On the Langlands Functoriality for Automorphic Forms

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Contents

1 Introduction

This is extended version of my lecture notes at The Institute of Mathematics, The Chinese Academy of Science, Beijing, over the past few years. My lectures were mainly concerned with a few aspects of my research in the modern theory of automorphic forms, in particular, in the Langlands program. It is by no means that this paper covers the whole theory of automorphic forms. For instance, we will not discuss the trace formula approach to the Langlands functoriality, and the significant arithmetic applications of automorphic representations. Instead, we will refer the readers to [A05] and [CHT06] and [Ty06]. Also we will refer the readers to the recent ICM invited lectures [CPS02], [Hr02], [Sh02], [Sd06], [Ty02], and [Ty04]. It is for sure that this paper will have certain overlap with my previous paper [Jng04], but I will try to discuss more recent progress after [Jng04].

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2 Automorphic Representations

2.1 Cuspidal automorphic forms

Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. For simplicity, take G to be a reductive k-split algebraic group. In most cases, G will be a k-split classical group, and in particular, take $G = \mathrm{SO}_{2n+1}$, which is given as follows. For example, one takes J_{2n+1} as defined inductively by

$$J_{2n+1} := \begin{pmatrix} & & 1 \\ & J_{2n-1} & \\ 1 & & \end{pmatrix},$$

and defines $\mathrm{SO}_{2n+1} = \{g \in \mathrm{GL}_{2n+1} \mid {}^t g J_{2n+1} g = J_{2n+1}, \det g = 1\}$. We denote by B a Borel subgroup of G. When $G = \mathrm{GL}_m$ or SO_{2n+1}, B can be taken to be the subgroup consisting of all upper-triangular matrices in G. Then there is a Levi decomposition

$$B = TU$$

where T is the maximal k-split torus and U is the unipotent radical of B. When $G = \mathrm{GL}_m$ or SO_{2n+1}, T is the diagonal subgroup and U consists of all upper-triangular matrices in G with all diagonal entries 1. Let $P = MN$ be a standard parabolic subgroup of G which contains B. When $G = \mathrm{GL}_m$, M is isomorphic to $\mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_r}$ with $m = m_1 + \cdots + m_r$, and N is given by the following unipotent elements of GL_m

$$\begin{pmatrix} I_{m_1} & * & * \\ & \ddots & * \\ & & I_{m_r} \end{pmatrix}.$$

For detailed discussions on algebraic groups we prefer to [Sp98].

It is known that the diagonal embedding of $G(k)$ into $G(\mathbb{A})$ has discrete image in $G(\mathbb{A})$ and the quotient $Z_G(\mathbb{A}) \cdot G(k) \backslash G(\mathbb{A})$ has finite volume with respect to the canonical Haar measure on the quotient space, where Z_G denotes the center of G. We consider the following L^2-space

$$L^2(G, \omega) = L^2(G(k) \backslash G(\mathbb{A}))_\omega,$$

which consists of all $\mathbb{C}$-valued square integrable functions f on $G(k) \backslash G(\mathbb{A})$, such that

$$f(zg) = \omega(z) f(g),$$

where $z \in Z_G(\mathbb{A})$ and ω is a character of $Z_G(k) \backslash Z_G(\mathbb{A})$. Naturally, the space $L^2(G, \omega)$ is a $G(\mathbb{A})$-module structure given by

$$(g \cdot f)(x) = f(xg)$$

for all $g, x \in G(\mathbb{A})$ and $f \in L^2(G, \omega)$. A function $f \in L^2(G, \omega)$ is called *cuspidal* if the following integral

$$\int_{N(k)\backslash N(\mathbb{A})} f(ng)dn$$

is zero for almost all $g \in G(\mathbb{A})$ and for the unipotent radical N of all parabolic subgroup $P = MN$ of G. If a smooth cuspidal function $f \in L^2(G, \omega)$ generates an irreducible $G(\mathbb{A})$-submodule in $L^2(G, \omega)$, then f is called a cuspidal automorphic form on $G(\mathbb{A})$. We prefer to [BrJ79] or [MW95] for a formal definition of cuspidal automorphic forms. Any irreducible $G(\mathbb{A})$-submodule of $L^2(G, \omega)$ generated by a cuspidal automorphic form is called an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Let (π, V_π) be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Then any function in V_π is cuspidal.

Let $L^2_c(G, \omega)$ be the submodule in $L^2(G, \omega)$ generated by all irreducible cuspidal $G(\mathbb{A})$-submodules in $L^2(G, \omega)$. Then it is a theorem of Gelfand and Piatetski-Shapiro that the cuspidal spectrum $L^2_c(G, \omega)$ decomposes into a Hilbert sum of irreducible cuspidal $G(\mathbb{A})$-modules with finite multiplicity. Any irreducible $G(\mathbb{A})$-submodule in $L^2(G, \omega)$ which is not cuspidal is called an irreducible non-cuspidal, square-integrable automorphic representation of $G(\mathbb{A})$. The submodule of $L^2(G, \omega)$ generated by all irreducible non-cuspidal, square-integrable automorphic representation of $G(\mathbb{A})$ is denoted by $L^2_r(G, \omega)$, which can be realized as residues of Eisenstein series on $G(\mathbb{A})$ following the Langlands theory of Eisenstein series ([L76] and [MW95]). The space $L^2_c(G, \omega) \oplus L^2_d(G, \omega)$ is often called the discrete spectrum of $G(\mathbb{A})$ and is denoted by $L^2_d(G, \omega)$. Any irreducible $G(\mathbb{A})$-submodule in $L^2_d(G, \omega)$ is called an irreducible discrete series automorphic representation of $G(\mathbb{A})$.

2.2 Satake parameters

Let Ω_k be the set of all local places of k and Ω_∞ (or Ω_f, resp.) be the subset of Ω_k consisting of all archimedean (or nonarchimedean, resp.) local places of k. Since G is k-split, for each $v \in \Omega_f$, $K_v = G(\mathbb{Z}_v)$ is a maximal compact subgroup, and for $v \in \Omega_\infty$, K_v is the maximal compact subgroup of $G(\mathbb{R})$ associated to a Cartan involution. Then $K = \prod_{v \in \Omega_k} K_v$ is a maximal compact subgroup of $G(\mathbb{A})$ with the Iwasawa decomposition

$$G(\mathbb{A}) = P(\mathbb{A}) \cdot K.$$

In the sense of Kirillov and Bernstein, the adelic group $G(\mathbb{A})$ is tame ([Cl06]). It follows that an irreducible unitary representation π of $G(\mathbb{A})$ can be written as a restricted tensor product $\pi = \otimes_{v\in\Omega_k}\pi_v$, where (π_v, V_{π_v}) is an irreducible unitary representation of $G(k_v)$, and for almost all finite places v, Y_{π_v} has K_v-invariant vectors, i.e. $V_{\pi_v}^{K_v} \neq 0$, where

$$V_{\pi_v}^{K_v} = \{u \in V_{\pi_v} \ : \ \pi_v(h)(u) = u, \textit{for all } h \in K_v\}.$$

From Satake's theory of p-adic spherical functions ([St63]), we have the following properties,

(1) $\dim \pi_v^{K_v} \leq 1$.

(2) If $\pi_v^{K_v} \neq 0$ (π_v is spherical), there is a unramified character χ_v of $T(\mathbb{Q}_v)$ s.t. π_v is the irreducible spherical constituent of $\mathrm{Ind}_{B(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\chi_v)$, where $B = TU$ is a Borel subgroup of G.

(3) The K_v-invariant vector of π_v is characterized by a semi-simple conjugacy class t_{π_v} in the Langlands dual group ${}^LG$ (which will be defined below), which is called the Satake parameter attached to π_v.

