Periods of Automorpic Forms

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Abstract. In this paper, we discuss periods of automorphic forms from the representation-theoretic point of view. It gives general theory of periods and some special and very interesting cases.

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1. Introduction

A classical result of Gelfand and Graev asserts that the Gelfand-Graev models are of multiplicity free over a finite field. For simplicity, we consider here for $\mathrm{GL}_2(\mathbb{F}_q)$, where $\mathbb{F}_q$ is a finite field. Let $B = TU$ be the standard Borel subgroup given by

$$B = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2\}$$

with the unipotent radical U given by

$$U = \{u(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2\}$$

Fix a nontrivial additive character ψ of $\mathbb{F}_q$, and then define

$$\psi_U(u(b)) := \psi(b).$$

The Gelfand-Graev representation in this case is given by the following induced representation

$$\mathrm{Ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)}(\psi_U).$$

It is not hard to check that such an induced representation is of multiplicity free ([**Bmp97**]).

This simple result was generalized to GL_2 over local field by Jacquet and Langlands and played an essential role in their classical work ([**JL70**]). It turned out that it holds for quasi-split reductive groups over local fields. This is the well-known uniqueness property of local Whittaker models ([**Shl74**]), which has been essential in the theory of automorphic L-functions, the theory of Langlands functoriality, the theory of harmonic analysis over local reductive groups, and the spectral theory of automorphic forms.

In this notes, we shall explain the periods of automorphic forms as generalization of the theory started from the Gelfand-Graev models. On the other hand,

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periods of automorphic forms also originated from the systematically study by Y. Manin ([**M72**] and [**M73**]) on modular symbols associated to classical modular forms. Some generalizations of modular symbols related to multiple values of zeta functions has been proposed by Y. Manin recently in [**M05**]. The adelic approach to study periods or modular symbols of automorphic forms was first considered by G. Harder in [**Hd87**] for GL_2. For more general groups and the study of periods or modular symbols of automorphic forms with application to Eisenstein cohomology of arithmetic groups, one may find more interesting discussions in [**Hd03**]. Some work on periods of automorphic forms and modular cycles may be found in [**HLR86**], [**Hr94**], [**Od01**], and [**Kd03**]. More general modular symbols have been studied in [**AB90**].

In the theory of automorphic representations, the periods play the role of automorphic quasi-invariant functionals over the space of automorphic representations. The theory of relative trace formulas, which was first studied by H. Jacquet, stimulated the recent study of periods of automorphic representations, which can be usually used to characterize certain cases of the Langlands functorial liftings of automorphic representations. A well prepared survey paper of H. Jacquet ([**Jcq97**]) provides some detailed discussions on the relative trace formula approach and applications in number theory.

It turns out that periods of automorphic forms are often natural and useful invariants to characterize certain family of automorphic representations, especially in the constructive theory of automorphic forms, such as the theta correspondences in terms of the Weil representations or more generally the automorphic minimal representations, and more recently automorphic descent construction [**GRS01**].

The discussion carried out in this notes is based on the author's own research interests related to this topic. It is clear that some very interesting progress will be omitted without being mentioned. The author's research related to the periods of automorphic forms has been supported in part over years by the NSF research postdoc fellowship, the A. Sloan research fellowship, the McKnight Land-Grant Professorship (UMN), and the regular NSF grants DMS-9896257, DMS-0098003, and DMS-0400414. I would like to thank Professor Stephen Rallis, Professor Ilya Piatetski-Shapiro, Professor Roger Howe, and Professor James Cogdell for their encouragement and support over years and to thank my long term collaborators, Professor David Ginzburg and Professor David Soudry, for stimulating discussions and supports related to the topics in this paper.

Finally, I would like to thank the organizers for inviting me to lecture in the conference. It is my great pleasure to submit this paper to the Proceedings. I spent my time from 1984 to 1989 in the Department of Mathematics, The East China Normal University, starting as a graduate student in Arithmetic Theory of Quadratic Forms, and then working as a junior faculty member. The stimulating research environment in the Department provided me lots of opportunities to learn advanced mathematics during my stay. I am grateful to the teachers in the Research Group of Algebra, especially Professor X.-H. Cao and late Professor Z.-F. Zhu for guiding me during the five years. I would also like to thank Professor H.-W. Lu for his Chinese translation of the well-known expository paper by Professor Stephen Gelbart, *An Elementary Introduction to the Langlands Program* ([**Glb84**]), which attracted my attention for the first time to the modern theory of automorphic forms. The topics included here have been presented in part in lectures at The Joint Number

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2. Periods

2.1. Automorphic forms and representations. Let G be a reductive algebraic group defined over a number field k. For simplicity, we may assume that G splits over k. Let $\mathbb{A}$ be the ring of adeles of k. Consider the space of square integrable functions over the quotient $Z_G(\mathbb{A})G(k)\backslash G(\mathbb{A})$, which is denoted by

$$L^2(G) := L^2(Z_G(\mathbb{A})G(k)\backslash G(\mathbb{A})).$$

Note that $Z_G(\mathbb{A})G(k)\backslash G(\mathbb{A})$ has finite volume with respect to the natural Haar measure. Following general spectral theory, one has

$$L^2(G) = L^2_{\text{disc}}(G) \oplus L^2_{\text{cont}}(G),$$

where $L^2_{\text{disc}}(G)$ is the discrete part of the spectrum, which is a Hilbert sum of irreducible subrepresentations with finite multiplicities, and $L^2_{\text{cont}}(G)$ is the continuous part of the spectrum. The discrete part of spectrum can be written as a direct sum of cuspidal spectrum and residual spectrum:

$$L^2_{\text{disc}}(G) = L^2_{\text{cusp}}(G) \oplus L^2_{\text{res}}(G).$$

An irreducible submodule of $L^2_{\text{disc}}(G)$ ($L^2_{\text{cusp}}(G)$, or $L^2_{\text{res}}(G)$, resp.) is called a discrete series (cuspidal, or residual, resp.) automorphic representation of $G(\mathbb{A})$. A general definition of automorphic forms (or representations) can be found in [**BJ79**]. However, the following theorem of Langlands provides a characterization of automorphic representations in terms of cuspidal automorphic representations of Levi subgroups of G, which is also convenient to be taken as an informal definition of general automorphic representations.

Theorem 2.1 (Langlands [**L79**]). *An irreducible (admissible) representation π of $G(\mathbb{A})$ is automorphic if and only if there exist a parabolic subgroup P with Levi subgroup M, and an irreducible cuspidal automorphic representation σ of $M(\mathbb{A})$ such that π is an irreducible constituent of the parabolically induced representation*

$$\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma).$$

A classical example of periods of automorphic forms comes from the Hecke theory. Let π be an irreducible cuspidal automorphic representation of $\text{GL}_2(\mathbb{A})$, and φ_π be a cusp form in π. Consider the following Mellin transform of φ_π

$$\int_{k^\times\backslash\mathbb{A}^\times} \varphi_\pi(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})|a|_{\mathbb{A}}^{s-\frac{1}{2}} d^\times a.$$

Since irreducible cuspidal automorphic representations of $\text{GL}_2(\mathbb{A})$ are generic, i.e. having a nonzero Whittaker-Fourier coefficient, and have the following Fourier expansion (The generalization to GL_n can be found in [**Shl74**].)

$$\varphi_\pi(g) = \sum_{\alpha\in k^\times} \mathcal{W}_{\varphi_\pi}^{\psi}(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g)$$

where $\mathcal{W}_{\varphi_\pi}(g)$ is the Whittaker-Fourier coefficient of φ_π with respect to a non-trivial additive character ψ, given by

$$\mathcal{W}^{\psi}_{\varphi_\pi}(g) := \int_{k\backslash\mathbb{A}} \varphi_\pi(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g)\psi^{-1}(x)dx.$$

Then the Mellin transform can be expressed as

$$\int_{k^\times\backslash\mathbb{A}^\times} \varphi_\pi(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix})|a|_{\mathbb{A}}^{s-\frac{1}{2}}d^\times a = \int_{\mathbb{A}^\times} \mathcal{W}^{\psi}_{\varphi_\pi}(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix})|a|_{\mathbb{A}}^{s-\frac{1}{2}}d^\times a.$$

By the Jacquet-Langlands version of Hecke theory, there exists a φ_π such that

$$\int_{\mathbb{A}^\times} \mathcal{W}^{\psi}_{\varphi_\pi}(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix})|a|_{\mathbb{A}}^{s-\frac{1}{2}}d^\times a = L(s,\pi).$$

Hence at $s = \frac{1}{2}$, the center of the symmetry from the functional equation of $L(s,\pi)$, we have

$$L(\frac{1}{2},\pi) = \int_{k^\times\backslash\mathbb{A}^\times} \varphi_\pi(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix})d^\times a.$$

The last integral gives the simplest example of the periods of automorphic forms, which will be discussed in detail in this paper. The general notion of periods of automorphic forms will be a natural generalization of both such simple periods and the Whittaker-Fourier coefficients. The model attached to it will be called the generalized Gelfand-Graev models of automorphic forms, the definition of which is based on the Dynkin-Kostant theory of nilpotent orbits. The generalized Gelfand-Graev representations of reductive groups over finite fields was well discussed in the expository paper by Kawanaka in [**Kwn87**].

