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Models for Certain Residual Representations of Unitary Groups

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We dedicate this paper to Stephen Gelbart on the occasion of his 60th birthday

Abstract. In this paper, we consider the generalized Gelfand-Graev models for automorphic forms on unitary groups, which characterize the existence of certain residual representations for unitary groups. As one of the consequences, we obtain results related to the Gross-Prasad type conjecture for unitary groups.

1. Introduction

One of the main problems in the modern theory of automorphic forms is to understand the discrete spectrum of the space of square integrable automorphic functions of a reductive algebraic group defined over a number field. The non-cuspidal part of the discrete spectrum is generated by the residues of Eisenstein series following the theory of Langlands ([**MW95**]). It is still an open problem to determine which residues of Eisenstein series do occur in the residual spectrum. The traditional approach follows from the Langlands theory of constant terms of Eisenstein series, which has been proved to be quite successful for certain families of Eisenstein series. For the general linear groups, the residual spectrum has been completely determined by the work of Moeglin and Waldspurger ([**MW89**]). The result turns out to be as conjectured by Jacquet ([**J84**]). Some lower rank cases have also been treated by this approach.

Starting with the work of Jacquet and Rallis ([**JR92**]), another approach to determine the residual spectrum has taken place. This approach is based on the inner structure of the cuspidal data, which one uses to build the Eisenstein series. The existence of the poles of such Eisenstein series is determined by the nonvanishing of certain period integral attached to the cuspidal data. This approach has also accomplished many interesting cases. We refer to [**Jng07**] and [**Jng08**] for more detailed accounts of this approach.

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In this paper, we study the residues of Eisenstein series of quasi-split unitary groups associated to maximal parabolic subgroup and generic cuspidal data. We show that the structure of the generic cuspidal data determines the location of the poles of the Eisenstein series and the structure of the corresponding residues. The structure of automorphic representations here are described in terms of the generalized Gelfand-Graev models attached to a certain family of nilpotent orbits in the corresponding Lie algebras. See §3 for the definition of the generalized Gelfand-Graev models and Theorem 4.6 for precise statement of our results. The relation between this new approach and the traditional approach is expressed by conjectures of the Gross-Prasad type. Some detailed remarks will be given at the end of this paper. For orthogonal groups, symplectic groups and metaplectic groups, this has been investigated in [**GJR04**] and [**GJR05**].

It is worthwhile to mention that for unitary groups of three variables, the discrete spectrum of the space of square integrable automorphic functions has been completely determined by the work of Rogawski ([**R90**]), the applications of which to arithmetic and number theory has been extensively discussed in [**LR92**] and [**BR94**]. The global Arthur packets in this case have been constructed by theta liftings and characterized by periods of automorphic forms by the work of Gelbart, Rogawski and Soudry in particular ([**GR91**] and [**GRS97**]), which forms typical examples for the development for the general unitary groups.

2. Eisenstein Series

We introduce the families of Eisenstein series on quasi-split unitary groups, which will be studied in the paper.

2.1. Quasi-split unitary groups. Let F be a number field and E be a quadratic extension of F, whose Galois group is denoted by $\Gamma_{E/F} = \{1, \iota\}$. Let J_m^ϵ be the ϵ-symmetric matrix of size $m \times m$ given inductively by

$$J_m^\epsilon = \begin{pmatrix} & & 1 \\ & J_{m-2}^\epsilon & \\ \epsilon & & \end{pmatrix} \tag{2.1}$$

with $J_2^\epsilon = \left(\begin{smallmatrix} & 1 \\ \epsilon & \end{smallmatrix}\right)$, $J_1^\epsilon = 1$ and $\epsilon = \pm$. Consider the $2m$-dimensional F-vector space $\mathcal{V}_m^\epsilon = E^m$ with the ϵ-Hermitian form attached to J_m^ϵ. It is a nondegenerate ϵ-Hermitian F-vector space with Witt index $2[\frac{m}{2}]$. Let U_m^ϵ be the F-quasisplit unitary group attached to $\mathcal{V}_m^\epsilon$. Let $\mathrm{R}_{E/F}(\mathrm{GL}_n)$ denote the Weil restriction of the GL_n from E to F. With respect to the choice of J_m^ϵ, we have a Borel subgroup $B = TU$ as an upper triangular matrix with maximal torus T, which consists of elements of type

$$\mathrm{diag}(t_1, \cdots, t_{\frac{m}{2}}, t_{\frac{m}{2}}^{-1}, \cdots, t_1^{-1}), \quad t_i \in E^\times,$$

if m is even, and of type

$$\mathrm{diag}(t_1, \cdots, t_{[\frac{m}{2}]}, 1, t_{[\frac{m}{2}]}^{-1}, \cdots, t_1^{-1}), \quad t_i \in E^\times,$$

if m is odd. The F-split maximal torus is $T_s = T \cap \mathrm{GL}_m(F)$.

For each integer $1 \le r \le [\frac{m}{2}]$, one has the polar decomposition of $\mathcal{V}_m^\epsilon$ as

$$\mathcal{V}_m^\epsilon = X_r \oplus \mathcal{V}_{m-2r}^\epsilon \oplus X_r^* \tag{2.2}$$

where X_r is the $2r$-dimensional totally isotropic F-subspace of $\mathcal{V}_m^\epsilon$ and X_r^* is the F-subspace dual to X_r. The stabilizer of X_r in U_m^ϵ is the standard maximal

F-parabolic subgroup, which is denoted by $P_r = P_{(r,m),\epsilon}$. Its Levi part M_r is isomorphic to $\mathrm{R}_{E/F}(\mathrm{GL}_r) \times \mathrm{U}^{\epsilon}_{m-2r}$. An element of M_r will be denoted by

$$m = m(a,h) = \begin{pmatrix} a & & \\ & h & \\ & & a^* \end{pmatrix} \in \mathrm{R}_{E/F}(\mathrm{GL}_r) \times \mathrm{U}^{\epsilon}_{m-2r} \subset \mathrm{U}^{\epsilon}_m. \tag{2.3}$$

The unipotent radical of P_r is denoted by N_r, whose elements are denoted by

$$n = n(x,z) = \begin{pmatrix} I_r & x & z \\ & I_{m-2r} & x^* \\ & & I_r \end{pmatrix} \in N_r \subset \mathrm{U}^{\epsilon}_m. \tag{2.4}$$

In the following we assume that $1 \leq r < [\frac{m}{2}]$. Let π be an irreducible unitary cuspidal automorphic representation of $\mathrm{R}_{E/F}(\mathrm{GL}_r)(\mathbb{A}_F)$, where $\mathbb{A}_F$ is the ring of adeles of F. We will denote by $\mathbb{A}_E$ the ring of adeles of E. Let σ be an irreducible unitary generic cuspidal automorphic representation of $\mathrm{U}^{\epsilon}_{m-2r}(\mathbb{A}_F)$. Attached to the generic cuspidal datum $(P_r, \pi \otimes \sigma)$, one has the Eisenstein series $E(g; \phi_{\pi\otimes\sigma}, s)$ on $\mathrm{U}^{\epsilon}_m(\mathbb{A}_F)$, which can be defined more precisely as follows.

We first realize the cuspidal automorphic representation $\pi \otimes \sigma$ in the space of square integrable automorphic functions $\mathrm{L}^2(Z_{M_r}(\mathbb{A}_F)M_r(F)\backslash M_r(\mathbb{A}_F))$, with Z_{M_r} being the center of M_r. Let K be the maximal compact subgroup of $\mathrm{U}^{\epsilon}_m(\mathbb{A}_F)$ such that

$$\mathrm{U}^{\epsilon}_m(\mathbb{A}_F) = P_r(\mathbb{A}_F)\mathrm{K}$$

is the Iwasawa decomposition. Let $\phi_{\pi\otimes\sigma}$ be a $\mathrm{K} \cap M_r(\mathbb{A}_F)$-finite automorphic form in space $V_{\pi\otimes\sigma}$, which is extended as a function of $\mathrm{U}^{\epsilon}_m(\mathbb{A}_F)$ ([**Sh88**, §2]), so that for $g = um\mathrm{k} \in \mathrm{U}^{\epsilon}_m(\mathbb{A}_F)$

$$\phi_{\pi\otimes\sigma}(g) = \phi_{\pi\otimes\sigma}(m\mathrm{k})$$

and for any fixed $\mathrm{k} \in \mathrm{K}$, the function

$$m \mapsto \phi_{\pi\otimes\sigma}(m\mathrm{k})$$

is a $\mathrm{K} \cap M_r(\mathbb{A}_F)$-finite automorphic form in space $V_{\pi\otimes\sigma}$. We define

$$\Phi(g, \phi_{\pi\otimes\sigma}, s) := \phi_{\pi\otimes\sigma}(g) \exp\langle s + \rho_{P_r}, H_{P_r}(g)\rangle \tag{2.5}$$

for $g \in \mathrm{U}^{\epsilon}_m(\mathbb{A}_F)$. As in [**Sh88**, §1], the parameter s is identified with $s\tilde{\alpha}_r$, where $\tilde{\alpha}_r$ is the co-root dual to the simple root α_r in the unipotent radical N_r with respect to the maximal F-split torus T_s. Note that α_r determines the standard parabolic subgroup $P_r = P^{\epsilon}_{r,m}$.

In our case we have

$$\exp\langle s + \rho_{P_r}, H_{P_r}(g)\rangle = |\det a|^{s+\frac{m-r}{2}}_{\mathbb{A}_E} \tag{2.6}$$

where we write $g = um(a,h)\mathrm{k} \in \mathrm{U}^{\epsilon}_m(\mathbb{A}_F)$ and $m(a,h)$ as in (2.3). Then the Eisenstein series is given by

$$E(g, \phi_{\pi\otimes\sigma}, s) = \sum_{\gamma \in P_r(F)\backslash \mathrm{U}^{\epsilon}_m(F)} \Phi(\gamma g, \phi_{\pi\otimes\sigma}, s), \tag{2.7}$$

which converges absolutely for the $Re(s)$ large, has meromorphic continuation to $\mathbb{C}$, and may have finitely many poles on the positive real axis, following the general theory of Eisenstein series ([**MW95**]).

2.2. Residues of Eisenstein series. We want to determine the location of the poles of the Eisenstein series $E(g,\phi_{\pi\otimes\sigma},s)$. It follows from the Langlands theory of Eisenstein series that the Eisenstein series $E(g,s,\phi_{\pi\otimes\sigma})$ has a pole at $s=s_0\in\mathbb{R}_{>0}$ if and only if its constant terms have a pole at $s=s_0$.