For example, if $G = \mathrm{GL}(m)$, the Satake parameter for

$$\mathrm{Ind}_{B(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\chi_v)$$

is

$$t_{\chi_v} = \mathrm{diag}(q^{-s_1}, q^{-s_2}, \cdots, q^{-s_m}) \in \mathrm{GL}_m(\mathbb{C}),$$

if $\chi_v(\mathrm{diag}(t_1, t_2, \cdots, t_m)) = |t_1|^{s_1} \ldots |t_m|^{s_m}$.

When π_v is spherical, the dimension of $V_{\pi_v}^{K_v}$ is one. We choose a nonzero vector u_v° in V_{π_v}. Take all finite subset S of local places in Ω_k, such that S contains all archimedean local places of k and for any local place v, which is not contained in S, the local component π_v is spherical. Consider all the factorizable vectors of the following form

$$(\otimes_{v\in S}u_v) \otimes (\otimes_{v\notin S}u_v^\circ).$$

Then the set of all factorizable vectors generates a dense subspace of the irreducible unitary representation (π, V_π) with $\pi = \otimes_v\pi_v$.

2.3 Complex dual groups

For k-split reductive algebraic groups G, take T to be a maximal k-split torus of G. Let $R(T,G)$ be the set of roots of G with respect to T and $R^\vee(T,G)$ be the set of coroots of G with respect to T. Let X be the $\mathbb{R}$-vector space generated by $R(T,G)$ and $X^\vee$ be the $\mathbb{R}$-vector space generated by $R^\vee(T,G)$. Finally, let Δ be the set of simple root in $R(T,G)$ with respect to a given Borel subgroup $B = TU$, and $\Delta^\vee$ be the dual of Δ in $R^\vee(T,G)$. Then $(X,\Delta;X^\vee,\Delta^\vee)$ is the root datum attached to (G,B,T). It follows from a standard theorem in the theory of linear algebraic groups ([Sp98]) that G is determined over k, up to isogeny, by a combinatorical datum, called the root datum attached to G.

The complex dual group of G is the complex algebraic group $G^\vee$ determined, uniquely up to isogeny, by the root datum dual to the one of G. The relations are given by the following diagram

$$\begin{tikzcd}
G \arrow[r, Leftrightarrow] \arrow[d, leftrightarrow] & (X,\Delta;X^\vee,\Delta^\vee) \arrow[d, leftrightarrow] \\
{}^L G \arrow[r, Leftrightarrow] & (X^\vee,\Delta^\vee;X,\Delta)
\end{tikzcd}$$

For example, we have the following table

G	$G^\vee$
$\mathrm{GL}(m)$	$\mathrm{GL}(m,\mathbb{C})$
$\mathrm{SL}(m)$	$\mathrm{PGL}(m,\mathbb{C})$
$\mathrm{SO}(2n+1)$	$\mathrm{Sp}(2n,\mathbb{C})$
$\mathrm{Sp}(2n)$	$\mathrm{SO}(2n+1,\mathbb{C})$
$\mathrm{SO}(2n)$	$\mathrm{SO}(2n,\mathbb{C})$
G_2	$G_2(\mathbb{C})$

The Langlands dual group ${}^L G$ is defined to be the semiproduct of the complex dual group ${}^L G$ and the absolute Galois group $\Gamma_k = \mathrm{Gal}(\bar{k}/k)$. When G is k-split, then the semiproduct is a direct product. Hence we may take the complex dual group as the Langlands dual group.

2.4 Automorphic L-functions

Automorphic L-functions in the sense of Langlands are defined as follows ([Br79]). Let $\pi = \otimes_v \pi_v$ be an irreducible unitary automorphic representation of $G(\mathbb{A})$ and ρ be a m-dimensional complex representation of the dual group $G^\vee$. For any unramified local component π_v ($v < \infty$), the representation $\pi_v, V_{\pi_v})$ is completely determined by the Satake parameter t_{π_v} of π_v, which is a semi-simple conjugacy class in the complex dual group $G^\vee$. Hence $\rho(t_{\pi_v})$ is a semi-simple conjugacy class in $\mathrm{GL}_m(\mathbb{C})$. Since a semi-simple conjugacy class in $\mathrm{GL}_m(\mathbb{C})$ is completely determined by its characteristic polynomial, it is natural to define a local L-factor to be

$$L(s, \pi_v, \rho) := [\det(I - \rho(t_{\pi_v})q_v^{-s})]^{-1}.$$

It is a meromorphic function in variable s attached to the datum (π_v, ρ). It is easy to see that the local L-factor $L(s, \pi_v, \rho)$ determines π_v unique up to isomorphism, when π_v is spherical. In other words, $L(s, \pi_v, \rho)$ is an invariant attached to π_v.

In order to form a global invariant attached to $\pi = \otimes_v \pi_v$, Langlands defines the Langlands automorphic L-function attached to (π, ρ) by

$$L(s, \pi, \rho) := \prod_{v \in S} L(s, \pi_v, \rho) \times \prod_{v \notin S} L(s, \pi_v, \rho) = \prod_{v \in S} L(s, \pi_v, \rho) \times L^S(s, \pi, \rho)$$

where S is a finite subset of Ω_k, including Ω_∞ such that π_v is unramified if $v \notin S$. The partial L-function $L^S(s, \pi, \rho)$ is defined by

$$L^S(s, \pi, \rho) := \prod_{v \notin S} L(s, \pi_v, \rho)$$

For $v \in S$, it is a difficult problem to define $L(s, \pi_v, \rho)$. One may define them in terms of the local Langlands correspondence conjecture and the local Langlands functorial transfer conjecture from G to the general linear groups.

Theorem 2.1 (Langlands)**.** *For any pair (π, ρ) as given above, the eulerian product $L(s, \pi, \rho)$ or the partial eulerian product $L^S(s, \pi, \rho)$ converges absolutely for* $\mathrm{Re}(s)$ *large.*

It follows that for $\mathrm{Re}(s)$ large, the partial L-functions $L^S(s, \pi, \rho)$ or the complete L-functions $L(s, \pi, \rho)$ if it can be defined, are invariants of π. Just as the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}},$$

Langlands made the following conjecture for basic analytic properties of the automorphic L-functions.

Conjecture 2.2 (Langlands)**.** *Automorphic L-functions $L(s,\pi,\rho)$ have meromorphic continuation to $\mathbb{C}$ with finitely many poles on the real line $\mathbb{R}$, and enjoy the functional equation*

$$L(s,\pi,\rho)=\epsilon(s,\pi,\rho)L(1-s,\pi^\vee,\rho).$$

By the Langlands theory of Eisenstein series ([L76] and [MW95]), the general Eisenstein series attached to a cuspidal datum (P,π) has a meromorphic continuation. In the case when the cuspidal datum is maximal, i.e. the standard parabolic subgroup P is maximal, the constant terms of the Eisenstein series along a standard parabolic subgroup are zero except that the standard parabolic is P. In this case, the constant term of the Eisenstein series along P is expressible in terms of certain automorphic L-functions as defined above. It is easy to see that the automorphic L-functions occurring in a constant term of relevant Eisenstein series have a meromorphic continuation to the whole complex plane $\mathbb{C}$. However, it remains to determine the functional equation, the finiteness and the location of the poles of these automorphic L-functions. The method developed by Shahidi based on the Langlands' theory on the constant terms of Eisenstein series, which is called the Langlands-Shahidi method, has proved the Langlands conjecture for many cases, which is known as the Shahidi list ([L71], [Sh88], and [GlS88]). On the other hand, there is a different approach, which is often called as the Rankin-Selberg method, also depending on the theory of Eisenstein series. The method has proved many cases of the Langlands conjecture, which even include cases not in the Shahidi list. D. Bump's paper summarizes the state of art for the Rankin-Selberg method ([Bm05]). In [Jng06a], a new case of the Langlands conjecture was proved based on the recent progress on the Langlands functoriality. This case is not in the Shahidi's list and also can not be reached by the current understanding of the Rankin-Selberg method.