2.2. Nilpotent orbits. In order to introduce the general notion of periods of automorphic forms, we have to recall some basic facts of nilpotent orbits. For simplicity, we refer to [**CM93**].

Let $\mathfrak{g}$ be the Lie algebra of G. Consider the theory of nilpotent orbits in $\mathfrak{g}$ over $\mathbb{C}$ or the algebraic closure $\bar{k}$ of k. According to the Dynkin-Kostant theory, for each nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}(\mathbb{C})$, there is a standard parabolic subgroup $P_{\mathcal{O}} = M_{\mathcal{O}}N_{\mathcal{O}}$, whose Lie algebra $\mathfrak{p}_{\mathcal{O}} = \mathfrak{m}_{\mathcal{O}} + \mathfrak{n}_{\mathcal{O}}$ has the following properties. First, the nilpotent radical $\mathfrak{n}_{\mathcal{O}}$ can be written as a graded $\mathfrak{m}_{\mathcal{O}}$-module

$$\mathfrak{n}_{\mathcal{O}} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3 \oplus \cdots,$$

and the intersection of $\mathcal{O}$ with $\mathfrak{n}_2$ is a single Zariski open dense $\mathfrak{m}_{\mathcal{O}}$-orbit. The stabilizer of this Zariski open dense $\mathfrak{m}_{\mathcal{O}}$-orbit is denoted by $\mathfrak{m}^\circ_{\mathcal{O}}$, up to isomorphism. If the nilpotent orbit $\mathcal{O}$ is even, then $\mathfrak{n}_{2i+1} = 0$ for all $i \geq 0$. The algebraic subgroup of G corresponding to $\mathfrak{m}^\circ_{\mathcal{O}}$ is denoted by $M^\circ_{\mathcal{O}}$. For any reductive algebraic subgroup H of $M^\circ_{\mathcal{O}}$, we form a semidirect product

$$J_{\mathcal{O},H} := H \times N_{\mathcal{O}}.$$

When $\mathfrak{n}_1 \neq 0$, the quotient $\mathfrak{n}_{\mathcal{O}}/\mathfrak{n}_3 \oplus \cdots$ has a structure of two-step nilpotent Lie algebra. In this case, we may consider the metaplectic cover of H and $J_{\mathcal{O},H}$, which are denoted by $\widetilde{H}$ and $\widetilde{J}_{\mathcal{O},H}$. In general, we only consider the cases when the subgroup H is relative big, so that the group $J_{\mathcal{O},H}$ has finitely many orbits acting on the flag variety $B\backslash G$ over $\mathbb{C}$ or $\bar{k}$, where B is a Borel subgroup of G. In such cases, the subgroup $J_{\mathcal{O},H}$ is usually called a *spherical subgroup* of G. Some detailed discussion of spherical varieties and spherical subgroups and related topics can be found in ([**Brn95**], [**Hw94**]).

Of course, for precise applications of the above discussion to the theory of automorphic forms, one have to discuss k-rational and k-stable structure of the theory nilpotent orbits and related topics ([**Ktw82**]). A natural formulation of such discussions is to use the theory of Galois cohomology. For this notes, we will avoid such a discussion, instead, we provide an example when $G = \mathrm{SO}(2n+1)$, the k-split odd orthogonal group.

Let k be a field of characteristic zero. Let $(V_{2n+1}, (\cdot,\cdot))$ be a nondegenerate quadratic vector space over k of dimension $2n+1$ with Witt index n. The symmetric bilinear form is given by

$$J_{2n+1}(d) = \begin{pmatrix} & & J_n \\ & d & \\ J_n & & \end{pmatrix} \tag{2.1}$$

where we define $J_n := \begin{pmatrix} & & 1 \\ & J_{n-2} & \\ 1 & & \end{pmatrix}$ inductively. We may choose a basis

$$\{e_1, \cdots, e_n, e_0, e_{-n}, \cdots, e_{-1}\} \tag{2.2}$$

such that

$$(e_i, e_j) = \begin{cases} 1 & \text{if } j = -i, \\ 0 & \text{if } j \neq -i, \\ d & \text{if } j = i = 0. \end{cases}$$

For each $r \in \{1, 2, \cdots, n\}$, we have the following partial polarization

$$V_{2n+1} = X_r \oplus V_{2(n-r)+1} \oplus X_r^* \tag{2.3}$$

where X_r is a totally isotropic subspace of dimension r and X_r^* is the dual of X_r with respect to the non-degenerate bilinear form $(\cdot,\cdot)$, and the subspace $V_{2(n-r)+1}$ is the orthogonal complement of $X_r \oplus X_r^*$. Without loss of generality, we may assume that X_r is generated by $e_1, \cdots, e_r$ and X_r^* is generated by $e_{-r}, \cdots, e_{-1}$.

We recall basic facts on nilpotent orbits of complex classical Lie algebras from [**CM93**]. For $\mathfrak{so}_{2n+1}(\mathbb{C})$, the nilpotent orbits or nilpotent conjugacy classes are in one to one correspondence with the partitions of $2n+1$ with even parts occurring with even multiplicity. Using the notation from [**CM93**], the set of such partitions is denoted by $\mathcal{P}_1(2n+1)$. We call such partitions *odd orthogonal partitions*.

For $r \in \{0, 1, 2, \cdots, n\}$, the partition $[2r+1, 1^{2(n-r)}]$ belongs to $\mathcal{P}_1(2n+1)$. By Theorem 6.3.7, [**CM93**], the partition $[2r+1, 1^{2(n-r)}]$ is special for every $r \in \{0, 1, 2, \cdots, n\}$ and also the involution of partitions stabilizes this family of special partitions. The nilpotent orbit corresponding to $[2r+1, 1^{2(n-r)}]$ is denoted by

$$\mathcal{O}_{n,r} = \mathcal{O}_{[2r+1, 1^{2(n-r)}]}.$$

The orbit corresponding to a special partition is called a special nilpotent orbit. It is easy to see that this family of special nilpotent orbits is linearly ordered. When $r = n$, $\mathcal{O}_{n,n} = \mathcal{O}_{[2n+1]}$ is the regular nilpotent orbit, and when $r = 0$, $\mathcal{O}_{n,0} = \mathcal{O}_{[1^{2n+1}]}$ is the zero nilpotent orbit.

The following discussion is trivial when $r = 0$. Hence we assume that $r \neq 0$. For each $r \in \{1, 2, \cdots, n\}$, there is a standard parabolic Lie subalgebra $\mathfrak{p}_n^r = \mathfrak{m}_n^r \mathfrak{n}_n^r$

(unique up to conjugation) with Levi subalgebra

$$\mathfrak{m}_n^r \cong (\mathfrak{gl}_1)^{\oplus r} \oplus \mathfrak{so}_{2(n-r)+1}$$

and nilpotent radical $\mathfrak{n}_n^r$, whose quotient modulo its derived subalgebra $[\mathfrak{n}_n^r, \mathfrak{n}_n^r]$ is given by

$$\mathfrak{n}_n^r/[\mathfrak{n}_n^r, \mathfrak{n}_n^r] \cong \mathbb{C}^{\oplus(r-1)} \oplus \mathbb{C}^{\oplus(2(n-r)+1)}.$$

Note that if $r = 0$, then $\mathfrak{p}_n^0 = \mathfrak{so}_{2n+1}$. Hence we may assume that the above space is zero when $r = 0$. When $r \neq 0$, the adjoint action of $\mathfrak{m}_n^r$ on $\mathfrak{n}_n^r$ induces an action of $(\mathrm{GL}_1)^r \times \mathrm{SO}_{2(n-r)+1}$ on the affine space $\mathbb{C}^{\oplus(r-1)} \oplus \mathbb{C}^{\oplus(2(n-r)+1)}$ such that for each

$$g = (t_1, t_2, \cdots, t_r; h) \in (\mathrm{GL}_1)^r \times \mathrm{SO}_{2(n-r)+1}$$

and each

$$v = (x_1, x_2, \cdots, x_{r-1}; u) \in \mathbb{C}^{\oplus(r-1)} \oplus \mathbb{C}^{\oplus(2(n-r)+1)},$$

we have

$$g \circ v = (t_1t_2^{-1}x_1, t_2t_3^{-1}x_2, \cdots, t_{r_1}t_r^{-1}x_{r-1}; t_r uh^{-1}). \tag{2.4}$$