The constant term of the Eisenstein series $E(g,\phi_{\pi\otimes\sigma},s)$ along a standard parabolic subgroup P is always zero unless $P=P_r$ ([**MW95**, II.1.7]). In this case, we have

$$\begin{aligned} E_{P_r}(g,\phi_{\pi\otimes\sigma},s) &= \int_{N_r(F)\backslash N_r(\mathbb{A}_F)} E(ng,\phi_{\pi\otimes\sigma},s)dn \\ &= \Phi(g,\phi_{\pi\otimes\sigma},s)+\mathcal{M}(w_r,s)(\Phi(\cdot,\phi_{\pi\otimes\sigma},s))(g). \end{aligned} \tag{2.8}$$

We denote here by w_r the longest Weyl element in the representatives of the double coset decomposition $W_{M_r}\backslash W_{\mathrm{U}_m^\epsilon}/W_{M_r}$ of the Weyl groups. The intertwining operator $\mathcal{M}(w_r,s)$ is defined by the following integral

$$\mathcal{M}(w_r,s)(\Phi(\cdot,\phi_{\pi\otimes\sigma},s))(g):=\int_{N_r(F)\backslash N_r(\mathbb{A}_F)}\Phi(w_r^{-1}ng,\phi_{\pi\otimes\sigma},s)dn$$

which is a $\mathrm{U}_m^\epsilon(\mathbb{A}_F)$-mapping from the unitarily induced representation

$$\mathrm{I}(s,\pi\otimes\sigma)=\mathrm{Ind}_{P_r(\mathbb{A}_F)}^{\mathrm{U}_m^\epsilon(\mathbb{A}_F)}(\pi\otimes\sigma\otimes\exp\langle s,H_{P_r}(\cdot)\rangle) \tag{2.9}$$

to $I(-s,w_r(\pi\otimes\sigma))$. It reduces to determine the location of poles of the term $\mathcal{M}(w_r,s)(\Phi(\cdot,\phi_{\pi\otimes\sigma},s))$ in (2.8) for some holomorphic (or standard) section $\Phi(g,\phi_{\pi\otimes\sigma},s)$ in $\mathrm{I}(s,\pi\otimes\sigma)$.

For factorizable sections

$$\Phi(\cdot,\phi_{\pi\otimes\sigma},s)=\otimes_v\Phi_v(\cdot,\phi_{\pi_v\otimes\sigma_v},s), \tag{2.10}$$

where $\Phi_v(\cdot,\phi_{\pi_v\otimes\sigma_v},s)$ is a section in $\mathrm{I}(s,\pi_v\otimes\sigma_v)$ and is unramified at almost all finite local places v, the term $\mathcal{M}(w_r,s)(\Phi(\cdot,\phi_{\pi\otimes\sigma},s))$ can be expressed as an infinite product

$$\mathcal{M}(w_r,s)(\Phi(\cdot,\phi_{\pi\otimes\sigma},s))=\prod_v\mathcal{M}_v(w_r,s)(\Phi_v(\cdot,\phi_{\pi_v\otimes\sigma_v},s)). \tag{2.11}$$

By the Langlands-Shahidi theory ([**L71**], [**Sh88**]), we have

$$\mathcal{M}(w_r,s)=\frac{L(s,\pi\times\sigma)L(2s,\pi,\mathrm{r}_A)}{L(s+1,\pi\times\sigma)L(2s+1,\pi,\mathrm{r}_A)}\cdot\prod_v\mathcal{N}_v(w_r,s), \tag{2.12}$$

where r_A denotes the Asai representation of the L-group of $\mathrm{R}_{E/F}(\mathrm{GL}_r)$, and $\mathcal{N}_v(w_r,s)$ is the normalized intertwining operator

$$\mathcal{N}_v(w_r,s)=\frac{1}{r(s,\pi_v,\sigma_v,w_r)}\cdot\mathcal{M}_v(w_r,s)$$

which defines a mapping from $\mathrm{I}(s,\pi_v\otimes\sigma_v)$ to $\mathrm{I}(-s,w_r[\pi_v\otimes\sigma_v])$. Here the function $r(s,\pi_v,\sigma_v,w_r)$ is equal to

$$\frac{L(s,\pi_v\times\sigma_v)L(2s,\pi_v,\mathrm{r}_A)}{L(s+1,\pi_v\times\sigma_v)L(2s+1,\pi_v,\mathrm{r}_A)\epsilon(s,\pi_v\times\sigma_v,\psi)\epsilon(2s,\pi_v,\mathrm{r}_A,\psi)}.$$

Proposition 2.1 ([**K05**]). *For the real part of s greater than or equal to $\frac{1}{2}$, the normalized local intertwining operator $\mathcal{N}_v(w_r,s)$ is holomorphic for all choice of data, and nonzero for some choice of data. More precisely, in that domain, for any holomorphic section $\Phi_v(\cdot,\phi_{\pi_v\otimes\sigma_v},s)$ in $\mathrm{I}(s,\pi_v\otimes\sigma_v)$, as a function in s,*

$\mathcal{N}_v(w_r,s)(\Phi_v(\cdot,\phi_{\pi_v\otimes\sigma_v},s))$ *is holomorphic for all choice of data and nonzero for some choice of data.*

Note that $\frac{1}{2}$ can be replaced by $-\frac{1}{m^2+1}$ in Proposition 2.1 according to Proposition 9.4 in [**KK05**] for m even and Proposition 5 in [**KK04**] for m odd. But $\frac{1}{2}$ is enough for the work in this paper. As a consequence, one has

PROPOSITION 2.2. *The Eisenstein series $E(g,\phi_{\pi\otimes\sigma},s)$ can possibly have a simple pole at $s=\frac{1}{2}$ or $s=1$. The existence of the pole at $s=\frac{1}{2}$ or $s=1$ of $E(g,\phi_{\pi\otimes\sigma},s)$ is equivalent to the existence of the pole at $s=\frac{1}{2}$ or $s=1$ of the product of L-functions*

$$L(s,\pi\times\sigma)L(2s,\pi,\mathrm{r}_A),$$

respectively.

PROOF. By the Langlands theory of constant terms of Eisenstein series, the Eisenstein series has a pole at $s=s_0$ if and only if the constant terms of the Eisenstein series has a pole at $s=s_0$. By (2.8), it is equivalent to the property that the global intertwining operator $\mathcal{M}(w_r,s)$ has a pole at $s=s_0\geq\frac{1}{2}$. From identity (2.12), if the global intertwining operator $\mathcal{M}(w_r,s)$ has a pole at $s=s_0$, then the quotient

$$\frac{L(s,\pi\times\sigma)L(2s,\pi,\mathrm{r}_A)}{L(s+1,\pi\times\sigma)L(2s+1,\pi,\mathrm{r}_A)}$$

must have a pole at $s=s_0$ since, by Proposition 2.1, the product $\prod_v\mathcal{N}_v(w_r,s)$ is holomorphic for all choice of the data when $s=s_0\geq\frac{1}{2}$. Now both L-functions $L(s,\pi\times\sigma)$ and $L(s,\pi,\mathrm{r}_A)$ are nonzero for the real part of s greater than one. It follows that the product

$$L(s,\pi\times\sigma)L(2s,\pi,\mathrm{r}_A)$$

must have a pole at $s=s_0\geq\frac{1}{2}$ if $\mathcal{M}(w_r,s)$ has a pole at $s=s_0\geq\frac{1}{2}$. Conversely, if the product of L-functions

$$L(s,\pi\times\sigma)L(2s,\pi,\mathrm{r}_A)$$

has a pole at $s=s_0$, then the global intertwining operator $\mathcal{M}(w_r,s)$ has a pole at $s=s_0$, because in (2.12), we can always choose a particular factorizable section $\Phi(\cdot,\phi_{\pi\otimes\sigma},s)$ as in (2.10), so that the product

$$\prod_v\mathcal{N}_v(w_r,s)(\Phi_v(\cdot,\phi_{\pi_v\otimes\sigma_v},s))$$

is nonzero at $s=s_0$.

It remains to determine the location of the poles of the product of L-functions $L(s,\pi\times\sigma)L(2s,\pi,\mathrm{r}_A)$.

Since both π and σ are unitary and generic, the Langlands functoriality transfers from the irreducible generic cuspidal automorphic representations of $\mathrm{U}_m^\epsilon(\mathbb{A}_F)$ to the general linear groups established in [**KK04**] and [**KK05**] imply that the automorphic L-function $L(s,\pi\times\sigma)$ converges absolutely and does not vanish for $Re(s)>1$, and has at most a simple pole at $s=1$.

The same statements hold for the Asai L-function $L(s,\pi,\mathrm{r}_A)$, i.e., it converges absolutely and does not vanish for $Re(s)>1$, and has at most a simple pole at $s=1$. This follows from the following identity

$$L(s,\pi\times\pi^\theta)=L(s,\pi,\mathrm{r}_A)L(s,\pi,\mathrm{r}_A\otimes\delta) \tag{2.13}$$

where θ is the nontrivial Galois element in the Galois groups of E/F and δ is the character associated to the quadratic extension E/F via the class field theory. Since $L(s, \pi \times \pi^\theta)$ is holomorphic and nonzero for $Re(s) > 1$ and has a possible simple pole at $s = 1$, and both $L(s, \pi, \mathrm{r}_A)$ and $L(s, \pi, \mathrm{r}_A \otimes \delta)$ are holomorphic for $Re(s) > 1$, one knows that both $L(s, \pi, \mathrm{r}_A)$ and $L(s, \pi, \mathrm{r}_A \otimes \delta)$ do not vanish for $Re(s) > 1$. (We would like to thank H. Kim for telling us this simple argument.)

This proves that the product of L-functions $L(s, \pi \times \sigma)L(2s, \pi, \mathrm{r}_A)$ has possible simple poles at either $s = \frac{1}{2}$ or $s = 1$. The proposition follows. □

We denote the residues at $s = \frac{1}{2}$ and at $s = 1$ of $E(g, s, \phi_{\pi\otimes\sigma})$ by

$$\mathcal{E}_{\frac{1}{2}}(g, \phi_{\pi\otimes\sigma}) := \mathrm{Res}_{s=\frac{1}{2}} E(g, \phi_{\pi\otimes\sigma}, s) \tag{2.14}$$

$$\mathcal{E}_1(g, \phi_{\pi\otimes\sigma}) := \mathrm{Res}_{s=1} E(g, \phi_{\pi\otimes\sigma}, s). \tag{2.15}$$

3. Various Models for Unitary groups

We introduce various models for automorphic forms on quasisplit unitary groups, which are the unitary group versions of the generalized Gelfand-Graev models or Bessel models for orthogonal groups ([**GJR05**]) or the Fourier-Jacobi models for symplectic or metaplectic groups ([**GJR04**]).

3.1. Hermitian type. In this section we consider the case where $\epsilon = +$, i.e., $\mathcal{V}_m^+$ is Hermitian. Then U_m^+ is a Hermitian F-quasisplit unitary group. As in [**GJR05**] for the orthogonal groups, we shall introduce the analogue of the generalized Gelfand-Graev models, or the generalized Bessel models for Hermitian unitary groups.

Let $\{e_1, \cdots, e_m\}$ be an E-basis for $\mathcal{V}_m^+$ so that

$$< e_i, e_j >= \begin{cases} 1 & \textit{if } j = m - i + 1, \\ 0 & \textit{if } j \neq m - i + 1, \end{cases}$$

where $< x, y >= {}^t y^\theta J_m^+ x$. Take e to be an integer with $1 \leq e < [\frac{m}{2}]$. Denote by $V_{m,e}$ the standard unipotent subgroup of U_m^+ consisting of elements of type:

$$v = v(u, x, z) = \begin{pmatrix} u & x & z \\ & I_{m-2e} & x^* \\ & & u^* \end{pmatrix} \in \mathrm{U}_m^+, \tag{3.1}$$

where $u = (u_{i,j}) \in U_e$, which is the standard maximal unipotent subgroup of $\mathrm{GL}_e(E)$, and $x = (x_{i,j})$ belongs to the additive group consisting of all $e \times (m - 2e)$-matrices over E.