3 The Langlands Functoriality

We will discuss the conjecture of the Langlands functoriality and its different variants.

3.1 The global conjectures

For k-split groups, the Langlands principle of functoriality (or the conjecture of the Langlands functoriality) can be formulated as follows.

Conjecture 3.1 (The Langlands functoriality)**.** *Let G and H be k-split reductive algebraic groups and let r be any group homomorphism*

$$r \ : \ H^\vee \to G^\vee$$

from the complex dual group $H^\vee$ to the complex dual group $G^\vee$. For any irreducible admissible automorphic representation σ of $H(\mathbb{A})$, there is an irreducible admissible automorphic representation π of $G(\mathbb{A})$ such that

$$L(s, \pi, \rho) = L(s, \sigma, \rho \circ r)$$

for all finite-dimensional complex representations ρ of $G^\vee$.

In such a case, π is called a global Langlands functorial lifting of σ or an image of σ under the Langlands functorial lifting.

For general discussion of the Langlands functoriality, we refer to [A02a] and [A02b]. As explicitly explained in [A02a], the Langlands functoriality conjecture follows from the conjectural existence of the automorphic Langlands group L_k associated to the number field k. One of the basic properties of the automorphic Langlands group L_k is that the equivalence classes of irreducible admissible automorphic representations of $G(\mathbb{A})$ are parametrized by the $G^\vee$-conjugacy classes of the L-homomorphisms from L_k to $G^\vee$, up to L-distinguishedness. In other words, for each $G^\vee$-conjugacy class ϕ of the L-homomorphisms from L_k to $G^\vee$, which is also called a global Langlands parameter, there exists a subset $\Pi(\phi)$, called the global L-packet attached to ϕ, of equivalence classes of irreducible admissible automorphic representations π of $G(\mathbb{A})$ such that

$$L(s, \rho \circ \phi) = L(s, \pi, \rho)$$

for all finite-dimensional complex representations ρ of $G^\vee$ and all $\pi \in \Pi(\phi)$. For more detailed discussion, we refer to [A02a].

In terms of global L-packets, the Langlands functoriality predicts a Langlands functorial map which takes global L-packets of $H(\mathbb{A})$ to global L-packets of $G(\mathbb{A})$. More precisely, let ϕ_H be a global Langlands parameter for $H(\mathbb{A})$. Then the composition of ϕ_H with the group homomorphism r from $H^\vee$ into $G^\vee$ gives a global Langlands parameter

$$\phi_G := r \circ \phi_H$$

for $G(\mathbb{A})$. If we denote by $\Pi(\phi_H)$ the global L-packet of automorphic representations of $H(\mathbb{A})$ attached to the global Langlands parameter ϕ_H and by $\Pi(\phi_G)$ the global L-packet of automorphic representations of $G(\mathbb{A})$ attached to the global Langlands parameter ϕ_G. Then the Langlands functoriality predicts that for any $\sigma \in \Pi(\phi_H)$ and $\pi \in \Pi(\phi_G)$ the following identity holds

$$L(s, \pi, \rho) = L(s, \sigma, \rho \circ r) \tag{3.1}$$

for all finite-dimensional complex representations ρ of $G^\vee$.

3.2 The Langlands functorial transfers

For a given irreducible unitary cuspidal automorphic representation π of $G(\mathbb{A})$, there is another family of automorphic L-functions which can be defined as follows. For any integer $m \geq 1$, let τ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$. One defines the Rankin-Selberg convolution of L-function $L(s, \pi \times \tau)$ or the partial L-function $L^S(s, \pi \times \tau)$ in the same way. One may form the following conjecture.

Conjecture 3.2 (The Langlands Functorial Transfer). *Let G and H be k-split reductive algebraic groups and let r be any group homomorphism*

$$r \ : \ H^\vee \to G^\vee$$

from the complex dual group $H^\vee$ to the complex dual group $G^\vee$. For any irreducible admissible automorphic representation σ of $H(\mathbb{A})$, there is an irreducible admissible automorphic representation π of $G(\mathbb{A})$ such that

$$L(s, \sigma \times \tau) = L(s, \pi \times \tau)$$

for all irreducible unitary cuspidal automorphic representations τ of $\mathrm{GL}_m(\mathbb{A})$ *with m being all positive integers.*

In such a case, π is called a global Langlands functorial transfer of π or an image of σ under the global Langlands functorial transfer.

It is a basic problem to show whether the Langlands functorial lifting conjecture (Conjecture 3.1) is equivalent to the Langlands functorial transfer conjecture (Conjecture 3.2).

There is a preliminary version of the global Langlands functorial transfer, which is called the weak global Langlands functorial transfer, and can be formulated as follows in terms of the Satake parameters.

Conjecture 3.3 (The Weak Langlands Transfer)**.** *Let G and H be k-split reductive algebraic groups and let r be any group homomorphism*

$$r \ : \ H^\vee \to G^\vee$$

from the complex dual group $H^\vee$ to the complex dual group $G^\vee$. For any irreducible admissible automorphic representation σ of $H(\mathbb{A})$, there is an irreducible admissible automorphic representation π of $G(\mathbb{A})$ such that at almost all local place v of k, the Satake parameter t_{π_v} attached to the spherical local component π_v and the image $r(t_{\sigma_v})$ of the Satake parameter t_{σ_v} attached to the spherical local component σ_v share the same semi-simple conjugacy class in $G^\vee$.

It is clear that the global Langlands functorial transfer implies the weak Langlands transfer. The relation between the global Langlands functorial transfer and the weak Langlands transfer is essentially the compatibility of the global Langlands functoriality with the local Langlands functoriality at every local place of k. We discuss the corresponding local conjectures in the following subsections.

3.3 The local conjectures

Fix a local place $v \in \Omega_k$, and denote by k_v the local field of k at v. We refer to [Br79] and [Vg93] (see also [Jng06b]) for the detailed discussion on the local Langlands parameters.

Let $\mathcal{W}_{k_v}$ be the local Weil group of k_v. Consider continuous homomorphisms ϕ_v from the Weil-Deligne group $\mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C})$ to $G^\vee$, which is admissible in the sense of [Br79]. The $G^\vee$-conjugacy class of such a homomorphism ϕ_v is called a local Langlands parameter. The set of local Langlands parameters is denoted by $\Phi(G/k_v)$. Let $\Pi(G/k_v)$ be the set of equivalence classes of irreducible admissible complex representations of $G(k_v)$.

The local Langlands conjecture for G over k_v asserts that for each local Langlands parameter $\phi_v \in \Phi(G/k_v)$, there should be a finite subset

$\Pi(\phi_v)$, which is called the local L-packet attached to ϕ_v such that the set $\{\Pi(\phi_v) \mid \phi_v \in \Phi(G/k_v)\}$ is a partition of $\Pi(G/k_v)$, among other required properties ([Br79]). The map $\phi_v \mapsto \Pi(\phi_v)$ is called the local Langlands correspondence or the local Langlands reciprocity law for G over k_v.

First, we recall the local Langlands conjecture for GL_n over k_v, which is proved by Harris-Taylor [HT01] and by Henniart [Hn00] for $v < \infty$, and by Langlands for v archimedean.

