$$

all the integers b_j must be even, i.e. $b_j = 2l_j$ for $j = 1, 2, \cdots, r$. In this case, σ can be realized as a quotient of the induced representation (4.3) such that the irreducible admissible representation τ belongs to the local L-packet $\Pi(\phi_{\psi_\tau})$, where the local Arthur parameter ψ_τ of τ can be expressed as

$$\psi_\tau = \oplus_{j=1}^{r}\psi_j = \oplus_{j=1}^{r}(\rho_j, a_j, l_j).$$

This proves the following relation between the generalized Shalika models and the local Arthur parameters, assuming that the local Arthur parametrization conjecture ([A05] and [M09]) holds.

Theorem 4.2. *Let* $\psi = \oplus_{j=1}^{r}(\rho_j, a_j, b_j)$ *be a local Arthur parameter of* SO_{4n}. *Let* $\Pi(\phi_\psi)$ *be the local L-packet associated to the local Langlands parameter* ϕ_ψ, *which is included in the local Arthur packet* $\Pi(\psi)$ *associated to the local Arthur parameter* ψ. *If there exists a member in* $\Pi(\phi_\psi)$ *having a nonzero generalized Shalika model, then in the local Arthur parameter* ψ, *the integers* b_j*'s are all even.*

Based on this, we can explain the result of Theorem 1.3 along the line of Conjecture 4.1, which leads the converse of Theorem 4.2. To this end, we state a conjecture for GL_{2n}.

Conjecture 4.3. *Let* ϕ_τ *be the local Langlands parameter associated to the irreducible admissible unitary representation* τ *of* GL_{2n}. *Then* τ *has a nonzero* $\theta_{\mathcal{N}_r}$*-model with* $0 \le r \le n$ *if and only if* ϕ_τ *is of symplectic type, i.e.*

$$\phi_\tau(\mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C})) \subset \mathrm{Sp}_{2n}(\mathbb{C}).$$

This conjecture is true when τ is supercuspidal, according to Theorem 1.1, [JNQ10-1]. Proposition 3.3 shows that when τ is generic, then if τ has a nonzero $\theta_{\mathcal{N}_r}$-model, then $r = n$ and τ has a nonzero Shalika model. It is expected that τ is of symplectic type. When $r = 0$, the symplectic models for the Speh representations of GL_{2n} was considered in [OS07].

Combining Theorem 1.2 with Conjecture 4.3, the result of Theorem 1.3 predicts that if an irreducible admissible representation σ of SO_{4n} has a local Arthur parameter ψ_σ and has a nonzero generalized Shalika model, then ψ_σ factors through the subgroup $G_X^\vee \times \mathrm{SL}_2(\mathbb{C})$ of $\mathrm{SO}_{4n}(\mathbb{C})$. This explanation of our new results in this paper supports Conjecture 4.1.

At this point, we do not have any more evidence to support our conjectures. However, it is not hard to see that assuming the local Arthur conjecture, the generalized Shalika models plays a role towards the endoscopy structure of the local Arthur packets via the endoscopy transfers. Such an investigation will lead to the converse of Theorem 4.2.

Another application of the generalized Shalika models to the endoscopy structures of irreducible generic supercuspidal representations of SO_{2n+1} was discussed in detail in [JNQ10-2]. We omit any further discussion here.

The whole theory discussed for orthogonal groups ([JQ07], [JNQ08], [JNQ10-1], and [JNQ10-2]) may be extended to other classical groups. We will go back to these topics in our future work.

References

[A05] Arthur, J., *An introduction to the trace formula.* Harmonic analysis, the trace formula, and Shimura varieties, 1–263, Clay Math. Proc., 4, Amer. Math. Soc., Providence, RI, 2005.

[BZ76] Bernstein, I. N.; Zelevinsky, A. V. *Representations of* $\mathrm{GL}(n, F)$*, where* F *is a non-archimedean local field.* Russ. Math. Surveys 31 (1976), No. 3, 1-68.

[BZ77] Bernstein, I. N.; Zelevinsky, A. V. *Induced representations of reductive p-adic groups. I.* Ann. Sci. École Norm. Sup. (4) 10 (1977), no. 4, 441–472.

[CM93] Collingwood, D.; McGovern, W. *Nilpotent orbits in semisimple Lie algebras.* Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[GN10] Gaitsgory, Dennis; Nadler, David *Spherical varieties and Langlands duality.* Mascow Math. J. Volume 10 (2010), no 1. (Special Issue: in honor of Pierre Deligne) 63–137.