Let ψ be a nontrivial character of $\mathbb{A}_F/F$. We define a character $\psi_{m,e}$ of $V_{m,e}(\mathbb{A}_F)$ by

$$\psi_{m,e}(v(u, x, z)) = \psi(\mathrm{tr}_{E/F}(u_{1,2} + \cdots + u_{e-1,e} + x_{e,\frac{m+1}{2}-e})) \tag{3.2}$$

if m is odd, and by

$$\psi_{m,e}(v(u, x, z)) = \psi(\mathrm{tr}_{E/F}(u_{1,2} + \cdots + u_{e-1,e} + x_{e,\frac{m}{2}-e} + x_{e,\frac{m}{2}-e+1})) \tag{3.3}$$

if m is even. It is clear that the character $\psi_{m,e}$ just defined is trivial on $V_{m,e}(F)$. The Levi subgroup of U_m^+ which normalizes $V_{m,e}$ is of form

$$\mathrm{R}_{E/F}(\mathrm{GL}_1)^e \times \mathrm{U}_{m-2e}^+.$$

The subgroup of $\mathrm{R}_{E/F}(\mathrm{GL}_1)^e \times \mathrm{U}^+_{m-2e}$ which stabilizes the character $\psi_{m,e}$ is U^+_{m-2e-1}, which is again an F-quasisplit Hermitian unitary group.

Let φ be an automorphic form on $\mathrm{U}^+_m(\mathbb{A}_F)$ and ϕ be an automorphic form on $\mathrm{U}_{m-2e-1}(\mathbb{A}_F)$. The period or generalized Gelfand-Graev model or generalized Bessel model of φ of type $(\mathrm{U}^+_{m-2e-1}, \phi, \psi_{m,e})$ is defined to be

$$\begin{aligned} \mathcal{P}_{\phi,\psi_{m,e}}(\varphi) &= \mathcal{P}(\varphi, \phi, \psi_{m,e}) \\ &:= \int_{\mathrm{U}^+_{m-2e-1}(F)\backslash \mathrm{U}^+_{m-2e-1}(\mathbb{A}_F)} \phi(h)\mathcal{F}_{\psi_{m,e}}(\varphi)(h)dh, \end{aligned} \tag{3.4}$$

where $\mathcal{F}_{\psi_{m,e}}(\varphi)(g)$ is the $(V_{m,e}, \psi_{m,e})$-Fourier coefficient defined by

$$\mathcal{F}_{\psi_{m,e}}(\varphi)(h) := \int_{V_{m,e}(F)\backslash V_{m,e}(\mathbb{A}_F)} \varphi(vh)\psi^{-1}_{m,e}(v)dv. \tag{3.5}$$

3.2. Skew-Hermitian type. In this section we consider the case where $\epsilon = -1$, i.e., $\mathcal{V}^-_m$ is skew-Hermitian. We denote by U^-_m the skew-Hermitian unitary group of $\mathcal{V}^-_m$. In this case we introduce the unitary group version of the Fourier-Jacobi models or periods. The symplectic or metaplectic version of the Fourier-Jacobi models or periods was considered in [**GJR04**]. We have to recall the Weil representations and theta functions. It is known that there are many good references for such topics. However, we refer to [**JngS07**] for details about the local and global calculations of the relevant cocycles. The basic facts about unitary groups and Weil representations are addressed in [**GR91**, §3].

We denote by $(\mathcal{V}^-_m, \mathrm{h})$ a nondegenerate skew-Hermitian E-vector space, which has E-dimension m. Let $W := \mathrm{R}_{E/F}(\mathcal{V}^-_m)$. Then $(W, \mathrm{tr}_{E/F}(\mathrm{h}))$ is a nondegenerate symplectic F-vector space of F-dimension 2m. Let

$$H_{2m+1} = H(W) = W \oplus F$$

be the Heisenberg group associated to $(W, \mathrm{tr}_{E/F}(\mathrm{h}))$. As in [**JngS07**], we may introduce both global metaplectic double cover, $\mathrm{Mp}(W)(\mathbb{A}_F)$, for $\mathrm{Sp}(W)(\mathbb{A}_F)$, where $\mathrm{Sp}(W)$ is the symplectic group associated to $(W, \mathrm{tr}_{E/F}(\mathrm{h}))$. By [**GR91**, Proposition 3.1.1], by means of the natural embeddings, the metaplectic double cover $\mathrm{Mp}(W)(\mathbb{A}_F)$ splits when restricted to the subgroup $\mathrm{U}^-_m(\mathbb{A}_F)$ and the F-rational points $\mathrm{U}^-_m(F)$ is in $\mathrm{Sp}(W)(F)$. Let

$$W = X + X^*$$

be the complete polarization of W. Then for each given nontrivial character ψ of $\mathbb{A}_F/F$, there is a unique (up to equivalence) Weil representation ω_ψ realized in the Schrödinger model $\mathcal{S}(X(\mathbb{A}_F))$. We refer to [**JngS07**, §2.1] for details of some basic formulas related to the Weil representation. For any $\varphi \in \mathcal{S}(X(\mathbb{A}_F))$, we define the theta function attached to φ by

$$\theta_{\varphi,\psi}(\widetilde{g}) = \theta(\widetilde{g}, \varphi, \psi) := \sum_{x\in X(F)} \omega_\psi(\widetilde{g})\varphi(x). \tag{3.6}$$

Since the Weil representation ω_ψ splits when restricted to $\mathrm{U}^-_m(\mathbb{A}_F)$, the theta function $\theta(\widetilde{g}, \varphi, \psi)$ is an automorphic function on $\mathrm{U}^-_m(\mathbb{A}_F)$.

As in §3.1, we introduce the standard unipotent subgroup $V_{m,e}$ (as (3.1)), which consists of elements of type:

$$v = v(u,x,z) = \begin{pmatrix} u & x & z \\ & I_{m-2e} & x^* \\ & & u^* \end{pmatrix} \in \mathrm{U}^-_m, \tag{3.7}$$

where $u = (u_{i,j}) \in U_e$, which is the standard maximal unipotent subgroup of $\mathrm{GL}_e(E)$, and $x = (x_{i,j})$ belongs to the additive group consisting of all $e \times (m-2e)$-matrices over E. We define a character $\psi^-_{m,e}$ of $V_{m,r}(\mathbb{A}_F)$ by

$$\psi^-_{m,e}(v(u,x,z)) = \psi_{U_e}(u) = \psi(\mathrm{tr}_{E/F}(u_{1,2} + \cdots + u_{e-1,e})), \tag{3.8}$$

which is the generic character of U_e. Since

$$V_{m,e}/[V_{m,e}, V_{m,e}] \cong E^{e-1} \oplus H_{2(m-2e)+1}(F),$$

we define the projection

$$\begin{aligned} \ell_{m,e} \quad &: \quad V_{m,e}(F) \to H_{2(m-2e)+1}(F) \\ \ell_{m,e} \quad &= \quad (x_{e,1}, \cdots, x_{e,m-2e}; z_{e,1}). \end{aligned} \tag{3.9}$$

It is clear that the Levi subgroup of U^-_m which normalizes $V_{m,e}$ is of form

$$\mathrm{R}_{E/F}(\mathrm{GL}_1)^e \times \mathrm{U}^-_{m-2e}$$

and U^-_{m-2e} stabilizes the character $\psi^-_{m,e}$ and the central character of $H_{2(m-2e)+1}$. As in [**GJR04**] and [**S05**, §2.2], we introduce the Fourier-Jacobi models or periods for (skew-Hermitian) unitary groups.

Let f be an automorphic form on $\mathrm{U}^-_m(\mathbb{A}_F)$ and ϕ be an automorphic form on $\mathrm{U}^-_{m-2e}(\mathbb{A}_F)$. The Fourier-Jacobi model or period of f of type $(\mathrm{U}^-_{m-2e}, \phi, \psi^-_{m,e})$ is defined to be

$$\begin{aligned} \mathcal{P}_{\phi,\psi^-_{m,e},\varphi}(f) \quad &= \quad \mathcal{P}(f, \phi, \psi^-_{m,e}, \varphi) \\ &:= \quad \int_{\mathrm{U}^-_{m-2e}(F)\backslash \mathrm{U}^-_{m-2e}(\mathbb{A}_F)} \phi(h)\mathcal{J}_{\psi^-_{m,e},\varphi}(f)(h)dh \end{aligned} \tag{3.10}$$

where $\mathcal{J}_{\psi^-_{m,e},\varphi}(f)(g)$ is the $(V_{m,e}, \psi^-_{m,e})$-Fourier-Jacobi coefficient of f defined by

$$\mathcal{J}_{\psi^-_{m,e},\varphi}(f)(g) := \int_{V_{m,e}(F)\backslash V_{m,e}(\mathbb{A}_F)} f(vg)\theta_{\varphi,\psi}(\ell_{m,e}(v)g)\psi^-_{m,e}(v)dv. \tag{3.11}$$

4. Periods of Residues: Hermitian Type

We shall calculate periods for a certain family of residues of Eisenstein series on unitary groups, which will be related to nonvanishing of the central value of the relevant L-functions.

4.1. Regularization of periods. As before we denote by U^+_m the F-quasisplit unitary group of Hermitian type, i.e., $\epsilon = 1$. Let τ be an irreducible unitary generic cuspidal automorphic representation of $\mathrm{U}^+_{2n+1}(\mathbb{A}_F)$, and σ be an irreducible unitary generic cuspidal automorphic representation of $\mathrm{U}^+_{2k}(\mathbb{A}_F)$. Thanks to [**KK04**] and [**KK05**], the Langlands functorial transfer of τ and σ to $\mathrm{R}_{E/F}(\mathrm{GL}_{2n+1})(\mathbb{A}_F)$ and $\mathrm{R}_{E/F}(\mathrm{GL}_{2k})(\mathbb{A}_F)$, respectively, exists. The image of τ and σ under the Langlands functorial transfer are denoted by $\pi(\tau)$ and $\pi(\sigma)$, respectively. The structure of the image has completely determined by [**KK04**] and [**KK05**], and by [**S05**].

We assume that both $\pi(\tau)$ and $\pi(\sigma)$ are cuspidal, although the general cases can be treated by the same ideas and methods, but need more complicated calculations.

We consider the following four types of Eisenstein series and their residues.

When $2n+1>2k$, we consider both $\mathrm{U}^+_{4n+2k+2}$ and U^+_{6n+3}. In $\mathrm{U}^+_{4n+2k+2}$, take the generic cuspidal datum $(P_{2n+1}, \pi(\tau)\otimes\sigma)$, where $P_{2n+1}=M_{2n+1}N_{2n+1}$ is the standard maximal parabolic subgroup, with its Levi subgroup M_{2n+1} isomorphic to the product $\mathrm{R}_{E/F}(\mathrm{GL}_{2n+1})\times\mathrm{U}^+_{2k}$. In U^+_{6n+3}, take the generic cuspidal datum $(Q_{2n+1}, \pi(\tau)\otimes\tau)$, where $Q_{2n+1}=L_{2n+1}V_{2n+1}$ is the standard maximal parabolic subgroup, with its Levi subgroup L_{2n+1} isomorphic to the product $\mathrm{R}_{E/F}(\mathrm{GL}_{2n+1})\times\mathrm{U}^+_{2n+1}$.