Theorem 3.4 ([HT01], [Hn00], [Hn93]). *Over a p-adic local field k_v, there is a unique collection of bijections*

$$\mathrm{rec}_v \ : \ \Pi(\mathrm{GL}_n/k_v) \rightarrow \Phi(\mathrm{GL}_n/k_v)$$

for every $n \geq 1$ such that

1. *for $\tau_v \in \Pi(\mathrm{GL}_1/k_v)$, $\mathrm{rec}_v(\tau_v) = \pi_v \circ \mathrm{Art}_v^{-1}$, where Art_v is the local Artin reciprocity map from $k_v^\times$ to $\mathcal{W}_{k_v}^{\mathrm{ab}}$;*
2. *for $\tau_{1,v} \in \Pi(GL_{n_1}/k_v)$ and $\tau_{2,v} \in \Pi(GL_{n_2}/k_v)$,*

   $$L(s, \tau_{1,v} \times \tau_{2,v}) = L(s, \mathrm{rec}_v(\tau_{1,v}) \otimes \mathrm{rec}_v(\tau_{2,v}))$$

   and

   $$\epsilon(s, \tau_{1,v} \times \tau_{2,v}, \psi_v) = \epsilon(s, \mathrm{rec}_v(\tau_{1,v}) \otimes \mathrm{rec}_v(\tau_{2,v}), \psi_v)$$

   where ψ_v is a given nontrivial character of k_v.
3. *for $\tau_v \in \Pi(\mathrm{GL}_n/k_v)$ and $\chi_v \in \Pi(\mathrm{GL}_1/k_v)$,*

   $$\mathrm{rec}_v(\tau_v \otimes (\chi_v \circ \det)) = \mathrm{rec}_v(\tau_v) \otimes \mathrm{rec}_v(\chi_v);$$

4. *for $\tau_v \in \Pi(\mathrm{GL}_n/k_v)$ with central character $\omega_{\pi_v} = \chi_v$,*

   $$\det \circ \mathrm{rec}_v(\tau_v) = \mathrm{rec}_v(\chi_v);$$

5. *for $\tau_v \in \Pi(\mathrm{GL}_n/k_v)$, $\mathrm{rec}_v(\tau_v^\vee) = \mathrm{rec}_v(\tau_v)^\vee$, where $\vee$ denotes the contragredient.*

We note that the uniqueness of the such maps is proved in [Hn93]. We refer to [HT01] for historical remarks on the proof of the local Langlands conjecture for $\mathrm{GL}_n(k_v)$.

One can define as in [JPSS83] the local γ-factors by

$$\gamma(s, \tau_{1,v} \times \tau_{2,v}, \psi_v) = \epsilon(s, \tau_{1,v} \times \tau_{2,v}, \psi_v) \cdot \frac{L(1-s, \tau_{1,v}^\vee \times \tau_{2,v}^\vee)}{L(s, \tau_{1,v} \times \tau_{2,v})}. \tag{3.2}$$

Note that for $\mathrm{GL}_n(F)$, the local L-packets always contains one member. This fact follows from [Hn93] and the Bernstein-Zelevinsky classification theory ([BZ77] and [Z80]). We refer to [Jng06b] for detailed discussion on the local gamma factors in general.

The local Langlands functoriality conjecture can be formulated as follows. Let G and H be k_v-split reductive algebraic groups defined. For a group homomorphism r_v from $H^\vee$ to $G^\vee$, there should be a functorial lifting from $\Pi(H/k_v)$ to $\Pi(G/k_v)$, which takes local L-packets of $H(k_v)$ to local L-packets of $G(k_v)$, and satisfies the following conditions.

1. For any local Langlands parameter $\phi_{H,v} \in \Phi(H/k_v)$,

   $$\phi_{G,v} := r_v \circ \phi_{H,v}$$

   is a local Langlands parameter in $\Phi(G/k_v)$, such that the functorial lifting ρ takes the local L-packet $\Pi(\phi_{H,v})$ to the local L-packet $\Pi(\phi_{G,v})$.

2. For any finite-dimensional complex representation ρ of $G^\vee$ and for $\sigma_v \in \Pi(\phi_{H,v})$ and $\pi_v \in \Pi(\phi_{G,v})$, one has

   $$L(s, \pi_v, \rho) = L(s, \sigma_v, \rho \circ r_v),$$

   and

   $$\epsilon(s, \pi_v, \rho, \psi_v) = \epsilon(s, \sigma_v, \rho \circ r_v, \psi_v).$$

It follows that $\gamma(s, \pi_v, \rho, \psi_v) = \gamma(s, \sigma_v, \rho \circ r_v, \psi_v)$. In this case, π_v is called a local Langlands functorial lifting of σ_v, or an image of σ_v under the local Langlands functorial lifting.

3.4 The local Langlands functorial transfer

From the formulation of the local Langlands conjecture for $\mathrm{GL}_n(k_v)$ (Theorem 3.4), one may formulate the local Langlands functorial transfer conjecture as follows.

Let r_v be a group homomorphism from $H^\vee$ to $G^\vee$. For an irreducible admissible representation σ_v of $H(k_v)$, there is an irreducible admissible representation π_v of $G(k_v)$ such that the following identities hold for all irreducible supercuspidal representations τ_v of $\mathrm{GL}_m(k_v)$ with m being all positive integers. For local twisted L-factors, it requires that

$$L(s, \pi_v \times \tau_v) = L(s, \sigma_v \times \tau_v); \tag{3.3}$$

for local twisted ϵ-factors, it requires that

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \epsilon(s, \sigma_v \times \tau_v, \psi_v); \tag{3.4}$$

and for local twisted gamma factors, it requires that

$$\gamma(s, \pi_v \times \tau_v, \psi_v) = \gamma(s, \sigma_v \times \tau_v, \psi_v). \tag{3.5}$$

In this case, π_v is called the local Langlands functorial transfer of σ_v, or an image of σ_v under the local Langlands functorial transfer. It is known that the local langlands functorial transfer holds for archimedean local places ([L89]), and for all unramified representations over all local fields ([St63]).

It is again very important to know whether the conjecture on the local Langlands functorial transfer is equivalent to the conjecture on the local Langlands functorial lifting. It is easy to see that for unramified representations, the local Langlands functorial lifting conjecture is equivalent to the local Langlands functorial transfer conjecture.

3.5 The local and global compatibility

We remark that the global Langlands conjecture requires automatically the compatibility with the corresponding local Langlands conjectures. Since for unramified representations, the local Langlands functorial lifting conjecture is equivalent to the local Langlands functorial transfer conjecture. It follows that if one only requires the local compatibility at unramified places, then the global Langlands functorial lifting is the same as the global Langlands functorial transfer.

The recent progress shows that one may first establish the weak Langlands functorial transfer, and then verify the compatibility with the local Langlands functorial transfer at all ramified places, in order to establish the global Langlands functorial transfer. It seems still beyond reach to establish the global Langlands functorial liftings for any interesting case.

4 The Langlands Functoriality: generic cases

We summarize below briefly the known cases of the global Langlands functorial transfer conjecture or its weak version.

4.1 From classical groups to the general linear group

The (global) weak Langlands functorial transfers from classical groups G to the general linear group have been established for irreducible generic cuspidal automorphic representations of $G(\mathbb{A})$, via a combination of the Converse Theorem of Cogdell and Piatetski-Shapiro and the refinement of the Langlands-Shahidi method for analytic properties of automorphic L-functions of certain type. We refer to [CPS02] and [Sh02] for wonderful survey of this approach, and to [Cg04] for wonderful, detailed account of the proof and the related problems.

We make the following list for the cases proved by the same method, but after [CPS02] and [Sh02].