Moreover, the nilpotent orbit $\mathcal{O}_{[2r+1,1^{2(n-r)}]}$ corresponds to the Zariski open orbit of $(\mathrm{GL}_1)^r \times \mathrm{SO}_{2(n-r)+1}$ on the affine space $\mathbb{C}^{\oplus(r-1)} \oplus \mathbb{C}^{\oplus(2(n-r)+1)}$. It is clear from the action given by (2.4) that we may choose the representatives of the Zariski open orbit to have the following form

$$v = (1, 1, \cdots, 1; u) \in \mathbb{C}^{\oplus(r-1)} \oplus \mathbb{C}^{\oplus(2(n-r)+1)},$$

that is, $x_1 = \cdots = x_{r-1} = 1$. Since the set of such vectors is stable under the the subgroup

$$(\mathrm{GL}_1)^{\Delta} \times \mathrm{SO}_{2(n-r)+1}$$

of $(\mathrm{GL}_1)^r \times \mathrm{SO}_{2(n-r)+1}$, where $(\mathrm{GL}_1)^{\Delta}$ is GL_1, diagonally embedded inside $(\mathrm{GL}_1)^r$, this induces further the action of $\mathrm{GL}_1 \times \mathrm{SO}_{2(n-r)+1}$ on the affine space $\mathbb{C}^{\oplus(2(n-r)+1)}$ as follows: for each $(t; h) \in \mathrm{GL}_1 \times \mathrm{SO}_{2(n-r)+1}$ and each $u \in \mathbb{C}^{\oplus(2(n-r)+1)}$, we have

$$(t; h) \circ u = tuh^{-1}. \tag{2.5}$$

Consider the partial polarization of V_{2n+1}

$$V_{2n+1} = X_r \oplus V_{2(n-r)+1} \oplus X_r^*$$

as in (2.3), with $V_{2(n-r)+1} \cong \mathbb{C}^{\oplus(2(n-r)+1)}$. By Witt's theorem, we know that the affine space $V_{2(n-r)+1}$ with the group action of $\mathrm{GL}_1 \times \mathrm{SO}_{2(n-r)+1}$ is a regular prehomogeneous vector space and decomposes over $\mathbb{C}$ (or $\bar{k}$) into three orbits

$$V_{2(n-r)+1} = \{0\} \cup Z_0 \cup Z_1 \tag{2.6}$$

where Z_0 is the set of all nonzero isotropic vectors with respect to the bilinear form and Z_1 is the set of all anisotropic vectors with respect to the bilinear form. It is clear that Z_1 is the Zariski open orbit.

It is important to remark that the above discussion works as well if one replaces the ground field $\mathbb{C}$ by the algebraic closure of a number field or a local field of characteristic zero.

In the sequel, we denote by k a number field or a local field of characteristic zero and by $\bar{k}$ the algebraic closure of k.

We will use the notion of k-stable orbits to discuss the k-rationality of the nilpotent orbits and parameterization. For a detailed discussion of k-stable orbits, we refer to [**Ktw82**]. The k-stable nilpotent orbits associated to the $\bar{k}$-nilpotent orbit $\mathcal{O}_{[2r+1,1^{2(n-r)}]}(\bar{k})$ are in one to one correspondence with the k-stable orbits associated to the $\bar{k}$-orbit $Z_1(\bar{k})$. Note that the k-rational orbit $Z_0(k)$ is k-stable, since Witt's theorem holds for number fields or local fields of characteristic zero.

If $r \neq n$, in the Zariski open $\bar{k}$-orbit $Z_1(\bar{k})$, one has the following parameterization of the k-stable orbits of $\mathrm{GL}_1(k) \times \mathrm{SO}_{2(n-r)+1}(k)$ in terms of the square classes of $k^\times$:

$$Z_1(k) = \cup_{\lambda \in k^\times/(k^\times)^2} Z_1^\lambda, \tag{2.7}$$

where the set Z_1^λ is defined by

$$Z_1^\lambda := \{v \in V_{2(n-r)+1}(k) \mid (v,v) = \lambda \mod (k^\times)^2\}.$$

It is clear that the set Z_1^λ is a single k-rational orbit of $\mathrm{GL}_1(k) \times \mathrm{SO}_{2(n-r)+1}(k)$. Note that $Z_0(k)$ is a single k-rational orbit of $\mathrm{GL}_1(k) \times \mathrm{SO}_{2(n-r)+1}(k)$.

If $r = n$, then $V_{2(n-r)+1}$ is one dimensional and has decomposition

$$V_1(k) = \{0\} \cup k^\times. \tag{2.8}$$

Hence $Z_1(k) = k^\times$ is a single $\mathrm{GL}_1(k)$-orbit. Therefore the k-stable regular nilpotent orbit $\mathcal{O}_{[2n+1]}(k)$ is a single k-rational orbit. Further discussion has been given in [**JS05**]

2.3. Fourier coefficients and periods. Let ψ_0 be a nontrivial additive character of $\mathbb{A}$ which is trivial on k. For each nilpotent orbit $\mathcal{O}$ in the Lie algebra $\mathfrak{g}$ of G, one can define a character $\psi_{\mathcal{O}}$ of $N_2(\mathbb{A})$, which is trivial on $N_2(k)$. When $\mathfrak{n}_1 = 0$, then we have $N_2 = N_{\mathcal{O}}$. In this case, $\psi_{\mathcal{O}}$ is a nontrivial character of $N_{\mathcal{O}}(\mathbb{A})$ and the subgroup $M_{\mathcal{O}}^\circ(\mathbb{A})$ stabilizes the character $\psi_{\mathcal{O}}$. When $\mathfrak{n}_1 \neq 0$, as explained explicitly in [**MW87**], the character $\psi_{\mathcal{O}}$ defines a character, which is denoted also by $\psi_{\mathcal{O}}$, on $N_2'(\mathbb{A})$, where N_2' is a subgroup of $N_{\mathcal{O}}$ such that $N_{\mathcal{O}}/\ker(\psi_{\mathcal{O}})$ is a Heisenberg group with the center $N_2'/\ker(\psi_{\mathcal{O}})$.

We will separate our discussion for these two cases to define Fourier coefficients and periods of automorphic forms φ on $G(\mathbb{A})$.

When $\mathfrak{n}_1 = 0$, this implies that $\mathfrak{n}_{2i+1} = 0$ for all $i \geq 0$, the Fourier coefficient of φ attached to the nilpotent orbit $\mathcal{O}$ or the $\mathcal{O}$-Fourier coefficient of φ is defined by the following integral

$$\mathcal{F}^{\psi_{\mathcal{O}}}(g,\varphi) := \int_{N_{\mathcal{O}}(k)\backslash N_{\mathcal{O}}(\mathbb{A})} \varphi(ng)\psi_{\mathcal{O}}^{-1}(n)dn. \tag{2.9}$$

It is easy to see that the $\mathcal{O}$-Fourier coefficient $\mathcal{F}^{\psi_{\mathcal{O}}}(g,\varphi)$ of φ has left invariant property with respect to the k-rational points $M_{\mathcal{O}}^\circ(k)$ of $M_{\mathcal{O}}^\circ$. Hence it defines an automorphic form on $M_{\mathcal{O}}^\circ(\mathbb{A})$.

Let H be a reductive closed subgroup of $M_{\mathcal{O}}^\circ$. Take an automorphic form ϕ on $H(\mathbb{A})$ and define formally the period integral

$$\mathcal{P}_{\mathcal{O},H}(\varphi,\phi,\psi_{\mathcal{O}}) := \int_{H(k)\backslash H(\mathbb{A})} \mathcal{F}^{\psi_{\mathcal{O}}}(h,\varphi)\phi(h)dh. \tag{2.10}$$

Note that this integral may be divergent. A general problem is to figure out, for a given φ on $G(\mathbb{A})$, what kind of automorphic forms ϕ on $H(\mathbb{A})$ will make the integral nonzero. If the integral is nonzero, one calls that the automorphic form φ is distinguished with respect to the data $(\mathcal{O}, H, \psi_{\mathcal{O}}, \phi)$, or has a generalized Gelfand-Graev model of type $(\mathcal{O}, H, \psi_{\mathcal{O}}, \phi)$.

When $\mathfrak{n}_1 \neq 0$, one define the $\mathcal{O}$-Fourier-Jacobi coefficient of an automorphic forms φ on $G(\mathbb{A})$ by the following integral

$$\mathcal{F}^{\psi_{\mathcal{O}}}(x, \varphi, \Phi) := \int_{N_{\mathcal{O}}(k)\backslash N_{\mathcal{O}}(\mathbb{A})} \varphi(nx)\theta_{\Phi}^{\psi_{\mathcal{O}}}(\bar{n}x)dn \tag{2.11}$$

Some modifications are needed here. It follows also that the $\mathcal{O}$-Fourier-Jacobi coefficient of φ is an automorphic form on $M_{\mathcal{O}}^{\circ}(\mathbb{A})$. Similarly, for an automorphic form ϕ on $H(\mathbb{A})$, one may formally define a period integral

$$\mathcal{P}_{\mathcal{O},H}(\varphi, \phi, \psi_{\mathcal{O}}, \Phi) := \int_{H(k)\backslash H(\mathbb{A})} \mathcal{F}^{\psi_{\mathcal{O}}}(h, \varphi, \Phi)\phi(h)dh. \tag{2.12}$$

Assume that the integral converges. If it is nonzero, then we call that the automorphic form φ is distinguished with respect to the data $(\mathcal{O}, H, \psi_{\mathcal{O}}, \phi)$, or has a generalized Gelfand-Graev model of type $(\mathcal{O}, H, \psi_{\mathcal{O}}, \phi)$.