When $2n+1<2k$, we consider both $\mathrm{U}^+_{4k+2n+1}$ and U^+_{6k}. In $\mathrm{U}^+_{4k+2n+1}$, take the generic cuspidal datum $(P_{2k}, \pi(\sigma)\otimes\tau)$, where $P_{2k}=M_{2k}N_{2k}$ is the standard maximal parabolic subgroup, with its Levi subgroup M_{2k} isomorphic to $\mathrm{R}_{E/F}(\mathrm{GL}_{2k})\times\mathrm{U}^+_{2n+1}$. In U^+_{6k}, take the generic cuspidal datum $(Q_{2k}, \pi(\sigma)\otimes\sigma)$, where $Q_{2k}=L_{2k}V_{2k}$ is the standard maximal parabolic subgroup, with its Levi subgroup L_{2k} isomorphic to $\mathrm{R}_{E/F}(\mathrm{GL}_{2k})\times\mathrm{U}^+_{2k}$.

When $2n+1>2k$, the Eisenstein series $E(g,\phi_{\pi(\tau)\otimes\sigma},s)$ has at most a simple pole at $s=\frac{1}{2}$. The simple pole at $s=\frac{1}{2}$ occurs if and only if the L-function $L(s,\pi(\tau)\times\sigma)$ does not vanish at $s=\frac{1}{2}$. On the other hand, the Eisenstein series $E(g,\phi_{\pi(\tau)\otimes\tau},s)$ has a simple pole at $s=1$. We denote the residues by

$$\mathcal{E}_{\pi(\tau)\otimes\sigma}(g):=\mathcal{E}_{\frac{1}{2}}(g,\phi_{\pi(\tau)\otimes\sigma}) \tag{4.1}$$

and

$$\mathcal{E}_{\pi(\tau)\otimes\tau}(g):=\mathcal{E}_{1}(g,\phi_{\pi(\tau)\otimes\tau}), \tag{4.2}$$

respectively. These are special cases of (2.14) and (2.15, respectively. Note that the residue $\mathcal{E}_{\pi(\tau)\otimes\tau}$ is nonzero, but the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma}$ may be zero.

When $2n+1<2k$, the Eisenstein series $E(g,\phi_{\pi(\sigma)\otimes\tau},s)$ has at most a simple pole at $s=\frac{1}{2}$. The simple pole at $s=\frac{1}{2}$ occurs if and only if the L-function $L(s,\pi(\sigma)\times\tau)$ does not vanish at $s=\frac{1}{2}$. On the other hand, the Eisenstein series $E(g,\phi_{\pi(\sigma)\otimes\sigma},s)$ has a simple pole at $s=1$. We denote the residues by

$$\mathcal{E}_{\pi(\sigma)\otimes\tau}(g):=\mathcal{E}_{\frac{1}{2}}(g,\phi_{\pi(\sigma)\otimes\tau}) \tag{4.3}$$

and

$$\mathcal{E}_{\pi(\sigma)\otimes\sigma}(g):=\mathcal{E}_{1}(g,\phi_{\pi(\sigma)\otimes\sigma}), \tag{4.4}$$

respectively. These are special cases of (2.14) and (2.15), respectively. Note that the residue $\mathcal{E}_{\pi(\sigma)\otimes\sigma}$ is nonzero, but the residue $\mathcal{E}_{\pi(\sigma)\otimes\tau}$ may be zero.

When $2n+1>2k$, we want to calculate the following period

$$\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau},\mathcal{E}_{\pi(\tau)\otimes\sigma},\psi_{6n+3,n-k}), \tag{4.5}$$

as defined in (3.4) with $m=6n+3$, $e=n-k$, $\varphi=\mathcal{E}_1(\cdot,\phi_{\pi(\tau)\otimes\tau})$, and $\phi=\mathcal{E}_{\frac{1}{2}}(\cdot,\phi_{\pi(\tau)\otimes\sigma})$.

When $2n+1<2k$, we want to calculate the following period

$$\mathcal{P}(\mathcal{E}_{\pi(\sigma)\otimes\sigma},\mathcal{E}_{\pi(\sigma)\otimes\tau},\psi_{6k,k-n-1}), \tag{4.6}$$

as defined in (3.4) with $m=6k$, $e=k-n-1$, $\varphi=\mathcal{E}_1(\cdot,\phi_{\pi(\sigma)\otimes\sigma})$, and $\phi=\mathcal{E}_{\frac{1}{2}}(\cdot,\phi_{\pi(\sigma)\otimes\tau})$.

As in [**GJR04**] and [**GJR05**], we have to use the Arthur truncation to regularize both periods in (4.5) and in (4.6). Recall from (3.4) that the period in (4.5), $\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_{\pi(\tau)\otimes\sigma}, \psi_{6n+3,n-k})$, is given by

$$\int_{\mathrm{U}_{4n+2k+2}(F)\backslash\mathrm{U}_{4n+2k+2}(\mathbb{A}_F)} \mathcal{E}_{\pi(\tau)\otimes\sigma}(h)\mathcal{F}_{\psi_{6n+3,n-k}}(\mathcal{E}_{\pi(\tau)\otimes\tau})(h)dh. \tag{4.7}$$

We will apply the Arthur truncation to the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma}(h)$. Similarly, the period in (4.6), $\mathcal{P}(\mathcal{E}_{\pi(\sigma)\otimes\sigma}, \mathcal{E}_{\pi(\sigma)\otimes\tau}, \psi_{6k,k-n-1})$, is given by

$$\int_{\mathrm{U}_{4k+2n+1}(F)\backslash\mathrm{U}_{4k+2n+1}(\mathbb{A}_F)} \mathcal{E}_{\pi(\sigma)\otimes\tau}(h)\mathcal{F}_{\psi_{6k,k-n-1}}(\mathcal{E}_{\pi(\sigma)\otimes\sigma})(h)dh. \tag{4.8}$$

We will have to apply the Arthur truncation to the residue $\mathcal{E}_{\pi(\sigma)\otimes\tau}(h)$.

We follow from [**GJR05**, §3.3], and [**GJR04**, §4.1]. Write $\phi = \phi_{\pi(\tau)\otimes\sigma}$, $w = w_{2n+1}$, $G = \mathrm{U}_{4n+2k+2}$ and $P = P_{2n+1}$. Following [**Ar78**] and [**Ar80**], the truncation of the Eisenstein series $E(g,\phi,s)$ is defined as follows:

$$\Lambda_c E(g,\phi,s) = E(g,\phi,s) - \sum_{\gamma\in P(F)\backslash G(F)} E_P(\gamma g,\phi,s)\tau_c(H(\gamma g)). \tag{4.9}$$

The constant term $E_P(g,\phi,s)$ of the Eisenstein series $E(g,\phi,s)$ along P can be expressed as (see (2.8) for definition and take $r = 2n+1$)

$$E_P(g,\phi,s) = \Phi(g,\phi,s) + \mathcal{M}(w,s)(\Phi(\cdot,\phi),s)(g).$$

We remark that the summation in (4.9) has only finitely many nonzero terms and converges absolutely ([**Ar78**, Corollary 5.2]). The truncated Eisenstein series can then be rewritten as follows

$$\begin{aligned}\Lambda_c E(g,\phi,s) &= \sum_{\gamma\in P(F)\backslash G(F)} \Phi(\gamma g,\phi,s)(1-\tau_c(H(\gamma g)))\\ &\quad - \sum_{\gamma\in P(F)\backslash G(F)} \mathcal{M}(w,s)(\Phi(\cdot,\phi,s))(\gamma g)\tau_c(H(\gamma g))\\ &:= \mathcal{E}_1(g) - \mathcal{E}_2(g).\end{aligned} \tag{4.10}$$

Let s_0 be a positive real number. Assume that the Eisenstein series $E(g,\phi,s)$ has a simple pole at $s = s_0$. We eventually show that $s_0 = \frac{1}{2}$. We denote by $E_{s_0}(g,\phi)$ the non-zero residue of $E(g,\phi,s)$. Then we have

$$\begin{aligned}\Lambda_c E_{s_0}(g,\phi) &= E_{s_0}(g,\phi) - \sum_{\gamma\in P(F)\backslash G(F)} \mathcal{M}(w,s)(\Phi(\cdot,\phi,s))_{s_0}(\gamma g)\tau_c(H(\gamma g))\\ &:= E_{s_0}(g,\phi) - \mathcal{E}_3(g).\end{aligned} \tag{4.11}$$

Following the argument in [**GJR04**, §4.1] and [**GJR05**, §3.3], we obtain from (4.10) and (4.11) the formula for the period $\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, E_{s_0}(g,\phi), \psi_{6n+3,n-k})$:

$$\begin{aligned}\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, &E_{s_0}(\cdot,\phi), \psi_{6n+3,n-k})\\ &= \mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_3, \psi_{6n+3,n-k}) + \mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \Lambda_c E_{s_0}(\cdot,\phi), \psi_{6n+3,n-k})\\ &= \mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_3, \psi_{6n+3,n-k})\\ &\quad + [\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_1, \psi_{6n+3,n-k}) - \mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_2, \psi_{6n+3,n-k})]_{s_0}\end{aligned}$$

Similarly, we have to prove the following proposition, which will be done in the next subsection.

Proposition 4.1. *For $i = 1, 2$, the following periods*

$$\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_i, \psi_{6n+3,n-k})$$

converge absolutely for $Re(s)$ large and have meromorphic continuation to the whole complex plane.

Since both $\Lambda_c E(g, \phi, s)$ and $\Lambda_c E_{s_0}(g, \phi)$ rapidly decay in the usual sense, the periods

$$\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \Lambda_c E(\cdot, \phi, s), \psi_{6n+3,n-k})$$

and

$$\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \Lambda_c E_{s_0}(\cdot, \phi), \psi_{6n+3,n-k})$$

converge absolutely. By Proposition 4.1, $\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_1, \psi_{6n+3,n-k})$ and $\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_2, \psi_{6n+3,n-k})$ converge absolutely for $Re(s)$ large and have meromorphic continuation to the complex plane. Hence we have

$$\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_2, \psi_{6n+3,n-k})_{s_0} = \mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_3, \psi_{6n+3,n-k}).$$

It follows that

$$\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, E_{s_0}(\cdot, \phi), \psi_{6n+3,n-k}) = \mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_1, \psi_{6n+3,n-k})_{s_0}. \tag{4.12}$$

This gives the regularization for the period $\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, E_{s_0}(\cdot, \phi), \psi_{6n+3,n-k})$. If $s_0 = \frac{1}{2}$, it is exactly the period defined in (4.7), which is the case when $2n + 1 > 2k$.