- Let G_n be one of the k-split classical groups: SO_{2n+1}, Sp_{2n}, and SO_{2n}. The complex dual group $G_n^\vee$ of G_n is $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{SO}_{2n+1}(\mathbb{C})$, or $\mathrm{SO}_{2n}(\mathbb{C})$, respectively. The natural embedding ι_n of $G_n^\vee$ to a general linear group is given by
$$\iota_n(G_n^\vee) \subset \mathrm{GL}_{2n}(\mathbb{C})$$
if G_n is SO_{2n+1} or SO_{2n}, and by
$$\iota_n(G_n^\vee) \subset \mathrm{GL}_{2n+1}(\mathbb{C})$$
if G_n is Sp_{2n}. This is proved in [CKPSS04] (see [CKPSS01] for the case that $G_n = \mathrm{SO}_{2n+1}$).

- Let G_n be either k-quasisplit unitary group $U(n,n)$ or $U(n+1,n)$. To define the group G_n, we need a quadratic extension F/k. Then the Langlands dual group ${}^L G_n$ when $G_n = U(n,n)$ is a semi-direct product $\mathrm{GL}_{2n}(\mathbb{C}) \rtimes \mathrm{Gal}(F/k)$ of the complex group $\mathrm{GL}_{2n}(\mathbb{C})$ and the Galois group $\mathrm{Gal}(F/k)$. The target group for $G_n = U(n,n)$ is $\mathrm{Res}_{F/k}(\mathrm{GL}_{2n})$, the Langlands dual group of which is
$$(\mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C})) \rtimes \mathrm{Gal}(F/k).$$

The Langlands functorial transfer for both cases were proved by H. Kim and M. Krishnamurthy in [KK04] and [KK05]. We refer to [KK04] and [KK05] for details.

- In [AS06a], M. Asgari and F. Shahidi established the weak Langlands functorial transfer from general spin groups GSpin_m to the general linear group for irreducible generic cuspidal automorphic representations. This completes the weak Langlands functorial transfers for the list of reductive k-split algebraic groups whose Langlands dual groups have classical derived groups. A particular case of this work provides the Langlands functorial transfer from GSp_4 to GL_4, which has been long expected. We refer to [AS06b] for more explicit results related to this Langlands transfer.

4.2 Refined properties of the Langlands functorial transfer

One of the refinements for the Langlands functorial transfer conjecture is to determine and to characterize the image of the Langlands functorial transfers. The key ingredient to so for the current known cases is from the Rankin-Selberg method for the automorphic L-functions involved in the Langlands functoriality. This is the work of Ginzburg, Rallis, and Soudry, generalizing the earlier work of Gelbart and Piatetski-Shapiro ([GPSR97], and [GlPSR87]. The following are the theorem for SO_{2n+1}.

Theorem 4.1 ([GRS01])**.** *Let π be an irreducible generic cuspidal automorphic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$, and let τ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$. Assume that the partial L-function $L^S(s,\pi\times\tau)$ has a pole at $s=1$. Then m is even, τ is self-dual, and the partial exterior square L-function of τ, $L^S(s,\tau,\Lambda^2)$ has a pole at $s=1$.*

From this theorem, we obtain the extra information for τ from the existence of the pole at $s=1$ of the tensor product L-function $L^S(s,\pi\times\tau)$. The following theorem indicates the significance of this extra information.

Theorem 4.2. *Assume that the partial exterior square L-function of τ, $L^S(s,\tau,\Lambda^2)$ has a pole at $s=1$. Then the following hold.*

1. *τ is self-dual, and m must be even, say, $m=2r$.*

2. *There is a unique irreducible generic cuspidal automorphic representation* σ *of* $\mathrm{SO}_{2r+1}(\mathbb{A})$*, such that* τ *is the Langlands functorial transfer from* σ*.*
3. *Write* $\tau = \otimes_v \tau_v$*. Each local component* τ_v *is a local Langlands functorial transfer from* $\mathrm{SO}_{2r+1}(k_v)$*.*

Part one was proved in [JS90] and [K00]. The existence of σ in part two was proved in [GRS01] by the automorphic descent method, and the uniqueness of σ in part two was proved in [JngS03] by the local converse theorem (see [Jng06b] for detailed discussion on the general version of the local converse theorem.). Part three was proved in [JngS04] by the local descent method and the local converse theorem. We refer to [Jng04] for more detailed discussion of the local theory.

It is expected that this theorem holds for other classical groups with suitable modification.

Based on these results, certain properties of the image of the Langlands functorial transfer can be determined as follows. We state below the theorem for SO_{2n+1} and refer to [CKPSS04] and [Sd05] for the statements for other classical groups.

Theorem 4.3 ([GRS01], [CKPSS01], [CKPSS04], [JngS03], [JngS04], and [Sd05])**.** *There is a unique one-to-one correspondence between the set* $\mathcal{B}_n$ *and the set* $\mathcal{A}_n$*, which is the Langlands functorial transfer from* $\mathrm{SO}_{2n+1}(\mathbb{A})$ *to* $\mathrm{GL}_{2n}(\mathbb{A})$*, where* $\mathcal{B}_n$ *is the set of the equivalence classes of irreducible generic cuspidal automorphic representations* σ *of* $\mathrm{SO}_{2n+1}(\mathbb{A})$*, and* $\mathcal{A}_n$ *is the set of equivalence classes of irreducible self-dual unitary automorphic representations* τ *of* $\mathrm{GL}_{2n}(\mathbb{A})$ *with the following properties:*

- *There is a partition* $n = \sum_{i=1}^{r} n_i$ *and for each* i*, there is an irreducible unitary self-dual cuspidal automorphic representation* τ_i *of* $\mathrm{GL}_{2n_i}(\mathbb{A})$ *such that*
$$\tau = \tau_1 \boxplus \cdots \boxplus \tau_r;$$
- *if* $i \neq j$*, then* $\tau_i \not\cong \tau_j$*;*
- *for all* i*,* $L^S(s, \tau_i, \Lambda^2)$ *has a pole at* $s = 1$*.*

Remark 4.1. *For* SO_{2n} *or for* Sp_{2n}*, the results are not as precise as in Theorem 4.3 for* SO_{2n+1}*, since the results in [JngS03] and [JngS04] for* SO_{2n+1}

have not been completely established for either SO_{2n} *or for* Sp_{2n}*. Also for* GSpin_m*, the automorphic descent has not been carried over. The analogue of Theorem 4.2 is not valid yet.*

It is very interesting to mention that Theorem 4.3 has applications to the Inverse Galois Problem recently by C. Khare, M. Larsen, and G. Savin ([KLS06]).

4.3 Endoscopy cases

The theory of endoscopy transfers of automorphic representations is fundamental to understand the Arthur conjecture on the basic structure of the discrete spectrum of square integrable automorphic forms. We refer to [KS99] and [A05] for general theory, and give below some interesting examples, where that the endoscopy transfers are characterized in terms of poles of certain automorphic L-functions. According to Langlands ([L04]), the theory may go well by assuming that the irreducible cuspidal automorphic representations under consideration are of Ramanujan type, i.e., it is tempered at every local place. By a conjecture of Shahidi ([Sh88]), one may further assume that the irreducible cuspidal automorphic representations are globally generic, i.e. it has a nonzero Whittaker Fourier coefficient.

The first example we want to discuss is the following twisted endoscopy case of GL_{2n}.