3. Some Problems

3.1. Global problems. One of the basic problems in general representation theory is the classification problem. A classification theorem may be very formal. This means that one can classify representations in terms of a certain set of invariants. However, in the modern theory of automorphic forms, representation theory works as a principle to conceptually understand problems from number theory and arithmetic.

Periods of automorphic forms provides the automorphic description of the existence of certain invariant functionals on the space of irreducible automorphic representations. The occurrence of the certain periods may imply certain arithmetic properties of automorphic representations. Jacquet's theory to characterize quadratic base change of irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_E)$ from $\mathrm{GL}_n(\mathbb{A}_F)$ is the original example, where E/F is a quadratic extension of fields. By combining the idea of Harder-Langlands-Rapoport ([**HLR86**]), Jacquet's theory asserts that an irreducible cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ is a base change from $\mathrm{GL}_n(\mathbb{A}_F)$ if and only if Π is distinguished with respect to a certain unitary subgroup of $\mathrm{GL}_n(E)$, that is, the representation Π has a nontrivial period defined over the unitary group ([**Jcq03**], [**JY90**]).

In general, we may propose one qualitative problem and one quantitative problem for periods of automorphic forms defined in the previous section.

Problem 3.1. *Find implications of the existence of nontrivial periods to the structure of the given irreducible discrete automorphic representations in the sense of Langlands functoriality or other types of lifting theories.*

Problem 3.2. *Find explicit formula for a given period as defined in Section 2.3 in terms of special values or the residue of a certain automorphic L-function or other types of invariants attached to the automorphic forms.*

More concretely, one may study the following problem. For a given irreducible discrete series automorphic representation π of $G(\mathbb{A})$, which may not be cuspidal, and for a k-rational nilpotent orbit $\mathcal{O}$, one can define the $\mathcal{O}$-Fourier coefficient $\mathcal{F}^{\psi_{\mathcal{O}}}(g, \varphi_\pi)$ or $\mathcal{O}$-Fourier-Jacobi coefficient $\mathcal{F}^{\psi_{\mathcal{O}}}(x, \varphi_\pi, \Phi)$ of an automorphic form φ_π in the space of π as given in the previous section. It is also a basic problem to understand the spectral decomposition of $\mathcal{F}^{\psi_{\mathcal{O}}}(g, \varphi_\pi)$ or $\mathcal{F}^{\psi_{\mathcal{O}}}(x, \varphi_\pi, \Phi)$ as automorphic representation of $M_{\mathcal{O}}^{\circ}(\mathbb{A})$.

Problem 3.3. *For a given irreducible discrete series automorphic representation* π *of* $G(\mathbb{A})$ *and a* k*-rational nilpotent orbit* $\mathcal{O}$*, find the discrete part of the spectral decomposition of* $\mathcal{F}^{\psi_{\mathcal{O}}}(g,\varphi_\pi)$ *or* $\mathcal{F}^{\psi_{\mathcal{O}}}(x,\varphi_\pi,\Phi)$ *as automorphic representation of* $M^\circ_{\mathcal{O}}(\mathbb{A})$*.*

Then one may ask how to use the information obtained from Problem 3.3 to characterize the structure of π in the sense of Problem 3.1. More practically, one may choose a particular irreducible discrete series automorphic representation σ of $H(\mathbb{A})$, where H is a big reductive algebraic subgroup of $M^\circ_{\mathcal{O}}$, so that the occurrence of σ in the spectral decomposition of $\mathcal{F}^{\psi_{\mathcal{O}}}(g,\varphi_\pi)$ or $\mathcal{F}^{\psi_{\mathcal{O}}}(x,\varphi_\pi,\Phi)$ when restricted to $H(\mathbb{A})$ should be enough to characterize a certain structure of π. In many cases, the particular representation σ can be chosen to be the trivial representation of $H(\mathbb{A})$ or one-dimensional automorphic character of $H(\mathbb{A})$.

Problem 3.4. *Let* π *be an irreducible discrete series automorphic representation of* $G(\mathbb{A})$*. For a given* k*-rational nilpotent orbit* $\mathcal{O}$*, find a reductive algebraic subgroup* H *of* $M^\circ_{\mathcal{O}}$ *and a particular irreducible automorphic representation* σ *of* $H(\mathbb{A})$*, such that the occurrence of* σ *in the spectral decomposition of* $\mathcal{F}^{\psi_{\mathcal{O}}}(g,\varphi_\pi)$ *or* $\mathcal{F}^{\psi_{\mathcal{O}}}(x,\varphi_\pi,\Phi)$ *when restricted to* $H(\mathbb{A})$ *can be used to characterize a certain structure of* π *in the sense of Problem 3.1.*

Remark 3.5. *After we introduce the notion of automorphic L-functions, we will discuss the relations between the periods of automorphic representations and the special values or residues of certain automorphic L-functions. Some connections to arithmetic theory will also be mentioned later.*

3.2. Local problems. One of the main features in the modern theory of automorphic forms and representations is to consider the basic local-global principle from number theory in the content of representations of local compact topological groups. As an important invariant for automorphic representations, it is natural to consider the local version of periods of automorphic forms.

It is a basic theorem that an irreducible automorphic representation π of $G(\mathbb{A})$ can be expressed as a restricted tensor product over all local places v of k of irreducible admissible representations π_v of $G(k_v)$. A period of automorphic representation π produces a certain quasi-invariant functional over the space of the local component π_v for all local places v of k. For simplicity, we consider only here the case when the nilpotent orbit $\mathcal{O}$ is even.

For a given irreducible discrete series automorphic representation π of $G(\mathbb{A})$ and for an even k-rational nilpotent orbit $\mathcal{O}$, if the $\mathcal{O}$-Fourier coefficient $\mathcal{F}^{\psi_{\mathcal{O}}}(g,\varphi_\pi)$ is nonzero, it produces a nontrivial functional in the following space

$$\mathrm{Hom}_{N_{\mathcal{O}}(\mathbb{A})}(\pi,\mathbb{C}_{\psi_{\mathcal{O}}})\cong\prod_v\mathrm{Hom}_{N_{\mathcal{O}}(k_v)}(\pi_v,\mathbb{C}_{\psi_{\mathcal{O},v}}).\tag{3.1}$$

It follows that for each local place v of k, there exists a nontrivial functional in the space

$$\mathrm{Hom}_{N_{\mathcal{O}}(k_v)}(\pi_v,\mathbb{C}_{\psi_{\mathcal{O},v}}).\tag{3.2}$$

This last space has a natural structure of $M^\circ_{\mathcal{O}}(k_v)$-module. The first local problem can be stated as follows.

Problem 3.6. *Let* π_v *be an irreducible admissible representation of* $G(k_v)$*, and let* σ_v *be an irreducible admissible representation of a big reductive subgroup* H *of*

$M^\circ_{\mathcal{O}}(k_v)$. Find the dimension of the space

$$\mathrm{Hom}_{N_{\mathcal{O}}(k_v)}(\pi_v, \mathbb{C}_{\psi_{\mathcal{O},v}})^{(H(k_v),\sigma_v)}$$

which is the $(H(k_v), \sigma_v)$-isotypic part of $\mathrm{Hom}_{N_{\mathcal{O}}(k_v)}(\pi_v, \mathbb{C}_{\psi_{\mathcal{O},v}})$.

It is clear that the dimension of the space in Problem 3.6 is independent of the choices of the characters $\psi_{\mathcal{O}}$ attached to the k-stable orbit of $\mathcal{O}$. If a k-rational nilpotent orbit $\mathcal{O}$ has the property that for an irreducible admissible representation σ_v of $H(k_v)$, the dimension of the space in Problem 3.6 is at most one for all irreducible admissible representations π_v of $G(k_v)$, then we say that the nilpotent orbit $\mathcal{O}$ has the uniqueness property of type $(H, \sigma_v$ or has (H, σ_v)-uniqueness. If such a uniqueness property holds for all σ_v, we say that the orbit $\mathcal{O}$ has the H-uniqueness. If such uniqueness holds only for unramified representations and characters, we say that $\mathcal{O}$ has the corresponding unramified uniqueness. Some conjectures of Gross and Prasad for local Bessel models of orthogonal groups were discussed in [**GP92**] and [**GP94**]. Some new evidences and examples for the local Gross-Prasad conjecture can be found in [**GR05**].

Problem 3.7. *Classify all nilpotent orbits $\mathcal{O}$ with $M^\circ_{\mathcal{O}}$-unramified uniqueness, that is, for all unramified irreducible admissible representations π_v of $G(k_v)$, all unramified irreducible admissible representations of $M^\circ_{\mathcal{O}}(k_v)$, and for unramified characters $\psi_{\mathcal{O},v}$, the dimension of the following space*

$$\mathrm{Hom}_{N_{\mathcal{O}}(k_v)}(\pi_v, \mathbb{C}_{\psi_{\mathcal{O},v}})^{(M^\circ_{\mathcal{O}}(k_v),\sigma_v)}$$

is at most one.