By the same argument, we have the regularization for the period in (4.8) when $2n + 1 < 2k$.

$$\mathcal{P}(\mathcal{E}_{\pi(\sigma)\otimes\sigma}, E_{s_0}(\cdot, \phi_{\pi(\sigma)\otimes\tau}), \psi_{6k,k-n-1}) = \mathcal{P}(\mathcal{E}_{\pi(\sigma)\otimes\sigma}, \mathcal{E}_1, \psi_{6k,k-n-1})_{s_0}. \tag{4.13}$$

If $s_0 = \frac{1}{2}$, it recovers the period in (4.8).

Corollary 4.2. *The periods defined in (4.5) and (4.6) are regularized as in (4.12) and (4.13), respectively.*

4.2. Period identity. We shall establish an identity for the period of the residues and as consequence, we prove Proposition 4.1.

First, the convergence for the real part of s large of periods

$$\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_i, \psi_{6n+3,n-k})$$

with $i = 1, 2$ follows as in the proof of Proposition 3.1, [**GJR01**]. The meromorphic continuation of both periods will follow from the explicit calculation.

We shall investigate the period $\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_1, \psi_{6n+3,n-k})$ in detail, while the conclusion for the period $\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_2, \psi_{6n+3,n-k})$ follows from the same argument. By (4.7), the period $\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_1, \psi_{6n+3,n-k})$ is given by the following integral

$$\int_{G(F)\backslash G(\mathbb{A}_F)} \mathcal{E}_1(h)\mathcal{F}_{\psi_{6n+3,n-k}}(\mathcal{E}_{\pi(\tau)\otimes\tau})(h)dh. \tag{4.14}$$

where $G = \mathrm{U}_{4n+2k+2}$ as before, and

$$\mathcal{E}_1(h) = \sum_{\gamma\in P(F)\backslash G(F)} \Phi(\gamma h, \phi, s)(1 - \tau_c(H(\gamma h))),$$

with $P = P_{2n+1}$ in G. In the following we set

$$\Phi^c(h, s) := \Phi(h, \phi, s)(1 - \tau_c(H(h))).$$

Then, for the real part of s large, (4.14) is absolutely convergent, and so we can unfold $\mathcal{E}_1$ in (4.14):

(4.15)
$$\int_{P(F)\backslash G(\mathbb{A}_F)} \Phi^c(h) \int_{V_{6n+3,n-k}(F)\backslash V_{6n+3,n-k}(\mathbb{A}_F)} \mathcal{E}_{\pi(\tau)\otimes\tau}(vh)\psi_{6n+3,n-k}(v)dvdh.$$

Recall that N is the unipotent radical of the parabolic subgroup $P = MN$. Factoring the measure $N(F)\backslash N(\mathbb{A}_F)$, we obtain the following integral as an inner integration in (4.15)

$$\int_{N(F)\backslash N(\mathbb{A}_F)} \int_{V_{6n+3,n-k}(F)\backslash V_{6n+3,n-k}(\mathbb{A}_F)} \mathcal{E}_{\pi(\tau)\otimes\tau}(vnh)\psi_{6n+3,n-k}(v)dvdn. \tag{4.16}$$

We first investigate the Fourier coefficient given in (4.16). Consider the Weyl group element of U_{6n+3} represented by

$$w = \begin{pmatrix} 0 & I_{2n+1} & & & \\ I_{n-k} & 0 & & & \\ & & I_{2k+1} & & \\ & & & 0 & I_{n-k} \\ & & & I_{2n+1} & 0 \end{pmatrix}.$$

In (4.16), since $\mathcal{E}_{\pi(\tau)\otimes\tau}(g)$ is automorphic, we may replace $\mathcal{E}_{\pi(\tau)\otimes\tau}(xg)$ by $\mathcal{E}_{\pi(\tau)\otimes\tau}(wxw^{-1}wg)$, and obtain

$$\int_{[N\times V_{6n+3,n-k}]} \mathcal{E}_{\pi(\tau)\otimes\tau}(z(v',y_1,y_2,y_3,x)wh)\psi_{6n+3,n-k}(v')\prod_i dy_i dv' dx. \tag{4.17}$$

We have to explain the notations used in (4.17):

$$[N\times V_{6n+3,n-k}] = (N\times V_{6n+3,n-k})(F)\backslash(N\times V_{6n+3,n-k})(\mathbb{A}_F),$$

the unipotent element

$$v' = \begin{pmatrix} v_1 & v_2 & v_3 \\ & I_{2k+1} & v_2^* \\ & & v_1^* \end{pmatrix},$$

and the element $z(v',y_1,y_2,y_3,x)$ is defined as the following product

$$\begin{pmatrix} I_{2n+1} & 0 & y_1 & y_2 & y_3 \\ & I_{n-k} & 0 & 0 & y_2^* \\ & & I_{2k+1} & 0 & y_1^* \\ & & & I_{n-k} & 0 \\ & & & & I_{2n+1} \end{pmatrix} \cdot \begin{pmatrix} I_{2n+1} & & & & \\ & v_1 & v_2 & v_3 & \\ & & I_{2k+1} & v_2^* & \\ & & & v_1^* & \\ & & & & I_{2n+1} \end{pmatrix} \cdot \ell^-(x)$$

where

$$\ell^-(x) = \begin{pmatrix} I_{2n+1} & 0 & & & \\ x & I_{n-k} & & & \\ & & I_{2k+1} & & \\ & & & I_{n-k} & 0 \\ & & & x^* & I_{2n+1} \end{pmatrix}.$$

Note that $v' \in V_{2n+1,n-k}$, and it is embedded in $\mathrm{U}_{6n+3)}$ as described in the matrix $z(v',y_1,y_2,y_3,x)$. Hence we have

$$\psi_{6n+3,n-k}(v') = \psi_{2n+1,n-k}(v'). \tag{4.18}$$

The variable y_1 is in $\mathrm{R}_{E/F}(\mathrm{Mat}_{2n+1\times 2k+1})$ with the condition that the $k+1$-st column is zero. The variables y_2,y_3 and x have no restriction except that the element $z(v', y_1, y_2, y_3, x)$ belongs to $\mathrm{U}_{6n+3)}$. All variables are integrated over the quotients of the $\mathbb{A}_F$-points of the corresponding affine spaces modulo the F-points.

Let $R = \mathrm{R}_{E/F}(\mathrm{Mat}_{2n+1\times 1})$ denote column vector. We embed R into $\mathrm{R}_{E/F}(\mathrm{Mat}_{2n+1\times 2n+1})$ by identifying it with the $n+1$-st column. Via this embedding, we view R as a subgroup of U_{6n+3} by considering the group of all matrices

$$r \in R \to \begin{pmatrix} I_{2n+1} & r & * \\ & I_{2n+1} & r^* \\ & & I_{2n+1} \end{pmatrix} \in \mathrm{U}_{6n+3}.$$

We now take the Fourier expansion of the residue $\mathcal{E}_{\pi(\tau)\otimes\tau}(g)$ in (4.17) along the group R and obtain

$$\sum_{\xi} \int_{R(F)\backslash R(\mathbb{A}_F)} \mathcal{E}_{\pi(\tau)\otimes\tau}(rg)\psi(\xi r)dr. \tag{4.19}$$

Here the sum is over all characters of the group $R(F)\backslash R(\mathbb{A}_F)$, which can be identified with $R(F)$. By ξr we denote the inner product of ξ with r where we identify each element as a column vector.

We are going to use the same argument as in [**GJR05**, §3.4], to calculate the Fourier coefficient in (4.17). We prefer to give here necessary technical steps of the argument and refer to [**GJR05**, §3.4] for more detailed explanation of the calculation.

Denote by X the subgroup of U_{6n+3} consisting of all matrices $\{m(x)\}$ as defined above. Let X_1 denote the subgroup of X which consists of all matrices $\{m(x)\}$ in X such that all rows, except the bottom row, are zero. Let X^1 denote the group of all matrices $m(x)$ in X such that the last row of x is zero. Hence we have that

$$X = X_1 \cdot X^1.$$

By replacing in (4.17) the residue $\mathcal{E}_{\pi(\tau)\otimes\tau}(g)$ by its Fourier expansion (4.19), we obtain

$$\int \sum_{\xi} \int_{[R]} \mathcal{E}_{\pi(\tau)\otimes\tau}(rz(v', y_1, y_2, y_3, x)wh)\psi_{2n+1,n-k}(v')\psi(\xi r)dr \prod_i dy_i dv' dx \tag{4.20}$$

where the first integration is over $[N \times V_{6n+3,n-k}]$.

Choose a suitable element $x(\xi) \in X_1(F)$ and rewrite the residue as

$$\mathcal{E}_{\pi(\tau)\otimes\tau}(x(\xi)rz(v', y_1, y_2, y_3)x(-\xi)x(\xi)x)wh).$$

By calculating the conjugation $x(\xi)rz(v', y_1, y_2, y_3)x(-\xi)$ and by changing the variables and collapsing the summation with integration for $x(\xi)x$ in (4.20), we obtain

$$\int \int_{X^1(F)\backslash X(\mathbb{A}_F)} \mathcal{E}_{\pi(\tau)\otimes\tau}(z(v', y_1, y_2, y_3, x)wg)\psi_{2n+1,n-k}(v') \prod_i dy_i dv' dx \tag{4.21}$$

where all variables are integrated as before except y_1 where now it is integrated over all $\mathrm{R}_{E/F}(\mathrm{Mat}_{2n+1\times 2k+1})$.

Define the subgroup Y of U_{6n+3} consisting of all unipotent element of the form

$$\ell^+(y) = \begin{pmatrix} I_{2n+1} & y & & & \\ 0 & I_{n-k} & & & \\ & & I_{2k+1} & & \\ & & & I_{n-k} & y^* \\ & & & 0 & I_{2n+1} \end{pmatrix}$$

where $y \in \mathrm{R}_{E/F}(\mathrm{Mat}_{2n+1 \times n-k})$. Let Y_1 denote the subgroup of Y which consists of all matrices $l(y)$ with all columns, except the last one, being zero. We are going to use the argument by means of Fourier expansion along $Y_1(F)\backslash Y_1(\mathbb{A}_F)$ to calculate the integral in (4.21).

Let X_2 denote the subgroup of X which consists of all matrices $m(x)$ in X with all rows, except the $(n-k-1)$-st row, being all zero. Let X^2 denote the group of all matrices $m(x)$ of X such that the last two rows of x are zero. Hence we have that

$$X^1 = X_2 \cdot X^2.$$

Arguing as before, using a suitable F-rational element, say, $x_2(\xi) \in X_2(F)$, the integral in (4.21) equals

$$\int\int_{X^2(F)\backslash X(\mathbb{A}_F)} \mathcal{E}_{\pi(\tau)\otimes\tau}(\ell^+(y)z(v',y_1,y_2,y_3,x)wg)\psi_{2n+1,n-k}(v')dy\prod_i dy_i dv' dx \tag{4.22}$$

where all variables are integrated as before. The variable y is integrated over $Y_1(F)\backslash Y_1(\mathbf{A})$.