[GN09] Gaitsgory, Dennis; Nadler, David *Hecke operators on quasimaps into horospherical varieties.* Doc. Math. 14 (2009), 19–46.

[G95] Ginzburg, David *On spin L-functions for orthogonal groups.* Duke Math. J. 77 (1995), no. 3, 753–798.

[HT01] Harris, M.; Taylor, R. *The geometry and cohomology of some simple Shimura varieties.* Annals of Math. Studies, 151, 2001, Princeton University Press.

[H00] Henniart, G. *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps* p*-adique.* (French) Invent. Math. 139 (2000), no. 2, 439–455.

[HR90] Heumos, M.; Rallis, S. *Symplectic-Whittaker models for* GL_n. Pacific J. Math. 146(1990), no. 2, 247–279.

[JR96] Jacquet, H.; Rallis, S.: *Uniqueness of linear periods.* Compositio Math. **102**, no. 1, 65–123 (1996)

[JS90] Jacquet, H.; Shalika, J.: *Exterior square* L*-functions.* Automorphic forms, Shimura varieties, and L-functions **II** (Ann Arbor, MI, 1988), 143–226, Perspect. Math., 11, Academic Press, Boston, MA (1990)

[JNQ08] Jiang, Dihua; Nien, Chufeng; Qin, Yujun *Local Shalika models and functoriality.* Manuscripta Math. 127(2008), 187-217.

[JNQ10-1] Jiang, Dihua; Nien, Chufeng; Qin, Yujun *On symplectic supercuspidal representations of* GL_{2n} *over* p*-adic fields.* Pacific J. of Math. 245 (2010)(2), 273–313.

[JNQ10-2] Jiang, Dihua; Nien, Chufeng; Qin, Yujun *Symplectic Supercuspidal Representations and Related Problems.* Science China Math. Vol. 53, No. 3, (2010), 533–546.

[JQ07] Jiang, Dihua; Qin, Yujun *Residues of Eisenstein series and generalized Shalika models for* $\mathrm{SO}(4n)$. J. of the Ramanujan Math. Soc. 22, No. 2 (2007) 1–33.

[JR92] Jacquet, H.; Rallis, S. *Kloosterman integrals for skew symmetric matrices.* Pacific J. Math. **154**, No. 2, 265-283 (1992)

[K96] Knop, Friedrich *Automorphisms, root systems, and compactifications of homogeneous varieties.* J. Amer. Math. Soc. 9 (1996), no. 1, 153–174.

[M09] Moeglin, C. *Comparaison des paramétres de Langlands et des exposants à l'intérieur d'un paquet d'Arthur.* Journal of Lie Theory, 19 (2009)(4), 797–840.

[MW87] Moeglin, C.; Waldspurger, J.-L. *Modéles de Whittaker dégénérés pour des groupes* p*-adiques.* (French) Math. Z. 196 (1987), no. 3, 427–452.

[N09-1] Nien, Chufeng *Klyachko models for general linear groups of rank 5 over a* p*-adic field.* Canad. J. Math. 61 (2009), no. 1, 222–240.

[N09-2] Nien, Chufeng *Uniqueness of Shalika models.* Canadian Journal of Mathematics, V61, No 6, 2009, 1325-1340.

[N10] Nien, Chufeng *Local uniqueness of generalized Shalika models for* $\mathrm{SO}(4n)$. Journal of Algebra 323, 2010, 437–457.

[O06] Offen, Omer *On symplectic periods of the discrete spectrum of GL(2n).* Israel Journal of Mathematics, 154, 253-298, 2006.

[OS07] Offen, Omer; Sayag, Eitan *On unitary representations of* GL_{2n} *distinguished by the symplectic group.* J. of Number Theory, 125(2):344-355, 2007.

[OS08-1] Offen, Omer; Sayag, Eitan *Global mixed periods and local Klyachko models for the general linear group.* Int. Math. Res. Not. IMRN 2008, no. 1, Art. ID rnm 136, 25 pp.

[OS08-2] Offen, Omer; Sayag, Eitan *Uniqueness and disjointness of Klyachko models.* J. of Funct. Anal., 254(11):2846-2865, 2008.

[PSP08] Prasad, Dipendra; Schulze-Pillot, Rainer *Generalised form of a conjecture of Jacquet and a local consequence.* J. Reine Angew. Math. 616 (2008), 219–236.

[S08] Sakellaridis, Yiannis *On the unramified spectrum of spherical varieties over p-adic fields.* Compos. Math. 144 (2008), no. 4, 978–1016.

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