Of course, it is an interesting problem to classify the nilpotent orbits with the unramified uniqueness property, but without the uniqueness property in general. From the point of view of the global theory, it is natural to consider the following problem.

Problem 3.8. *For a given k-rational nilpotent orbit $\mathcal{O}$ with the unramified uniqueness property, and for a given particular irreducible admissible representation σ_v of $M^\circ_{\mathcal{O}}(k_v)$, characterize all irreducible admissible representation π_v of $G(k_v)$ with the property that the dimension of the space*

$$\mathrm{Hom}_{N_{\mathcal{O}}(k_v)}(\pi_v, \mathbb{C}_{\psi_{\mathcal{O},v}})^{(M^\circ_{\mathcal{O}}(k_v),\sigma_v)}$$

is one.

If $\mathcal{O}$ is a k-rational nilpotent orbit with the uniqueness property, then at unramified local place, it is expected that the nontrivial functional attached to $\mathcal{O}$ does not vanish at the unramified vectors of π_v. In this case one says that the unramified vectors are the test vectors for the nontrivial functional attached to the nilpotent orbit $\mathcal{O}$ at the unramified local place. However, at the ramified local places, it is highly desired to develop he theory of local new vectors as test vectors for the nonvanishing of the local functionals. Some significant applications of the theory of local new vectors can be found in the study of special values of automorphic L-functions. See, for instance, the recent development of the Gross-Zagier formula by S.-W. Zhang ([**Z01**]) for some detailed discussion. B. Roberts and R. Schmidt has developed a theory of local new vectors for certain ramified representations of $\mathrm{GSp}_4(k_v)$ ([**RS03**]).

The uniqueness of local Whittaker models has been used by F. Shahidi ([**Sh88**] and [**Sh90**]) to construct his local coefficients, which can be related to the local gamma factors of generic representations and to the local Plancherel measures, and to some other problems in representations of p-adic groups. This work has been generalized by S. Friedberg and D. Goldberg ([**FG99**]) to deal with some nongeneric representations of p-adic groups by introducing the local Bessel models. It is expected that the whole theory should have the same conclusion for the other models attached to the nilpotent orbits with the uniqueness property.

For applications to automorphic L-functions, the uniqueness of Whittaker models has played an indispensable role in the Rankin-Selberg method to establish the basic analytic properties as conjectured by Langlands for families of automorphic L-functions ([**Bmp05**]). The uniqueness of Bessel models attached to the subregular nilpotent orbit of of $\mathfrak{so}_{2n+1}$ was first introduced in [**N75**] to study the standard automorphic L-functions for irreducible generic cuspidal automorphic representations of SO_{2n+1}. This work was generalized in [**G90**] and [**S93**] to the Bessel models attached to nilpotent orbits of $\mathfrak{so}_{2n+1}$ parameterized by partitions $[2r+1, 1^{2(n-r)}]$ with $r = 0, 1, \cdots, n$, for irreducible generic cuspidal automorphic representations of SO_{2n+1}. In [**GPSR97**], the whole theory was established without assuming the irreducible cuspidal automorphic representations being generic. The uniqueness of Bessel models for classical groups has been considered by S. Rallis. Some of his work in this direction can be found in [**GPSR97**].

3.3. Local-global principle.

One of the main concerns in the modern theory of automorphic forms is the local-global principle for certain properties which can be formulated for automorphic representations and for representations at all local places.

Let k be a number field and k_v be the local field at a local place v of k. Let π be an irreducible discrete series automorphic representation of $G(\mathbb{A})$. For simplicity, consider an even k-rational nilpotent orbit $\mathcal{O}$ with the uniqueness property, and an irreducible automorphic representation σ of $M^{\circ}_{\mathcal{O}}(\mathbb{A})$. The condition for the nonvanishing of the period $\mathcal{P}_{\mathcal{O}}(\varphi_\pi, \phi_\sigma, \psi_{\mathcal{O}})$ as defined in (2,10) with $H = M^{\circ}_{\mathcal{O}}$ is called a global period condition, and is denoted by **(gpc)**. The nonzero period $\mathcal{P}_{\mathcal{O}}(\varphi_\pi, \phi_\sigma, \psi_{\mathcal{O}})$ defines a nonzero element $\ell_{\mathcal{O}}$ in the following space

$$\mathrm{Hom}_{N_{\mathcal{O}}(\mathbb{A})}(\pi, \mathbb{C}_{\psi_{\mathcal{O}}})^{(M^{\circ}_{\mathcal{O}}(\mathbb{A}),\sigma)}. \tag{3.3}$$

By the uniqueness property for $\mathcal{O}$, the nonzero element $\ell_{\mathcal{O}}$ is unique up to scalar. By (3.1), the nonzero element $\ell_{\mathcal{O}}$ define for each local place v of k a nonzero element $\ell_{\mathcal{O},v}$, which is unique up to scalar, in the following space

$$\mathrm{Hom}_{N_{\mathcal{O}}(k_v)}(\pi_v, \mathbb{C}_{\psi_{\mathcal{O},v}})^{(M^{\circ}_{\mathcal{O}}(k_v),\sigma_v)}. \tag{3.4}$$

The condition for the nonvanishing of the space in (3.4) with the assumption that at all unramified local places v, the element $\ell_{\mathcal{O},v}$ does not vanish at the unramified vectors in π_v, is call a local period condition, and is denoted by **(lpc)**.

Problem 3.9 (Local-Global Principle). *For a k-rational nilpotent orbit $\mathcal{O}$, find a set of conditions depending on π and σ, such that* **(lpc)** *implies* **(gpc)***, i.e.*

$$(\mathbf{lpc}) \oplus \ \textbf{what conditions} \cong (\mathbf{gpc}).$$

In reality, it is useful to find the *sufficient conditions* for

$$(\mathbf{lpc}) \oplus \ \textbf{what conditions} \rightarrow (\mathbf{gpc}).$$

We remark that the conditions which make the equivalence in Problem 3.9 hold are expected to have relation with special values or poles of certain automorphic L-functions. We will discuss some typical cases after we introduce the notion of automorphic L-functions in the next section.

More precise relation between the local periods and the global periods is expected to be an identity which shows that the global period and a certain normalized product of local periods over all the local places are proportional. The proportional constant are expected to be a certain special value or residue of a certain automorphic L-function, which is related to the conditions in the local-global principle. Some interesting examples will be discussed later. It is well-known that the uniqueness of Whittaker models implies that the global Whittaker function attached to an irreducible generic automorphic representation can be expressed as a product of the corresponding local Whittaker functions over all local places.

4. Automorphic L-functions

4.1. L-functions. The simplest L-function is the well-known Riemann zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

It is almost elementary to show that $\zeta(s)$ converges absolutely for the real part of s greater than one, has a meromorphic continuation to the whole complex plane $\mathbb{C}$, and satisfies a functional equation which relates s to $1-s$. Any information about $\zeta(s)$ in the vertical strip $0\le Re(s)\le 1$ will be sensitive, mysterious, important, etc. A recent survey by S. Gelbart and S. Miller ([**GM04**]) provides a good source of discussion on this topics.

R. Langlands introduced the notion of automorphic L-functions attached to automorphic representations as follows. Let ${}^LG$ be the Langlands dual group of the reductive group G. Let π be an irreducible, unitary, cuspidal, automorphic representation of $G(\mathbb{A})$ and ρ be a finite-dimensional complex representation of ${}^LG$. The automorphic L-function attached to the pair (π,ρ) is given by the following Euler product

$$L(s,\pi,\rho)=\prod_v L(s,\pi_v,\rho)$$

if we write $\pi=\otimes_v\pi_v$. For almost all local finite places, the local components π_v are unramified. By Satake's theory, the unramified π are uniquely up to isomorphism determined by a semisimple conjugate class t_{π_v} in the dual group ${}^LG$. At the unramified place v, the local L-factor $L(s,\pi_v,\rho)$ is defined to be

$$L(s,\pi_v,\rho):=\det(I-\rho(t_{\pi_v})N_v^{-s})^{-1},$$

where N_v is the cardinal of the residual field at v. For the other local places, the local L-factor may be defined via the local Langlands correspondence, or via the Langlands-Shahidi method, or via the Rankin-Selberg method. See [**GSh88**] for more discussions.