Continuing this process column by column in Y, we eventually obtain that (4.22) equals

$$\int\int_{X(\mathbb{A}_F)} \mathcal{E}_{\pi(\tau)\otimes\tau}(\ell^+(y)z(v',y_1,y_2,y_3,x)wg)\psi_{2n+1,n-k}(v')dy\prod_i dy_i dv' dx \tag{4.23}$$

where now y is integrated over the quotient of the $\mathbb{A}_F$-rational points of the affine space $\mathrm{R}_{E/F}(\mathrm{Mat}_{2n+1\times n-k})(\mathbb{A}_F)$ modulo the F-rational points, with the first column being zero.

Let Y_{n-k} denote the subgroup of Y which consists of all matrices $l(y)$ with all columns, except the first one, being all zeros. We take the Fourier expansion along the group Y_{n-k} for the residue $\mathcal{E}_{\pi(\tau)\otimes\tau}(g)$.

$$\mathcal{E}_{\pi(\tau)\otimes\tau}(g) = \sum_{\xi}\int_{Y_{n-k}(F)\backslash Y_{n-k}(\mathbb{A}_F)} \mathcal{E}_{\pi(\tau)\otimes\tau}(yg)\psi(\xi y)dy. \tag{4.24}$$

It is clear that the group of all characters of $Y_{n-k}(F)\backslash Y_{n-k}(\mathbb{A}_F)$ can be identified with $Y_{n-k}(F)$, and the adjoint action of the group $\mathrm{R}_{E/F}(\mathrm{GL}_{2n+1})(F)$ on $Y_{n-k}(F)$ has two $\mathrm{R}_{E/F}(\mathrm{GL}_{2n+1})(F)$-orbits. Hence we can write the summation (4.24) as a sum of two summations according to the two orbits.

By replacing the residue $\mathcal{E}_{\pi(\tau)\otimes\tau}(g)$ by its Fourier expansion (4.24) in the integral (4.23), we deduce that the summation over the Zariski open orbit must be zero and the contribution from the closed orbit, i.e., the zero orbit gives what we wanted. More precisely, we can calculate as in [**GJR05**, pp. 177-178], and by the cuspidality of $\pi(\tau)$, we obtain the integrals of the following type as inner integration in each summand over the Zariski open orbit:

$$\int_{[V_{6n+3,3n-k+1}]}\mathcal{E}_{\pi(\tau)\otimes\tau}(v)\psi_{6n+3,3n-k+1}(v)dv. \tag{4.25}$$

By the analogue of Proposition 3.6, [**GJR05**], we deduce that the residue $\mathcal{E}_{\pi(\tau)\otimes\tau}(g)$ has no nonzero Fourier coefficients with respect to

$$(V_{6n+3,3n-k+1}, \psi_{6n+3,3n-k+1}).$$

Hence the contribution from the Zariski open orbit must be all zero.

This implies that (4.23) equals

$$\int\int_{X(\mathbb{A}_F)}\mathcal{E}_{\pi(\tau)\otimes\tau}(\ell^+(y)z(v',y_1,y_2,y_3,x)wg)\psi_{2n+1,n-k}(v')dy\prod_i dy_i dv'dx \tag{4.26}$$

where now y is integrated over the quotient of the $\mathbb{A}_F$-rational points of $\mathrm{R}_{E/F}(\mathrm{Mat}_{2n+1\times n-k})$ modulo the F-rational points.

Note that the unipotent subgroup of U_{6n+3} generated by all elements of type $l(y)z(0,y_1,y_2,y_3,0)$ is exactly the unipotent radical V_{2n+1} of the standard maximal parabolic subgroup $Q_{2n+1} = L_{2n+1}V_{2n+1}$ of U_{6n+3}, as first given in §4.1. Hence the partial integration over $V_{2n+1}(F)\backslash V_{2n+1}(\mathbb{A}_F)$ in (4.26) is the constant term of the residue $\mathcal{E}_{\pi(\tau)\otimes\tau}(g)$, which will be denoted by

$$\mathcal{E}^{V_{2n+1}}_{\pi(\tau)\otimes\tau}(g).$$

Therefore, (4.26) can be written as

$$\int_{[V_{2n+1,n-k}]}\int_{X(\mathbb{A}_F)}\mathcal{E}^{V_{2n+1}}_{\pi(\tau)\otimes\tau}(z(v',x)wg)\psi_{2n+1,n-k}(v')dv'dx \tag{4.27}$$

where $z(v',x) = z(v',0,0,0,,x)$. We summarize the above calculation as

Proposition 4.3. *Integral (4.15) equals*

$$\int\Phi^c(h)\int_{[V_{2n+1,n-k}]}\int_{X(\mathbb{A}_F)}\mathcal{E}^{V_{2n+1}}_{\pi(\tau)\otimes\tau}(z(v',x)wh)\psi_{2n+1,n-k}(v')dv'dxdh, \tag{4.28}$$

where the integration on the variable h is over $M(F)N(\mathbb{A}_F)\backslash\mathrm{U}_{4n+2k+2}(\mathbb{A}_F)$.

Next, we are going to simplify the integral in (4.28). The constant term $\mathcal{E}^{V_{2n+1}}_{\pi(\tau)\otimes\tau}$ of the residue $\mathcal{E}_{\pi(\tau)\otimes\tau}$ can be simplified as follows:

$$\mathcal{E}^{V_{2n+1}}_{\pi(\tau)\otimes\tau}(g) = \phi_{\pi(\tau)\otimes\tau}(g)\cdot\exp < -1+\rho_{Q_{2n+1}}, H_{Q_{2n+1}}(g) > . \tag{4.29}$$

If we write as the Iwasawa decomposition $\mathrm{U}_{6n+3}(\mathbb{A}_F) = Q_{2n+1}(\mathbb{A}_F)\mathrm{K}'$ that $g = vl(a,b)\mathrm{k}'$, then we have

$$\exp < -1+\rho_{Q_{2n+1}}, H_{Q_{2n+1}}(g) >= |\det a|^{2n}_{\mathbb{A}_E}.$$

By the Iwasawa decomposition for $\mathrm{U}_{4n+2k+2}(\mathbb{A}_F)$ with respect to $P = MN$, we write $M(F)N(\mathbb{A}_F)\backslash\mathrm{U}_{4n+2k+2}(\mathbb{A}_F)$ as

$$(\mathrm{GL}_{2n+1})(E)\backslash\mathrm{GL}_{2n+1}(\mathbb{A}_E))\times(\mathrm{U}_{2k}(F)\backslash\mathrm{U}_{2k}(\mathbb{A}_F))\cdot\mathrm{K},$$

and it follows that if we write $h = nm(a,b)\mathrm{k}$, then from the definition of Φ^c, we have

$$\Phi^c(h) = \phi_{\pi(\tau)\otimes\sigma}(m(a,b)k)\cdot|\det a|^{s+n+k+\frac{1}{2}}_{\mathbb{A}_E}\cdot(1-\tau_c(H(a))).$$

Also we have to change the order of integrations in (4.28) by

$$z(v',x)wh = z(v')\ell(x)wm(a,b)\mathrm{k} \mapsto z(v')m(a,b)\ell^{-}(x)w\mathrm{k}.$$

It follows that (4.28) equals

$$\int_{\mathrm{K}}\int_{X(\mathbb{A}_F)}\int_{[\mathrm{R}_{E/F}(\mathrm{GL}_{2n+1}]} |\det a|_{\mathbb{A}_E}^{s-\frac{1}{2}} \cdot (1-\tau_c(H(a))) \int_{[\mathrm{U}_{2k}]} \phi_{\pi(\tau)\otimes\sigma}(m(a,b)\mathrm{k})$$

$$\int_{[V_{2n+1,n-k}]} \phi_{\pi(\tau)\otimes\tau}(z(v')m(a,b)\ell^{-}(x)w\mathrm{k})\psi_{2n+1,n-k}(v')dv'dmdxd\mathrm{k}. \tag{4.30}$$

We denote the $(V_{2n+1,n-k},\psi_{2n+1,n-k})$-Fourier coefficient in (4.30) by

$$\mathcal{F}^{\psi_{2n+1,n-k}}(\phi_{\pi(\tau)\otimes\tau})(g) := \int_{[V_{2n+1,n-k}]} \phi_{\pi(\tau)\otimes\tau}(z(v')g)\psi_{2n+1,n-k}(v')dv'. \tag{4.31}$$

We consider the Langlands decomposition for $\mathrm{GL}_{2n+1}(\mathbb{A}_E)$

$$\mathrm{GL}_{2n+1}(\mathbb{A}_E) = \mathrm{GL}_{2n+1}(\mathbb{A}_E)^1 \cdot A^{+},$$

and set $\mathrm{PM}_{2n+1,2k} = \mathrm{R}_{E/F}(\mathrm{PGL}_{2n+1}) \times \mathrm{U}_{2k}$. Let d be the degree of the number field E over $\mathbb{Q}$. Then (4.30) equals

$$\mathrm{vol}(\mathbb{A}_E^1/E^{\times}) \int_{\mathbb{R}^{+}} |t|^{(2n+1)d(s-\frac{1}{2})}(1-\tau_c(t))dt^{\times} \int_{\mathrm{K}}\int_{X(\mathbb{A}_F)} \tag{4.32}$$

$$\int_{[\mathrm{PM}_{2n+1,2k}]} \phi_{\pi(\tau)\otimes\sigma}(m\mathrm{k})\mathcal{F}^{\psi_{2n+1,n-k}}(\phi_{\pi(\tau)\otimes\tau})(m\ell^{-}(x)w\mathrm{k})dmdxd\mathrm{k}.$$

It is easy to see that

$$\int_{\mathbb{R}^{+}} |t|^{(2n+1)d(s-\frac{1}{2})}(1-\tau_c(t))dt^{\times} = \frac{c^{(2n+1)d(s-\frac{1}{2})}}{(2n+1)d(s-\frac{1}{2})}$$

which has a simple pole at $s=\frac{1}{2}$. In (4.32) we are left with

$$\int_{\mathrm{K}\times X(\mathbb{A}_F)\times[\mathrm{PM}_{2n+1,2k}]} \phi_{\pi(\tau)\otimes\sigma}(m\mathrm{k})\mathcal{F}^{\psi_{2n+1,n-k}}(\phi_{\pi(\tau)\otimes\tau})(m\ell^{-}(x)w\mathrm{k})dmdxd\mathrm{k} \tag{4.33}$$

which is holomorphic in s. Hence we obtain from the above explicit calculation and (4.12) the main identity for this case.

THEOREM 4.4. *When $s_0 \neq \frac{1}{2}$, the period*

$$\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, E_{s_0}(\cdot,\phi_{\pi(\tau)\otimes\sigma}), \psi_{6n+3,n-k})$$

is identically zero. When $s_o = \frac{1}{2}$, the period

$$\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_{\pi(\tau)\otimes\sigma}, \psi_{6n+3,n-k})$$

is equal to

$$c\cdot\int_{\mathrm{K}\times X(\mathbb{A}_F)\times[\mathrm{PM}_{2n+1,2k}]} \phi_{\pi(\tau)\otimes\sigma}(m\mathrm{k})\mathcal{F}^{\psi_{2n+1,n-k}}(\phi_{\pi(\tau)\otimes\tau})(m\ell^{-}(x)w\mathrm{k})dmdxd\mathrm{k}$$

where $\mathrm{PM}_{2n+1,2k} = \mathrm{R}_{E/F}(\mathrm{PGL}_{2n+1}) \times \mathrm{U}_{2k}$ *and the constant* $c = \frac{\mathrm{vol}(\mathbb{A}_E^1/E^{\times})}{(2n+1)d}$.