Theorem 4.4. *Let* π *be an irreducible, self-dual, unitary, cuspidal automorphic representation of* $\mathrm{GL}_{2n}(\mathbb{A})$*. The exterior square L-function* $L(s, \pi, \Lambda^2)$ *has a pole at* $s = 1$ *if and only if* π *is the Langlands functorial transfer from an irreducible generic cuspidal automorphic representation* σ *of* $\mathrm{SO}_{2n+1}(\mathbb{A})$*.*

The proof of this theorem is given by the automorphic descent construction ([GRS01]) and by the refined local theory ([JngS03] and [JngS04]). In [Sd05], other twisted endoscopy cases have been discussed in details.

In the following, we discuss in some details a case of standard elliptic endoscopy transfer for SO_{2n+1}. See [Jng06a] for complete proofs.

We recall from [A04] and [A05] the basic structure of all standard elliptic endoscopy groups of SO_{2n+1}. Let $n = n_1 + n_2$ with $n_1, n_2 > 0$. Take a semisimple element

$$s_{n_1,n_2} = \begin{pmatrix} -I_{n_1} & & 0 \\ & I_{2n_2} & \\ 0 & & -I_{n_1} \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{C}).$$

Then the centralizer of s_{n_1,n_2} in $\mathrm{Sp}_{2n}(\mathbb{C})$ is given by

$$H^\vee_{[n_1,n_2]} = \mathrm{Cent}_{\mathrm{Sp}_{2n}(\mathbb{C})}(s_{n_1,n_2}) = \mathrm{Sp}_{2n_1}(\mathbb{C}) \times \mathrm{Sp}_{2n_2}(\mathbb{C}).$$

The standard elliptic endoscopy group associated to the partition $n = n_1+n_2$ is

$$H_{[n_1,n_2]} = \mathrm{SO}_{2n_1+1} \times \mathrm{SO}_{2n_2+1},$$

and the groups $H_{[n_1,n_2]}$ exhaust all standard elliptic endoscopy groups of SO_{2n+1}, in the sense of [KS99].

More generally, for any nontrivial partition $n = n_1 + n_2 + \cdots + n_r$ with nonzero n_i's,

$$\mathrm{SO}_{2n_1+1} \times \mathrm{SO}_{2n_2+1} \times \cdots \times \mathrm{SO}_{2n_r+1}$$

is an elliptic endoscopy group of SO_{2n+1} and all elliptic endoscopy groups of SO_{2n+1} have this form. Set

$$H_{[n_1\cdots n_r]} := \mathrm{SO}_{2n_1+1} \times \mathrm{SO}_{2n_2+1} \times \cdots \times \mathrm{SO}_{2n_r+1} \tag{4.1}$$

where $n_j > 0$ and $n = \sum_{j=1}^r n_j$. It follows that the number r has a range from 1 to n. It is clear that the complex dual group of $H_{[n_1\cdots n_r]}$ is

$$H^\vee_{[n_1\cdots n_r]}(\mathbb{C}) = \mathrm{Sp}_{2n_1}(\mathbb{C}) \times \mathrm{Sp}_{2n_2}(\mathbb{C}) \times \cdots \times \mathrm{Sp}_{2n_r}(\mathbb{C}). \tag{4.2}$$

In order to characterize the endoscopy transfer for this case, we need to introduce the second fundamental L-function for SO_{2n+1}.

We denote by ρ_2 the second fundamental complex representation of $\mathrm{Sp}_{2n}(\mathbb{C})$, which has dimension equal to $2n^2 - n - 1$ and may be constructed explicitly through the exterior square representation of $\mathrm{GL}_{2n}(\mathbb{C})$. One may also call ρ_2 the exterior square representation of $\mathrm{Sp}_{2n}(\mathbb{C})$.

Let ι be the natural embedding of $\mathrm{Sp}_{2n}(\mathbb{C})$ into $\mathrm{GL}_{2n}(\mathbb{C})$. Let Λ^2 be the exterior square representation of $\mathrm{GL}_{2n}(\mathbb{C})$ on the vector space $\Lambda^2(\mathbb{C}^{2n})$, which has dimension $2n^2 - n$. The composition $\Lambda^2 \circ \iota$ of Λ^2 with ι is a complex representation of $\mathrm{Sp}_{2n}(\mathbb{C})$ and decomposes into a direct sum of two irreducible representations of $\mathrm{Sp}_{2n}(\mathbb{C})$:

$$\Lambda^2 \circ \iota = \rho_2 \oplus \mathbf{1}_{\mathrm{Sp}_{2n}} \tag{4.3}$$

where ρ_2 is the second fundamental complex representation of $\mathrm{Sp}_{2n}(\mathbb{C})$, which is irreducible and has dimension $2n^2-n-1$, and $\mathbf{1}_{\mathrm{Sp}_{2n}}$ is the trivial representation of $\mathrm{Sp}_{2n}(\mathbb{C})$. First we show that the the second fundamental L-function $L(s, \pi, \rho_2)$ has the following analytic properties.

Theorem 4.5 ([Jng06a]). *Let π be an irreducible generic cuspidal automorphic representation of* $\mathrm{SO}_{2n+1}(\mathbb{A})$ *and ρ_2 be the second fundamental complex representation of the complex dual group* $\mathrm{Sp}_{2n}(\mathbb{C})$ *of* SO_{2n+1}*. Then the second fundamental automorphic L-function $L(s,\pi,\rho_2)$ enjoys following properties.*

(1) There exists an irreducible admissible automorphic representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$ *such that*

$$L(s,\pi,\rho_2) = \frac{L(s,\tau,\Lambda^2)}{\zeta_k(s)}$$

holds for the real part of s large.

(2) The eulerian product defining the L-function $L(s,\pi,\rho_2)$ converges absolutely for the real part of s greater than or equal to one, has meromorphic continuation to the whole complex plane, and satisfies the functional equation

$$L(s,\pi,\rho_2) = \epsilon(s,\tau,\rho_2)L(1-s,\pi,\rho_2)$$

with $\epsilon(s,\pi,\rho_2) = \epsilon(s,\tau,\Lambda^2)$, the ϵ-factor for the exterior square L-function $L(s,\tau,\Lambda^2)$.

(3) The L-function $L(s,\pi,\rho_2)$ has possible poles at $s = 0, 1$, besides other possible poles in the open interval $(0,1)$.

This theorem shows that the Langlands conjecture (Conjecture 2.2) holds for $L(s,\pi,\rho_2)$. We remark that this theorem for $L(s,\pi,\rho_2)$ can not be proven via either the Langlands-Shahidi method or by any currently known integral representation of the Rankin-Selberg type. In general, one experts to prove the above theorem for a new family of automorphic L-functions through the Langlands functoriality.

The theorem for standard elliptic endoscopy transfer in this case can be stated as follows.

Theorem 4.6 ([Jng06a]). *Let π be an irreducible generic cuspidal automorphic representation of* $\mathrm{SO}_{2n+1}(\mathbb{A})$ *and ρ_2 be the second fundamental complex representation of* $\mathrm{Sp}_{2n}(\mathbb{C})$.

(1) The partial second fundamental L-function $L^S(s,\pi,\rho_2)$ has a pole of order $r-1$ at $s = 1$ if and only if there exists a partition $n = \sum_{j=1}^r n_j$

with $n_j > 0$ *such that* π *is an endoscopy lifting from an irreducible generic cuspidal automorphic representation*

$$\pi_1 \otimes \cdots \otimes \pi_r$$

of $H_{[n_1 \cdots n_r]}(\mathbb{A})$.

(2) *The partition* $[n_1 \cdots n_r]$ *is uniquely determined by the irreducible generic cuspidal automorphic representation* π. *More precisely, the set of positive integers*

$$\{n_1, n_2, \cdots, n_r\}$$

consists of all positive integers m *such that there exists an irreducible cuspidal automorphic representation* τ *of* $\mathrm{GL}_m(\mathbb{A})$ *such that the tensor product L-function* $L(s, \pi \times \tau)$ *has a pole at* $s = 1$.