Theorem 4.1 (Langlands ([**GSh88**])). *For any given pair (π,ρ) given as above, the automorphic L-function $L(s,\pi,\rho)$ converges absolutely for the real part of s large.*

Conjecture 4.2 (Langlands ([**GSh88**])). *For any given pair (π, ρ) given as above, the automorphic L-function $L(s, \pi, \rho)$ has meromorphic continuation to the whole complex plane $\mathbb{C}$ and satisfies a functional equation which related to s to $1-s$.*

It follows that there is a positive real number S_0, such that $L(s, \pi, \rho)$ will be sensitive, mysterious, and important when s lies in the vertical strip

$$\frac{1}{2} - s_0 \leq Re(s) \leq \frac{1}{2} + s_0.$$

Remark 4.3. (1) *The Langlands conjecture has been verified for many cases, by either the Langlands-Shahidi method or by the Rankin-Selberg method. See* [**GSh88**] *and* [**Bmp05**] *for detailed discussions.*

(2) *If π is generic, i.e. has a nonzero Whittaker-Fourier coefficient, then one expects that $\frac{1}{2} + s_0 = 1$. Shahidi proved in* [**Sh88**] *for all automorphic L-functions in the Langlands-Shahidi list have the property that the range of the absolute convergence can be made for the real part of s greater than two, which leads a weaker version of the generalized Ramanujan conjecture.*

(3) *The special values or the poles of automorphic L-functions at real value s satisfying*

$$\frac{1}{2} \leq s \leq \frac{1}{2} + s_0$$

should have something to do with the functorial structure of or the arithmetic significance of π.

We will discuss in more detail about the nonvanishing of special values or the existence of poles of automorphic L-functions $L(s, \pi, \rho)$ in terms of the functorial structure of π or the nonvanishing of periods of the automorphic forms in π. One of the most natural links to accomplish this is to use the Langlands theory of Eisenstein series, which was used by Langlands and then by Shahidi to prove the validity of the Langlands conjecture for a big family of automorphic L-functions.

4.2. Residual representations. From Langlands' theory of Eisenstein series ([**L76**] and [**MW95**]), the nonvanishing condition of certain automorphic L-functions is equivalent to the condition for the existence of a residue of certain Eisenstein series.

For simplicity of discussion, we may assume that G is quasi-split or even split over k. Let $P = MN$ be a standard maximal parabolic k-subgroup of G. Let $K = \prod_v K_v$ be a standard maximal compact subgroup of $G(\mathbb{A})$ such that

$$G(\mathbb{A}) = P(\mathbb{A})K$$

is the Iwasawa decomposition of $G(\mathbb{A})$.

Let σ be an irreducible cuspidal automorphic representation of $M(\mathbb{A})$. Consider V_σ-valued smooth functions ϕ_σ over $G(\mathbb{A})$ satisfying following properties:

1. for a fixed $g \in G(\mathbb{A})$, $k \mapsto \phi_\sigma(gk)$ generates a finite-dimensional representations of K;
2. for a fixed $g \in G(\mathbb{A})$, $m \mapsto \phi_\sigma(mg)$ is a smooth vector in V_σ and the function ϕ_σ satisfies $\phi_\sigma(ng) = \phi_\sigma(g)$ for all $n \in N(\mathbb{A})$.

Define

$$\Phi(g; s, \phi_\sigma) := \phi_\sigma(g) \exp < s + \rho_P, H_P(g) >$$

where ρ_P is the half of the sum of all positive roots with respect to the center of M, the parameter s is normalized as in [**Sh88**], and H_P is the Harish-Chandra map with respect to P. Then we define Eisenstein series of $G(\mathbb{A})$ attached to the cuspidal datum (P,σ) as

$$E(g;s,\phi_\sigma) := \sum_{\gamma\in P(k)\backslash G(k)} \Phi(\gamma g;s,\phi_\sigma). \tag{4.1}$$

THEOREM 4.4 (Langlands ([**L76**] and [**MW95**]). *For any given cuspidal datum (P,σ), the Eisenstein series $E(g;s,\phi_\sigma)$ converges absolutely for the real part of s large, has meromorphic continuation to the whole complex plane $\mathbb{C}$ (with finitely many poles at the positive real line after suitable normalization), and satisfies a functional equation relating the value s to the value $-s$.*

REMARK 4.5. *It is expected that the structure of the residual representations of $E(g;s,\phi_\sigma)$ at a real number $s_0>0$ should have direct relations with the structure of the cuspidal datum (P,σ). This turns out to be quite useful.*

From the Langlands theory of the constant terms of Eisenstein series, the poles of Eisenstein series can be detected in terms of explicit calculation of the constant terms of the Eisenstein series. The constant term of the Eisenstein series $E(g;s,\phi_\sigma)$ along a standard parabolic subgroup P' is always zero unless $P'=P$ ([**MW95**], II.1.7). In this case the constant term can be expressed as

$$\begin{aligned} E_P(g;s,\phi_\sigma) &= \int_{N(k)\backslash N(\mathbb{A})} E(ug;s,\phi_\sigma)dn \\ &= \Phi(g;s,\phi_\sigma)+\mathcal{M}(w_0,s)(\Phi(\cdot;s,\phi_\sigma))(g) \end{aligned} \tag{4.2}$$

where w_0 the longest Weyl element in the representatives of the double coset decomposition $W_M\backslash W_G/W_M$ of the Weyl groups.

It follows from the Langlands theory of Eisenstein series that $E(g;s,\phi_\sigma)$ has a pole at $s=s_0$ if and only if the term $\mathcal{M}(w_0,s)(\Phi(\cdot;s,\phi_\sigma))$ above has a pole at $s=s_0$ for some holomorphic (or standard) section $\Phi(g;s,\phi_\sigma)$ as defined before. Following the standard argument, if we consider factorizable sections

$$\Phi(\cdot;s,\phi_\sigma)=\otimes_v\Phi_v(\cdot;s,\phi_{\sigma_v}),$$

then we have

$$\mathcal{M}(w_0,s)(\Phi(\cdot;s,\phi_\sigma))=\prod_v\mathcal{M}_v(w_0,s)(\Phi_v(\cdot;s,\phi_{\sigma_v})).$$

By the Langlands-Shahidi theory ([**L71**] and [**Sh88**]) we have

$$\mathcal{M}(w_0,s)=\frac{\prod_l L(ls,\sigma,r_l)}{\prod_l L(ls+1,\sigma,r_l)}\cdot\prod_v\mathcal{N}_v(w_0,s), \tag{4.3}$$

where $\mathcal{N}_v(w_0,s)$ is the normalized intertwining operator

$$\mathcal{N}_v(w_0,s)=\frac{1}{r(s,\pi_v,\sigma_v,w_0)}\cdot\mathcal{M}_v(w_0,s).$$

Here the function $r(s,\pi_v,\sigma_v,w_0)$ is equal to

$$\frac{\prod_l L(ls,\sigma_v,r_l)}{\prod_l L(ls+1,\sigma_v,r_l)\prod_l\epsilon(ls,\sigma_v,r_l,\psi_v)}.$$

By a conjecture of Shahidi [**Sh90**], it is expected that $\mathcal{N}_v(w_0,s)(\Phi_v(\cdot;s,\phi_{\sigma_v}))$ is holomorphic and nonzero for $Re(s)\geq 0$ if σ_v is tempered. It has been checked for many cases when σ_v is tempered and generic, and when G is a classical group ([**CKPSS04**]).

By assuming Shahidi's conjecture on $\mathcal{N}_v(w_0, s)(\Phi_v(\cdot; s, \phi_{\sigma_v}))$, we know that the existence of a pole at $s = s_0$ of $E(g; s, \phi_\sigma)$ is equivalent to the existence of a pole at $s = s_0$ of the product of L-functions

$$\prod_l L(ls, \sigma, r_l).$$

Following from Shahidi ([**Sh88**]), if σ is generic, then the L-function $L(s, \sigma, r_l)$ is holomorphic for $Re(s) > 2$, and by the generalized Ramanujan conjecture, $L(s, \sigma, r_l)$ is holomorphic for $Re(s) > 1$. If we write

$$\prod_l L(ls, \sigma, r_l) = L(s, \sigma, r_1)L(2s, \sigma, r_2) \prod_{l \geq 3} L(ls, \sigma, r_l).$$

It is expected that $\prod_{l \geq 3} L(ls, \sigma, r_l)$ should be holomorphic for $Re(s) \geq \frac{1}{2}$. Then the existence of a pole at $s = s_0$ of the product of L-functions

$$\prod_l L(ls, \sigma, r_l)$$

should be equivalent to either that $L(s, \sigma, r_1)$ has a pole at $s = 1$ (Case (1)) or that $L(s, \sigma, r_1)$ does not vanish at $s = \frac{1}{2}$ and $L(s, \sigma, r_2)$ has a pole at $s = 1$ (Case ($\frac{1}{2}$)). Note that the so called (Case (0)) is the one corresponding to the holomorphicity of the product $\prod_l L(ls, \sigma, r_l)$.

Remark 4.6. *One expects that Case* (1) *and Case* ($\frac{1}{2}$) *should not occur at the same time, but it is possible for both not to occur (when Case* (0) *occurs. For classical groups and generic σ, this follows from the explicit Langlands functorial lifting from classical groups to the general linear group. In particular, the structures of σ related to these two cases are essentially different.*

4.3. Model-comparison. In the previous subsection, the existence of the residual representation

$$\mathcal{E}_{s_0}(g; \phi_\sigma) := \mathrm{Res}_{s=s_0} E(g; s, \phi_\sigma) \tag{4.4}$$

is expected to be characterized in terms of the existence of a pole at $s = s_0$ of the product of L-functions

$$L(s, \sigma, r_1)L(2s, \sigma, r_2) \tag{4.5}$$

assuming that σ is generic. The question in this subsection is to ask how to describe the structure of the residual representation $\mathcal{E}_{s_0}(g; \phi_\sigma)$ in terms of the structure of the cuspidal support σ, or vise versa. From a representation-theoretic point of view, structures of representations are usually characterized by means of certain invariants attached to the representations. For automorphic representations, it is natural to consider period integrals in order to distinguish automorphic representation from each other. By reciprocity, that an automorphic representation has a certain nonzero period is equivalent to that the automorphic representation has a certain model. For example, it has a Whittaker model if the representation is generic.