REMARK 4.5. From formulas (4.32) and (4.33), the period

$$\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \mathcal{E}_1, \psi_{6n+3,n-k})$$

has meromorphic continuation to the whole complex plane.

This completes the proof of Proposition 4.1 for this period. The statement for the period

$$\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \mathcal{E}_2, \psi_{6n+3,n-k})$$

follows from the same argument (as in [**GJR05**, §3.4] and [**GJR01**]). Hence the proof for Proposition 4.1 is now completed. Therefore the case when $2n+1 > 2k$ is done. It is clear now that the other case when $2n+1 < 2k$ can be done in the exactly same way. We omit the details here.

4.3. Nonvanishing in terms of the periods. Recall from §2.2 the definition of a family of Eisenstein series $E(g, \phi_{\pi\otimes\sigma}, s)$ on the Hermitian unitary group $\mathrm{U}_m(\mathbb{A}_F)$ attached to the generic cuspidal data $(P_r, \pi\otimes\sigma)$, where π is the irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_r(\mathbb{A}_E)$ and σ is an irreducible generic unitary cuspidal automorphic representation of $\mathrm{U}_{m-2r}(\mathbb{A}_F)$. From Proposition 2.2, the location of the poles of the Eisenstein series $E(g, \phi_{\pi\otimes\sigma}, s)$ is determined by that of the product

$$L(s, \pi\times\sigma)L(2s, \pi, \mathrm{r}_A). \tag{4.34}$$

We want to get more precise information about the location of poles of the Eisenstein series $E(g, \phi_{\pi\otimes\sigma}, s)$ in terms of the cuspidal datum $(P_r, \pi\otimes\sigma)$.

When the pole is at $s = 1$, then $L(s, \pi\times\sigma)$ has a pole at $s = 1$. By the Langlands functorial transfer from $\mathrm{U}_{m-2r}(\mathbb{A}_F)$ to $\mathrm{GL}_{m-2r}(\mathbb{A}_E)$ ([**KK04**], [**KK05**], and [**S05**]), we know that r and m must be in the same parity and π must be the image of an irreducible generic unitary cuspidal automorphic representation of $\mathrm{U}_r(\mathbb{A}_F)$ and also must be one of the isobaric factors of the image $\pi(\sigma)$ of σ under the Langlands functorial transfer. The converse follows also easily from the explicit Langlands functorial transfer from F-quasisplit unitary groups to the general linear groups ([**KK04**], [**KK05**], and [**S05**]). However, we conjecture that if r and m are in the same parity, the possible pole can only be at $s = 1$.

When the possible pole is at $s = \frac{1}{2}$, we conjecture that it can only occur in in the case when r and m are not in the same parity. If this pole occurs, then the Asai L-function $L(s, \pi, \mathrm{r}_A)$ must have a pole at $s = 1$, and hence $\pi = \pi(\tau)$ is the image of an irreducible generic unitary cuspidal automorphic representation of $\mathrm{U}_r(\mathbb{A}_F)$ under the Langlands functorial transfer ([**KK04**], [**KK05**], and [**S05**]). For simplicity, we assume that σ is stable, i.e., the image of σ under the Langlands functorial transfer is cuspidal. This is a technical assumption undertaken in the previous section, although we conjecture the general case can be established by using the same arguments as well.

Theorem 4.6. *Assume that r and m are not in the same parity. If $3r > m$, and if the period $\mathcal{P}(\phi_\tau, \phi_\sigma, \psi_{r,\frac{3r-m-1}{2}})$ does not vanish for some choice of data, then the Eisenstein series $E(g, \phi_{\pi(\tau)\otimes\sigma}, s)$ has a simple pole at $s = \frac{1}{2}$, and the L-function $L(s, \tau\times\sigma)$ does not vanish at $s = \frac{1}{2}$. Similarly, if $3r < m$, and if $\mathcal{P}(\phi_\sigma, \phi_\tau, \psi_{r,\frac{m-3r-1}{2}})$ does not vanish for some choice of data, then the Eisenstein series $E(g, \phi_{\pi(\sigma)\otimes\tau}, s)$ has a simple pole at $s = \frac{1}{2}$, and the L-function $L(s, \tau\times\sigma)$ does not vanish at $s = \frac{1}{2}$.*

By the definition of the models or periods of automorphic forms for the Hermitian type unitary groups in §3, when $3r > m$, the period is given by $\mathcal{P}(\phi_\tau, \phi_\sigma, \psi_{r,\frac{3r-m-1}{2}})$ and when $3r < m$, the period is given by $\mathcal{P}(\phi_\sigma, \phi_\tau, \psi_{r,\frac{m-3r-1}{2}})$.

Suppose that m is even, then r is odd. We write $r = 2n+1$ and $m - 2r = 2k$. Then $3r > m$ is the same as $2n+1 > 2k$. In this case the period is given by

$$\mathcal{P}(\phi_\tau, \phi_\sigma, \psi_{2n+1,n-k}) = \int_{[\mathrm{U}_{2k}]} \phi_\sigma(h) \int_{[V_{2n+1,n-k}]} \phi_\tau(vg)\psi_{2n+1,n-k}(v)dvdh. \tag{4.35}$$

We consider the following generic cuspidal forms

$$\phi_{\pi(\tau)\otimes\sigma} = \phi_{\pi(\tau)} \otimes \phi_\sigma \in V_{\pi(\tau)\otimes\sigma}; \tag{4.36}$$

$$\phi_{\pi(\tau)\otimes\tau} = \overline{\phi}_{\pi(\tau)} \otimes \phi_\tau \in V_{\pi(\tau)\otimes\tau}, \tag{4.37}$$

where $\overline{\phi}_{\pi(\tau)}$ is the complex conjugate of $\phi_{\pi(\tau)}$. Recall from (2.5) and §4.1 that

$$\begin{aligned}\Phi(g, \phi_{\pi(\tau)\otimes\sigma}, s) &= \phi_{\pi(\tau)\otimes\sigma}(g)\exp\langle s + \rho_{P_{2n+1}}, H_{P_{2n+1}}(g)\rangle \\ \Phi(g, \phi_{\pi(\tau)\otimes\tau}, s) &= \phi_{\pi(\tau)\otimes\tau}(g)\exp\langle s + \rho_{Q_{2n+1}}, H_{Q_{2n+1}}(g)\rangle.\end{aligned}$$

We need the following lemma, the proof of which is the same as that of Proposition 5.3 in [**GJR04**]. We omit the details here.

LEMMA 4.7. *If the period*

$$\mathcal{P}(\phi_\tau, \phi_\sigma, \psi_{2n+1,n-k})$$

does not vanish for some given $\phi_\sigma \in V_\sigma$ *and* $\phi_\tau \in V_\tau$*, then integral* (4.33)

$$\int_{\mathrm{K}\times X(\mathbb{A}_F)\times[\mathrm{PM}_{2n+1,2k}]} \phi_{\pi(\tau)\otimes\sigma}(m\mathrm{k})\mathcal{F}^{\psi_{2n+1,n-k}}(\phi_{\pi(\tau)\otimes\tau})(m\ell^-(x)w\mathrm{k})dmdxd\mathrm{k}$$

does not vanish for the corresponding data defined in (4.36) *and* (4.37)*, and for* $\phi_{\pi(\tau)} \in V_{\pi(\tau)}$*, where* $\mathcal{F}^{\psi_{2n+1,n-k}}(\phi_{\pi(\tau)\otimes\tau})$ *is defined as in* (4.31).

By Lemma 4.7 and Theorem 4.4, if the period

$$\mathcal{P}(\phi_\tau, \phi_\sigma, \psi_{2n+1,n-k})$$

does not vanish for some given $\phi_\sigma \in V_\sigma$, $\phi_\tau \in V_\tau$, and $\phi_{\pi(\tau)} \in V_{\pi(\tau)}$, then the period

$$\mathcal{P}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_{\pi(\tau)\otimes\sigma}, \psi_{6n+3,n-k})$$

does not vanish for the corresponding data. In particular, this implies that the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma}(h)$ does not vanish for the given data. This shows that the Eisenstein series $E(g, \phi_{\pi(\tau)\otimes\sigma}, s)$ ($2n+1 > 2k$, or $3r > m$) has a pole at $s = \frac{1}{2}$. By Proposition 2.2, the central value $L(\frac{1}{2}, \pi(\tau)\times\sigma)$ must be nonzero. By the Langlands functorial transfer ([**KK04**], [**KK05**], and [**S05**]), we have

$$L(s, \tau\times\sigma) = L(s, \pi(\tau)\times\sigma).$$

Hence we obtain the result that if the period $\mathcal{P}(\phi_\tau, \phi_\sigma, \psi_{2n+1,n-k})$ does not vanish, then the L-function $L(s, \tau\times\sigma)$ does not vanish at $s = \frac{1}{2}$. This proves the theorem for the case when $3r > m$ or equivalently $2n+1 > 2k$.

When $3r < m$, by the above notation, we have $2n+1 < 2k$. We can use the Eisenstein series $E(g, \phi_{\pi(\sigma)\otimes\tau}, s)$ and the same argument to show that the following L-functions

$$L(s, \sigma\times\tau) = L(s, \pi(\sigma)\times\tau) = L(s, \pi(\tau)\times\sigma)$$

dose not vanish at $s = \frac{1}{2}$. The definition of these L-functions are valid via the Langlands functorial transfer ([**KK04**], [**KK05**], and [**S05**]). Hence we still obtain that the product

$$L(s, \pi(\tau)\times\sigma)L(2s, \pi(\tau), \mathrm{r}_A)$$

has a pole at $s = \frac{1}{2}$. By Proposition 2.2, the Eisenstein series $E(g, \phi_{\pi(\tau)\otimes\sigma}, s)$ also has a pole at $s = \frac{1}{2}$. This completes the proof for the Eisenstein series $E(g, \phi_{\pi(\tau)\otimes\sigma}, s)$. It is clear now that the proof for the other Eisenstein series $E(g, \phi_{\pi(\sigma)\otimes\tau}, s)$ is the same. This finishes the proof of Theorem 4.6.

5. Final Remarks

5.1. Various models. In §3, following [**S05**], we introduced the models of the Gelfand-Graev type and of the Fourier-Jacobi type for Hermitian unitary groups and for skew-Hermitian unitary groups separately. However, one can easily find that both models can be introduced for either Hermitian unitary groups or skew-Hermitian unitary groups.

5.2. Periods of the Fourier-Jacobi type. It is easy to see that the most of the work in §4 can be carried over for the periods of the Fourier-Jacobi type introduced in §3.2. This can be compared to the work ([**GJR04**] and [**GJR05**]) for the periods of the Gelfand-Graev type for orthogonal groups and the periods of the Fourier-Jacobi type for the symplectic or metaplectic groups. We omit the discussion here. However, there is an important difference for unitary groups from these for orthogonal, symplectic, or metaplectic groups. This can be explained in terms of global Langlands parameters as follows. For simplicity, we consider Hermitian unitary group only, which is denoted by U_m, and the corresponding Hermitian form is given by $J_m = J_m^+$.