(3) *The set* $\{\pi_1, \cdots, \pi_r\}$ *of irreducible generic cuspidal automorphic representations of* $\mathrm{SO}_{2n_i+1}(\mathbb{A})$ *is completely determined by the irreducible generic cuspidal automorphic representation* π, *up to equivalence. Namely, it is the set of irreducible generic cuspidal automorphic representations* π' *(up to equivalence) of* $\mathrm{SO}_{2l+1}(\mathbb{A})$ *such that the tensor product L-function* $L(s, \pi \times \tau(\pi'))$ *has a pole at* $s = 1$, *where* $\tau(\pi')$ *is the Langlands functorial transfer of* π' *to* $\mathrm{GL}_{2l}(\mathbb{A})$ *and is irreducible and cuspidal.*

The proof of Theorem 3.2 uses explicit results about the Langlands functorial transfer from irreducible generic cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ to $\mathrm{GL}_{2n}(\mathbb{A})$, which have been established through [CKPSS04], [GRS01], [JngS03] and [JngS04]. The following are essentially the summary of those results. For other classical groups, it is the work in progress of D. Soudry and the author.

(1) An irreducible automorphic representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$ is the Langlands functorial transfer of an irreducible generic cuspidal automorphic representation π of $\mathrm{SO}_{2n+1}(\mathbb{A})$ if and only if τ is equivalent to the following isobaric representation

$$\tau \cong \tau_1 \boxplus \cdots \boxplus \tau_r \tag{4.4}$$

where τ_j for $j = 1, 2, \cdots, r$, is an irreducible cuspidal automorphic representation of $GL_{2n_j}(\mathbb{A})$ with the properties that

   (a) $n = \sum_{j=1}^{r} n_j$ is a partition of n with $n_j > 0$;
   (b) $\tau_i \ncong \tau_j$ if $i \neq j$;
   (c) the partial exterior square L-function $L^S(s, \tau_j, \Lambda^2)$ has a pole at $s = 1$.

(2) If $\tau = \tau(\pi)$ is the Langlands functorial transfer of π, then π is uniquely determined by τ and for every irreducible cuspidal automorphic representation τ' of $\mathrm{GL}_l(\mathbb{A})$ (any $l \geq 1$), one has

$$L(s, \pi \times \tau') = L(s, \tau(\pi) \times \tau').$$

(3) The functorial transfer from π to $\tau(\pi)$ is compatible with the local Langlands functorial principle.

(4) The set $\{\tau_1, \cdots, \tau_r\}$ consists of all irreducible cuspidal automorphic representations τ' of $\mathrm{GL}_l(\mathbb{A})$ with $l = 1, 2, \cdots$, such that the tensor product L-function $L(s, \pi \times \tau')$ has a pole (of order one) at $s = 1$.

(5) For each τ_j of $\mathrm{GL}_{2n_j}(\mathbb{A})$ with $j = 1, 2, \cdots, r$, there exists a unique up to equivalence irreducible generic cuspidal automorphic representation π_j of $\mathrm{SO}_{2n_j+1}(\mathbb{A})$ such that the Langlands functorial transfer of π_j from $\mathrm{SO}_{2n_j+1}(\mathbb{A})$ to $\mathrm{GL}_{2n_j}(\mathbb{A})$ is τ_j.

Remark 4.2. *In a special case when $r = n$, Theorem 3.2 was part of the conjecture stated in [GJng00]. One notices that in the conjecture stated in [GJng00], it is expected that there should be a period condition characterizing the endoscopy structure of the irreducible generic cuspidal automorphic representation π of $\mathrm{SO}_{2n+1}(\mathbb{A})$. It will be a very interesting problem to find a period condition in general.*

One may establish similar theorems for other classical groups, by replacing the second fundamental automorphic L-functions by the symmetric square L-functions. We remark that the uniqueness results as asserted in Part (2) and Part (3) in Theorem 3.2 is not currently valid. See [JngS03] and [JngS04] for the detailed discussion for SO_{2n+1} case. It is very likely that such uniqueness properties should be reasonably modified.

We believe that when π is an irreducible generic cuspidal automorphic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$, the algebraic subgroup ${}^L H_\sigma$ of the complex dual group $\mathrm{Sp}_{2n}(\mathbb{C})$ should be

$$\mathrm{Sp}_{2n_1}(\mathbb{C}) \times \cdots \times \mathrm{Sp}_{2n_r}(\mathbb{C})$$

for some partition $n = \sum_{i=1}^{r} n_i$. From the proof, it is easy to see that the number r of the factors $\mathrm{Sp}_{2n_i}(\mathbb{C})$ is completely determined by the second fundamental representation ρ_2. However, the sizes of the factors, namely, $2n_i$'s are determined by the existence of the pole at $s = 1$ of tensor product L-functions for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_{2m}$ for all possible positive integers m. This seems not quite fit into the Langlands conjecture. On the other hand, we develop local versions of the Langlands conjecture in [Jng06b]. Some local evidence in [Jng06b] suggests the following conjecture.

Conjecture 4.7. *For a given irreducible generic cuspidal automorphic representation π of $\mathrm{SO}_{2n+1}(\mathbb{A})$, the structure of the algebraic subgroup*

$$^{L}H_{\sigma} = \mathrm{Sp}_{2n_1}(\mathbb{C}) \times \cdots \times \mathrm{Sp}_{2n_r}(\mathbb{C})$$

is completely determined by the even fundamental representations

$$\rho_2, \rho_4, \cdots, \rho_{2[\frac{n}{2}]}$$

of the complex dual group $\mathrm{Sp}_{2n}(\mathbb{C})$.

Remark 4.3. *There are some more recent progress on the construction of endoscopy transfers for classical groups via the refined automorphic descent method, which are the work in progress of D. Ginzburg, D. Jiang, and D. Soudry. We will report these progress in other occasion in future.*

5 Other Instances

There are many interesting and important instances of the Langlands functorial transfers having been established case by case. We will give a brief review below.

5.1 Symmetric powers

The symmetric power liftings for GL_2 is a fundamental problem in the modern theory of automorphic forms.

Let Sym^n be the $(n+1)$-dimensional complex representation of $\mathrm{GL}_2(\mathbb{C})$. The Langlands functorial transfer conjecture predicts that for any irreducible cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$, there should be an irreducible automorphic representation Π of $\mathrm{GL}_{n+1}(\mathbb{A})$ such that

$$L(s, \pi, \mathrm{Sym}^n) = L(s, \Pi).$$

Then $\Pi = \mathrm{Sym}^n(\pi)$ is called the n-th symmetric power lifting of π. As discussed in [Sh94] and [Mt04], the important applications of the existence of the n-th symmetric power lifting of π for all n include the Ramanujan Conjecture for π and the Sato-Tate conjecture. See [CHT06] and [Ty06] for the recent progress on the Sato-Tate conjecture based more geometric arguments in arithmetic theory of Shimura varieties.

The existence of the n-th symmetric power liftings of π has been established for $n = 2$ by S. Gelbart and H. Jacquet ([GlJ78]), for $n = 3$ by H. Kim and F. Shahidi ([KSh02] and for $n = 4$ by H. Kim ([K03]). For $n \geq 5$, it is still an open problem, which seems unreachable within the current understanding of the theory of automorphic forms, in general, although some special cases have been studied in [CHT06] and [Ty06].