When consider two automorphic representations which are related via a certain kind of transfers, say, Langlands functoriality, theta correspondence, or endoscopy lifting, it is natural to expect the two automorphic representations should share certain compatible models. This should be viewed as a version of the Langlands functoriality principle. In reality, one may expect an explicit formula which relates a certain model for one representation to a certain model for the other representation. This is what we called *Model-comparison identity*. Such an idea is not really new. One can find such model-comparison identities in the constructive theory of automorphic forms, such as the Langlands theory of constant terms of Eisenstein series, the Langlands-Shahidi method and the Rankin-Selberg method to study automorphic L-functions, explicit theta correspondences, the descent method (backward liftings) of automorphic forms, relative trace formula method, etc.

We consider in the following the relation between the cuspidal datum (P, σ) and the residual representation $\mathcal{E}_{s_0}(g; \phi_\sigma)$ of $G(\mathbb{A})$. It is the simplest case of Langlands functoriality from (M, σ) to $(G, \mathcal{E}_{s_0}(g; \phi_\sigma))$. The problem is: for a given (M, σ) and the value s_0, to find a suitable spherical subgroup H of G, an automorphic representation τ of $M \cap H$, and an automorphic representation $\mathcal{E}_0(h; \phi_\tau)$ of $H(\mathbb{A})$, such that there exists an identity which relates the period (called the outer period) of $\mathcal{E}_{s_0}(g; \phi_\sigma)$

$$\int_{H(k)\backslash H(\mathbb{A})} \mathcal{E}_{s_0}(h; \phi_\sigma)\mathcal{E}_0(h; \phi_\tau)dh \tag{4.6}$$

and the period (called the inner period) of σ

$$\int_{M\cap H(k)\backslash M\cap H(\mathbb{A})} \phi_\sigma(x)\phi_\tau(x)dx. \tag{4.7}$$

A Theorem one wishes to prove is

THEOREM 4.7. *The nonvanishing of outer period (4.6) is equivalent to the nonvanishing of inner period (4.7).*

REMARK 4.8. *Since the Langlands functoriality from σ to $\mathcal{E}_{s_0}(g; \phi_\sigma)$ is nothing but the 'parabolic induction', it is natural to find the data $(H, \tau, \mathcal{E}_0(h; \phi_\tau))$ through the Frobenius reciprocity law. More precisely, one has to study the orbital decomposition*

$$P\backslash G/H$$

*and possible quasi-invariant distributions attached to each of the orbits. Note that the Rankin-Selberg method ([**Bmp05**]) for automorphic L-functions uses the Zariski open orbit! However, in the model-comparison study, one pays more attention to the other lower dimensional orbits, the closed orbits in particular.*

*Of course, one has to regularize the period integral to deal with the convergence problem. One can apply Arthur's truncation method to one of the residual automorphic forms in the integrand. Other methods of regularization may also apply, see [**JLR99**] and [**LR04**] for instance.*

REMARK 4.9. *Some very interesting cases have been treated by this approach and have some striking consequences. A detailed discussion has be carried out in author's recent survey paper [**Jng05**].*

*From the above arguments, it follows that the nonvanishing of the inner period in (4.7) implies the existence of the pole at s_0 of the product of L-functions in (4.5). The converse to this statement is known as a version of the conjecture of Gross-Prasad type ([**GP92**] and [**GP94**]). We will discuss this issue in the next section.*

5. Examples

5.1. Special values of the Rankin-Selberg L-functions. In this section we discuss briefly the work joint with D. Ginzburg and S. Rallis on nonvanishing of the central value of the Rankin-Selberg L-functions ([**GJR04**] and [**GJR**]). The main idea is to use the *Model-comparison identity* approach to link the periods of the generalized Gelfand-Graev type to the central value of the Rankin-Selberg L-functions. More detailed discussions and survey of related work has been written in [**Jng05**].

We may assume that $n \geq r$. We denote that $G_n := \mathrm{SO}_{2n+1}$ and $H_r := \mathrm{SO}_{2r}$. Let $\mathcal{O}_{n,r}$ be the nilpotent orbit of $\mathfrak{so}_{2n+1}$ corresponding to the odd orthogonal partition $[2r+1, 1^{2(n-r)}]$ in $\mathcal{P}_1(2n+1)$. Then we have

$$N_{n,r} = N_{\mathcal{O}_{n,r}} := \{v(u,x,z) = \begin{pmatrix} u & x & z \\ & I_{2r+1} & x^* \\ & & u^* \end{pmatrix}\} \subset G_n$$

where $u \in U_{n-r}$, which is the standard upper triangular unipotent subgroup of GL_{n-r}. Define a character $\psi_{n,r}$ of $N_{n,r}(\mathbb{A})$ by

$$\psi_{n,r}(v) := \psi_0(u_{1,2} + \cdots + u_{r-l-1,r-l})\psi_0(x_{r-l,l+1}).$$

It is clear that $\psi_{n,r}$ is trivial on $N_{n,r}(k)$. Denote the normalizer of $N_{n,r}$ in G_n by

$$M := N_{G_n}(N_{n,r}) \cong \mathrm{GL}_1^{n-r} \times \mathrm{SO}_{2r+1}.$$

Then the connected component of the stabilizer of $\psi_{n,r}$ in M is isomorphic to H_r.

When $n \leq l$, similar notations can be introduced, and we omit the details.

For $\varphi_\sigma \in V_\sigma$ and $\varphi_\tau \in V_\tau$, define $\psi_{n,r}$-Fourier coefficient of φ_σ to be

$$\mathcal{F}^{\psi_{n,r}}(\varphi_\sigma)(h) := \int_{N_{n,r}(k)\backslash N_{n,r}(\mathbb{A})} \varphi_\sigma(vh)\psi_{n,r}(v)^{-1}dv,$$

and the period is defined as follows:

$$\mathcal{P}_{n,r}(\varphi_\sigma, \varphi_\tau, \psi_{n,r}) := \int_{H_r(k)\backslash H_r(\mathbb{A})} \mathcal{F}^{\psi_{n,r}}(\varphi_\sigma)(h)\varphi_\tau(h)dh. \tag{5.1}$$

The following is one of the main results proved in [**GJR04**],[**GJR**].

THEOREM 5.1 ([**GJR04**],[**GJR**]). *If the period $\mathcal{P}_{r,l}(\varphi_\sigma, \varphi_\tau, \psi_{r,l})$ is not identically zero on the space $V_\sigma \otimes V_\tau$, then $L(\frac{1}{2}, \pi_1 \times \pi_2)$ is nonzero.*

In order to explain the model-comparison identity approach, we have to introduce the following two Eisenstein series:

1. the Eisenstein series $E(g; s, \phi_{\pi_1 \otimes \tau})$ on SO_{4r+2l} attached to the cuspidal data $\pi_1 \otimes \tau$ of $\mathrm{GL}_{2r} \times \mathrm{SO}_{2l}$;
2. the Eisenstein series $E(g; s, \phi_{\pi_1 \otimes \sigma})$ on SO_{6r+1} attached to the cuspidal data $\pi_1 \otimes \sigma$ of $\mathrm{GL}_{2r} \times \mathrm{SO}_{2r+1}$.

PROPOSITION 5.2 ([**GJR04**],[**GJR**]). *The Eisenstein series $E(g; s, \phi_{\pi_1 \otimes \tau})$ has a possible simple pole at $s = \frac{1}{2}$. The residue, denoted by $\mathcal{E}_{\frac{1}{2}}(g; \phi_{\pi_1 \otimes \tau})$, is nonzero if and only if $L(\frac{1}{2}, \pi_1 \times \tau)$ is nonzero. The Eisenstein series $E(g; s, \phi_{\pi_1 \times \sigma})$ has a simple pole at $s = 1$, and the residue is denoted by $\mathcal{E}_1(g; \phi_{\pi_1 \times \sigma})$.*

Then the 'outer' period is given by

$$\mathcal{P}_{3r,2r+l}(\mathcal{E}_1(\cdot;\phi_{\pi_1\times\sigma}),\mathcal{E}_{\frac{1}{2}}(\cdot;\phi_{\pi_1\otimes\tau}),\psi_{3r,2r+l}).$$

The 'specialization' of Theorem 4.7 to this case is the following theorem.

THEOREM 5.3 ([**GJR04**],[**GJR**]). *The nonvanishing of outer period*

$$\mathcal{P}_{3r,2r+l}(\mathcal{E}_1(\cdot;\phi_{\pi_1\times\sigma}),\mathcal{E}_{\frac{1}{2}}(\cdot;\phi_{\pi_1\otimes\tau}),\psi_{3r,2r+l})$$

is equivalent to the nonvanishing of the inner period $\mathcal{P}_{r,l}(\varphi_\sigma,\varphi_\tau,\psi_{r,l})$.

More detailed discussions on this Theorem and the relation to the Gross-Prasad conjecture have been given in [**Jng05**]. The analogue for unitary groups is the work in progress of Ginzburg and Jiang. The following example is a special case of the Gross-Prasad conjecture, which was proposed first by H. Jacquet.

5.2. Local-global principle. The most significant lower rank case of the Gross-Prasad conjecture is the case when $G = \mathrm{SO}_4$ and $H = \mathrm{SO}_3$. This can be viewed as the period associated to the zero orbit. In this case, it is a conjecture of Jacquet.

Let π_1, π_2, and π_3 be irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ with the product of the central characters being trivial, i.e.

$$\omega_{\pi_1}\cdot\omega_{\pi_2}\cdot\omega_{\pi_3}=1.$$

The conjecture of Jacquet asserts that the central value of the triple product L-function

$$L(\frac{1}{2},\pi_1\times\pi_2\times\pi_3)$$

is nonzero if and only if there is a unique quaternion algebra D over k such that the trilinear period integral

$$\int_{Z_{\mathrm{D}^\times}(\mathbb{A})\mathrm{D}^\times(k)\backslash\mathrm{D}^\times(\mathbb{A})}\varphi_{\pi_1^{\mathrm{D}}}(x)\varphi_{\pi_2^{\mathrm{D}}}(x)\varphi_{\pi_3^{\mathrm{D}}}(x)dx$$

is nonzero for a certain choice of $\varphi_{\pi_i^{\mathrm{D}}}$, $i=1,2,3$, where π_1^{D} is the image of π_i under the Jacquet-Langlands correspondence.

The conjecture of Jacquet was completely proved in [**HK04**]. In [**Jng98b**] and [**Jng01**], a different approach has been applied to the split period case of Jacquet's conjecture, which was reformulated in the form of the local-global principle. More recently, Boecherer, Furusawa, and Schultze-Pillot proved some special case for the relevant pair $(\mathrm{SO}_5,\mathrm{SO}_4)$ ([**BFSP04**]), and Ichino and Ikeda proved some special case for the relevant pair $(\mathrm{SO}_6,\mathrm{SO}_5)$ ([**II04**]). In those new cases, they obtain precise formula for the special value of the relevant automorphic L-functions.

THEOREM 5.4 ([**HK04**], [**Jng98b**], and [**Jng01**]). *Let* π_i *(*$i=1,2,3$*) be irreducible cuspidal automorphic representations of* $GL_2(\mathbf{A})$ *with central character* ω_i *(*$i=1,2,3$*). Assume that the product* $\prod_{i=1}^3\omega_i=1$. *Then the following are equivalent.*

(1) *For each local place* v, $Hom_{GL_2(F_v)}(\pi_{1,v}\otimes\pi_{2,v}\otimes\pi_{3,v},1)\neq 0$, *and the central value of the triple product L-function* $L(\frac{1}{2},\pi_1\otimes\pi_2\otimes\pi_3)$ *is nonzero.*

(2) *The (global) trilinear period*

$$\int_{Z(\mathbb{A})GL_2(F)\backslash GL_2(\mathbb{A})} \phi_{\pi_1}(x)\phi_{\pi_2}(x)\phi_{\pi_3}(x)dx$$

is nonzero for certain choice of cusp forms $\phi_{\pi_i} \in \pi_i$, $i = 1, 2, 3$.

This is a perfect example of the local-global principle for periods. It also happens that the conditions we found is sufficient to make **(lpc)** imply **(gpc)**. The following is a good example.

The genericity of automorphic forms has played an indispensable roles in the theory of automorphic forms and L-functions. Generic automorphic forms have nonzero Whittaker-Fourier coefficients, which are the Fourier coefficients associated to the regular nilpotent orbit, and produce the global Whittaker models for the automorphic forms and the automorphic representations. The uniqueness of local Whittaker models proved by J. Shalika [**Shl74**] means that the regular nilpotent orbit has the uniqueness property. The significance of this uniqueness property has been discussed in [**Bmp05**] and [**GSh88**].

It is known that all irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ are generic. The existence of non-generic cuspidal automorphic representations for groups which is not GL_n yields counter-examples to the generalized Ramanujan conjecture ([**HPS79**]). The local-global relation of the genericity has been a very sensitive problem in the theory. From Arthur's conjecture ([**Ar89**], [**Ar04**], and [**Cl**]), it is expected that for a given irreducible cuspidal automorphic representation $\pi = \otimes\pi_v$, if one of the local component π_v is generic, then every local component is generic. Some results for $\mathrm{SO}(2n+1)$ have been proved in [**JS05**].

The local-global principle of genericity asks the condition for an irreducible cuspidal automorphic representation π with all local components π_v being generic to be globally generic, i.e. to have a nonzero Whittaker-Fourier coefficient. The following is a theorem proved for Sp(4).

Theorem 5.5 ([**KRS92**]). *Let* $\pi = \otimes\pi_v$ *be an irreducible cuspidal automorphic representation of* $\mathrm{Sp}_4(\mathbb{A})$. *Assume that all local components* π_v *are generic, and the partial degree 5 L-function attached to* π, $L(s, \pi, \mathrm{St})$, *has a pole at* $s = 1$. *Then* π *is globally generic.*

In fact, if this is the case, then π is not only just globally generic, but also an image of theta lifting from an even orthogonal groups. More discussions can be found in [**GRS97**].

5.3. Functoriality. The characterization of the image of a certain functorial transfer or lifting in terms of automorphic invariants, i.e. periods of automorphic forms is one of the original motivation of the current study of periods of automorphic forms. The theory of quadratic base change of irreducible cuspidal automorphic representations of GL_n was established by Arthur and Clozel using twisted version of the Arthur-Selberg trace formula ([**AC89**]). By Jacquet's relative trace formula, the theory can be formulated in terms of periods. Let E/k be a quadratic extension of number fields. An irreducible cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ is a base change from $\mathrm{GL}_n(\mathbb{A}_k)$ if and only if there exists a unitary group U_n in $\mathrm{GL}_n(E)$ such that the period $\mathcal{P}_{\{0\},U_n}(\varphi_\Pi, 1)$ is not identically zero ([**Jcq03**], [**JY90**]).

Another example of characterization of the image of Langlands functorial lifting in terms of periods has been completed via recent work on the Langlands functorial transfer from classical groups to the general linear group. More precisely, let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$. Then the following three statements are equivalent.

(1) π is the image of Langlands functorial lifting from SO_{2n+1};
(2) π has a nonzero Shalike period;
(3) The exterior square L-function $L(s,\pi,\Lambda^2)$ has a simple pole at $s=1$.

The theory has been established over years through the work of many authors in [**JS90**], [**Bmp90**], [**CKPSS04**], [**GRS01**], [**K02**], [**JS03**].

Some results of this type can also be found in [**GJ00**], [**K00**], [**GJ01**], [**KR94**], [**GRS97**], to mention a few.

6. Final Remarks

In connection with Problem 3.2, one may think of Deligne's conjecture on special values of motivic L-functions ([**D79**]). According the Langlands philosophy, these motivic L-functions are expected to be automorphic. From this point of view, one expects an expression of special values of automorphic L-functions in terms of periods of automorphic forms. If the automorphic forms are motivic, then one expects that the periods of motivic automorphic forms can be expressed as periods in the sense of the Deligne conjecture. In the Rankin-Selberg method approach to automorphic L-functions, one has a natural expression of special values of certain automorphic L-functions in terms of a certain integral. Such integrals may be transferred to periods of automorphic forms, which may be closer to what required by the Deligne conjecture. The well-known Siegel-Weil formula, especially the one extended by Kudla and Rallis, has been the useful tool to obtain the 'right' periods. We refer to [**Hr94**] for detailed discussion. The relative trace formula approach to establish such explicit identities is also very powerful. The generalization of the well-known Waldspurger formula to general cuspidal automorphic forms of GL_2 is obtained in [**BM03**], and the generalization of the Waldspurger formula to high rank group cases are expected by using relative trace formula.

Another aspect connected to Problem 3.2 is to calculate explicitly certain periods of automorphic forms. If the periods can be expressed in terms of something of L-functions, then the analytic method on automorphic L-functions may be used to estimate the periods. Then nice applications to arithmetic may follow. In the recent work [**JLZ05**], one finds that the equidistribution property for certain cycles in the Hilbert modular varieties can be established by explicit calculation of certain periods. It is expected that the method formulated in [**JLZ05**] may be applicable to many other interesting cases.

It is very worthwhile to point out that even the periods of automorphic forms may not have local uniqueness property, one may still obtain a formula which expresses the global period of automorphic forms as a eulerian product of local periods (or functionals) with a natural choice at each local place. An very interesting example has been discussed in [**Jcq01**].

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