5.2 Tensor products

The tensor product of two automorphic representations is a basic problem in establishing the Tanakian properties for irreducible automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ for all n. The basic version of the problem can be formulated as follows. Let π_1 and π_2 be irreducible cuspidal automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A})$ and $\mathrm{GL}_{n_2}(\mathbb{A})$, respectively. Then there should exists an irreducible automorphic representation Π of $\mathrm{GL}_{n_1 \times n_2}(\mathbb{A})$ such that

$$L(s, \Pi) = L(s, \pi_1 \times \pi_2).$$

Then $\Pi = \pi_1 \pi_2$ is called the tensor product lifting of π_1 and π_2.

The case when $n_1 = n_2 = 2$ was established by D. Ramakrishnan ([Rm00]), and the case when $n_1 = 2$ and $n_2 = 3$ was established by H. Kim and F. Shahidi ([KSh02]). Besides these cases, it is a widely open problem. A twisted version of the case when $n_1 = n_2 = 2$ was proved by [Kr03], which is stated as follows: let E/k be a quadratic extension of algebraic number fields, and let π be an irreducible cuspidal automorphic representation of $\mathrm{Res}_{E/k}(\mathrm{GL}_2)(\mathbb{A})$. Then there is an irreducible automorphic representation Π of $\mathrm{GL}_4(\mathbb{A})$ such that

$$L(s, \pi, \mathrm{Asai}) = L(s, \Pi),$$

where $\mathrm{Res}_{E/k}(\mathrm{GL}_2)$ is the Weil restriction of GL_2 from E to the base field k, and $L(s, \pi, \mathrm{Asai})$ is the Asai L-function attached to π.

We refer to [Rm94] for more detailed discussion of this problem and related applications.

5.3 Exceptional group of type G_2

By using the exceptional theta correspondences, the theory of automorphic representations of G_2 has been much better understood recently. We only give a very brief review on this topic.

Based on the construction of the minimal representations of reductive groups over a local field ([Kz90], [KzS90], and [GnS05]), automorphic theta functions has been determined in [Sv93] and [GRS97a]. The basic set-up for exceptional theta liftings related to G_2 was established in [GRS97b]. For instance, we refer to the following papers for discussions of the theory related to exceptional theta liftings: [RS89], [GRS97c], [Gn00a], [Gn00b], [GJng01], [GJngR01], [GJngR02], [GnGrS02], [GnGJ02], [G03], [GnS03], [Gn05], [GnG05a], [GnG05b], [G05], [GnG06], and [BFG06].

5.4 Generic case for GSp_6

This is an example concerning the Langlands functorial transfer from G_2, the k-split exceptional group of type G_2, to PGSp_6, corresponding to the the natural embedding of $G_2^\vee = G_2(\mathbb{C})$ into $\mathrm{GSpin}_7(\mathbb{C}) = \mathrm{PGSp}_6^\vee$. The result can be formulated as follows

Theorem 5.1 (Ginzburg-Jiang [GJng01])**.** *Let π be an irreducible generic cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbb{A})$. Then the following are equivalent.*

(1) π is the Langlands functorial transfer from an irreducible generic cuspidal automorphic representation σ of $G_2(\mathbb{A})$.

(2) The degree 8 automorphic L-function $L(s, \pi, \mathrm{Spin}_8)$ has a pole at $s = 1$.

(3) The generalized Shalika model for π does not vanish.

The proof of this theorem is based on [BG92a] for the integral representation for $L(s, \pi, \mathrm{Spin}_8)$, on [Vo97] for the local theory at ramified places, and on the explicit construction of the functorial transfer via the exceptional theta functions on GE_7.

One expects that for an irreducible, unitary, generic, cuspidal automorphic representation π of $\mathrm{PGSp}_6(\mathbb{A})$ with $L(s, \pi, \mathrm{Spin}_8)$ having a pole at $s = 1$, one of the candidates for ${}^L H_\pi$ should be $G_2(\mathbb{C})$.

There are some other known examples of this type, for instance, we have a same theorem for SL_3 to G_2 with irreducible generic cuspidal automorphic representations π of $G_2(\mathbb{A})$ and the standard L-function $L(s, \pi)$ having a pole at $s = 1$. This is Theorem 3.7 in [GJng01].

6 Beyond the Genericity

The existence of non-generic irreducible cuspidal automorphic representation for reductive groups which are not of A_n-type was first discovered by R. Howe and I. Piatetski-Shapiro [HPS79]. They provide the first examples of irreducible cuspidal automorphic representations whose local components are nontempered at almost all local places, i.e. the counter-examples of the generalized Ramanujan conjecture. It turns out that this is a general phenomenon. These cuspidal automorphic representations are called in [PS83] CAP automorphic representations, i.e. cuspidal automorphic representations associated to a certain parabolic subgroup. The reason for this is that these cuspidal automorphic representations locally look like the local components of a residual automorphic representation at almost all local places. The basic structure of the discrete spectrum becomes much more complicate when the group is not of A_n-type, because of the existence of the CAP automorphic representations. One has the following conjecture.

Conjecture 6.1 (The CAP Conjecture [JngS07a]). *Assume that G is k-quasisplit reductive group. Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. There exists a standard parabolic subgroup $P = MN$ of G and an irreducible generic cuspidal automorphic representation σ of $M(\mathbb{A})$ such that π is nearly equivalent to an irreducible constituent of the unitarily induced representation*

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma).$$

This conjecture has been long expected, but was for the first time precisely stated in general in [JngS07a]. A version of this conjecture for $G = \mathrm{Sp}(4)$ was stated in the work of I. Piatetski-Shapiro and D. Soudry ([Sd90] and [PSS87]). The point here is that the CAP conjecture above requires the cuspidal datum to be generic.

By the strong multiplicity one theorem for automorphic representations of $\mathrm{GL}(n)$ of H. Jacquet and J. Shalika ([JS81]), there are no CAP automorphic representations for $\mathrm{GL}(n)$. After the pioneer work of Piatetski-Shapiro on the Saito-Kurokawa lift ([PS83]), CAP automorphic representations have attracted a lot of attentions in the investigation of the basic structures of the discrete spectrum of automorphic forms, relating to the Arthur conjectures. Many more examples have been constructed by means of the theory of theta functions and more recently by other new methods.

We list below some references for the known CAP automorphic representations for each group.

- For $G = \mathrm{GSp}(4)$, [HPS79], [PS83], [PSS87], [Sd90], [Sch05], [Pt06], [Sch07].
- For $G = G_2$, the k-split exceptional group of type G_2, [RS89], [GRS97b], [GJng01], [GnGJ02], [GnS03], [Gn05], [GnG05a], [GnG05b], [G05], [GnG06].
- For $G = \mathrm{SO}_{2n}$, k-split special even orthogonal group, [GJngR02].
- For $G = \mathrm{SO}_{2n+1}$, k-split special old orthogonal group, [JngS07a], [JngS07b].
- For $G = \mathrm{Sp}_{2n}$, k-split symplectic group, [Ik01], [Sch03], [GRS05].
- For $G = U_m$, k-quasisplit unitary group, [GlRS97], [Ik].

We make the following remarks.

Remark 6.1. *It is important to point out that all the know CAP automorphic representations confirms the CAP conjecture.*

Since the CAP conjecture requires that the cuspidal datum in the conjecture is generic, it reduces the Langlands functorial transfer for general cuspidal automorphic representations to the case of irreducible generic cuspidal automorphic representations. For irreducible generic cuspidal automorphic representations, the Langlands functorial transfer for various groups have been or will be established by the Converse Theorem and L-function method, as discussed in §4.

Finally, we point out that the cuspidal datum in the CAP conjecture can be made very precise as just predicted by the Arthur Conjectures, but we will omit the details here.

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