The Langlands dual group ${}^L\mathrm{U}_m$ of U_m is $\mathrm{GL}_m(\mathbb{C}) \ltimes \Gamma_{E/F}$. Recall that $\Gamma_{E/F} = \{1, \iota\}$. The action of ι on $\mathrm{GL}_m(\mathbb{C})$ is given by

$$\iota(g) = \Phi_m \cdot {}^tg^{-1} \cdot \Phi_m^{-1}, \tag{5.1}$$

where Φ_m is an $m \times m$-matrix with $(\Phi_m)_{i,j} = (-1)^{i-1}\delta_{i,n-j+1}$, see [**BR94**, 1.8] for more explanation. A $2m$-dimensional complex representation ρ_{2m} of the Langlands dual group ${}^L\mathrm{U}_m$ is given by

$$\begin{aligned} g &\mapsto \begin{pmatrix} g & 0 \\ 0 & {}^tg^{-1} \end{pmatrix} \\ \iota &\mapsto \begin{pmatrix} 0 & \Phi_m \\ \Phi_m^{-1} & 0 \end{pmatrix} \end{aligned} \tag{5.2}$$

for any $g \in \mathrm{GL}_m(\mathbb{C})$ and $\iota \in \Gamma_{E/F}$. It is easy to check that the representation is well-defined. Following from this definition, one can calculate directly that the representation ρ_{2m} is of symplectic type if m is even, and is of orthogonal type if m is odd. This means that when m is even, the image

$$\rho_{2m}({}^L\mathrm{U}_m) = \rho_{2m}(\mathrm{GL}_m(\mathbb{C}) \ltimes \Gamma_{E/F})$$

is included in the symplectic subgroup $\mathrm{Sp}_{2m}(\mathbb{C})$ of $\mathrm{GL}_{2m}(\mathbb{C})$; otherwise it is included in the orthogonal subgroup $\mathrm{O}_{2m}(\mathbb{C})$.

Let L_F be the conjectural Langlands group attached to the number field F. The global Langlands parameter of U_m is an admissible homomorphism

$$\phi_m \ : \ L_F \rightarrow^L \mathrm{U}_m = \mathrm{GL}_m(\mathbb{C}) \ltimes \Gamma_{E/F}, \tag{5.3}$$

up to conjugation ([**BR94**]). Assume that an irreducible cuspidal automorphic representations π_m of $U_m(\mathbb{A}_F)$ is parametrized by a global Langlands parameter

ϕ_m. Then we expect that the following identity for L-functions holds:

$$L(s,\pi_m,\rho_{2m}) = L(s,\rho_{2m}\circ\phi_m). \tag{5.4}$$

For an irreducible cuspidal automorphic representations π_m of $U_m(\mathbb{A}_F)$ and an irreducible cuspidal automorphic representations π_n of $U_m(\mathbb{A}_F)$, the tensor product $\pi_m\otimes\pi_n$ is an irreducible cuspidal automorphic representation of $(U_m\times U_n)(\mathbb{A}_F)$. The Langlands dual group of $U_m\times U_n$ is

$$^L(U_m\times U_n) = (\mathrm{GL}_m(\mathbb{C})\times\mathrm{GL}_n(\mathbb{C}))\ltimes\Gamma_{E/F}. \tag{5.5}$$

It has the $2mn$-dimensional complex representation ρ_{2mn} given by

$$\begin{aligned} (g_m,g_n) &\mapsto \begin{pmatrix} g_m\otimes {}^tg_n^{-1} & 0 \\ 0 & {}^tg_m^{-1}\otimes g_n \end{pmatrix} \\ \iota &\mapsto \begin{pmatrix} 0 & \Phi_m\otimes\Phi_n \\ \Phi_m^{-1}\otimes\Phi_n^{-1} & 0 \end{pmatrix}. \end{aligned} \tag{5.6}$$

Then we expect to have the following identity of L-functions

$$L(s,\pi_m\otimes\pi_n,\rho_{2mn}) = L(s,\rho_{2mn}\circ(\phi_m\otimes\phi_n)). \tag{5.7}$$

The L-functions are of degree $2mn$. It is easy to check that ρ_{2mn} is of orthogonal type if m and n are in the same parity; and of symplectic type otherwise.

Recall from §3.1 that the periods defined in (3.4) deal with the different parity of unitary groups U_m and U_{m-2e-1}. In this case, the above discussion and the discussions in [**GJR04**] and [**GJR05**] suggest that the L-functions

$$L(s,\pi_m\otimes\pi_{m-2e-1},\rho_{2m(m-2e-1)})$$

should be of symplectic type, and the central value (at $s=\frac{1}{2}$) of the L-functions should be critical in the sense of Deligne. Theorem 4.6 suggests that the periods given in the theorem should be attached to the central critical value of the L-functions. Some precise conjectures of this nature, which are the refinement of the Gross-Prasad conjecture ([**GP92**] and [**GP94**]), were given recently by Ichino and Ikeda for some important cases of orthogonal groups ([**II06**]). There should be expected to have a unitary group version of such conjectural identities. While the periods defined in §3.2 deal with the same parity of unitary groups U_m and U_{m-2e}. In this case, the L-functions

$$L(s,\pi_m\otimes\pi_{m-2e},\rho_{2m(m-2e)})$$

should be of orthogonal type. The periods are expected to be attached to the pole at $s=1$ of such L-functions. This is compatible with the results proved in [**GJR04**] and [**GJR05**].

References

[Ar78] J. Arthur, *A trace formula for reductive groups. I. Terms associated to classes in $G(Q)$*, Duke Math. J. **45** (1978), no. 4, 911–952.

[Ar80] J. Arthur, *A trace formula for reductive groups. II. Applications of a truncation operator*, Compositio Math. **40** (1980), no. 1, 87–121.

[BR94] D. Blasius and J. Rogawski, *Zeta functions of Shimura varieties*, Motives (Seattle, WA, 1991), 525–571, Proc. Sympos. Pure Math. **55**, Part 2, Amer. Math. Soc., Providence, RI, 1994.

[GR91] S. Gelbart and J. Rogawski *L-Functions and Fourier-Jacobi coefficients for the Unitary group $U(3)$*, Invent. Math. **105**, (1991), 445-472 .

[GRS97] S. Gelbart, J. Rogawski and D. Soudry, *Endoscopy, theta-liftings, and period integrals for the unitary group in three variables*, Ann. of Math. (2) **145** (1997), no. 3, 419–476.

[GJR01] D. Ginzburg, D. Jiang and S. Rallis, *Nonvanishing of the central critical value of the third symmetric power L-functions*, Forum Math. **13** (2001), no. 1, 109–132.

[GJR04] D. Ginzburg, D. Jiang and S. Rallis, *Non-vanishing of the central value of the Rankin-Selberg L-functions. I.*, J. Amer. Math. Soc. **17** (2004), no. 3, 679–722.

[GJR05] D. Ginzburg, D. Jiang and S. Rallis, *On the nonvanishing of the central value of the Rankin-Selberg L-functions. II*, Automorphic Representations, L-functions and Applications: Progress and Prospects, 157–191, Ohio State Univ. Math. Res. Inst. Publ., 11, de Gruyter, Berlin, 2005.

[GP92] B. Gross and D. Prasad, *On the decomposition of a representation of* SO_n *when restricted to* SO_{n-1}, Canad. J. Math. **44** (1992), 974–1002.

[GP94] B. Gross and D. Prasad, *On irreducible representations of* $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$, Canad. J. Math. **46** (1994), no. 5, 930–950.

[II06] A. Ichino and T. Ikeda, *On the Restriction of Automorphic Forms on* $\mathrm{SO}(n+1)$ *to* $\mathrm{SO}(n)$ *and the Global Gross-Prasad Conjecture*, Lecture Notes at Fukuoka, 2006 (www.math.kyoto-u.ac.jp/ ikeda).

[J84] H. Jacquet, *On the residual spectrum of* $\mathrm{GL}(n)$, Lie Group Representations, II (College Park, Md., 1982/1983), 185–208, Lecture Notes in Math., **1041**, Springer, Berlin, 1984.

[JR92] H. Jacquet and S. Rallis, *Symplectic periods*, J. Reine Angew. Math. **423** (1992), 175–197.

[Jng07] D. Jiang, *Periods of automorphic forms*, Proceedings of the International Conference on Complex Geometry and Related Fields, Studies in Advanced Mathematics **39**, 2007, 125–148.

[Jng08] D. Jiang, *Residues of Eisenstein series and related problems*, Eisenstein Series and Applications, Progress in Math. **258** (2008), 187–204.

[JngS07] D. Jiang and D. Soudry, *On the genericity of cuspidal automorphic forms of SO_{2n+1}, II*, Compositio Math. **143** (2007), no. 3, 721–748.

[K08] H. Kim, *Langlands-Shahidi method and poles of automorphic L-functions III: Exceptional groups*, J. Number Theory **128** (2008), no. 2, 354–376.

[K05] H. Kim, *On local L-functions and normalized intertwining operators*, Canad. J. Math. **57** (2005), no. 3, 535–597.

[KK05] H. Kim and M. Krishnamurthy, *Stable base change lift from unitary groups to* GL_n, IMRP Int. Math. Res. Pap. 2005, no. 1, 1–52.

[KK04] H. Kim and M. Krishnamurthy, *Base change lift for odd unitary groups.* Functional analysis VIII, 116–125, Various Publ. Ser. (Aarhus), **47**, Aarhus Univ., Aarhus, 2004.

[L71] R. Langlands, *Euler products*, A James K. Whittemore Lecture in Mathematics given at Yale University, 1967, Yale Mathematical Monographs, 1, Yale University Press, New Haven, Conn.-London, 1971.

[LR92] R. Langlands and D. Ramakrishnan, *The zeta functions of Picard modular surfaces*, Edited by Robert P. Langlands and Dinakar Ramakrishnan, Université de Montréal, Centre de Recherches Mathématiques, Montreal, QC, 1992.

[MW89] C. Moeglin and J.-L. Waldspurger, *Le spectre residuel de* $\mathrm{GL}(n)$, (French) [The residual spectrum of $\mathrm{GL}(n)$] Ann. Sci. Ecole Norm. Sup. (4) **22** (1989), no. 4, 605–674.

[MW95] C. Moeglin and J.-L. Waldspurger, *Spectral Decomposition and Eisenstein Series*, Cambridge Tracts in Mathematics **113**, Cambridge University Press, Cambridge, 1995.

[R90] J. Rogawski, *Automorphic representations of unitary groups in three variables*, Annals of Mathematics Studies **123**, Princeton University Press, Princeton, NJ, 1990.

[Sh88] F. Shahidi, *On the Ramanujan conjecture and the finiteness of poles of certain L-functions*, Ann. of Math. (2) **127** (1988), 547-584.

[Sh90] F. Shahidi, F. *A proof of Langlands' conjecture on Plancherel measures; Complementary series for p-adic groups*, Annals of Math. **132** (1990), 273–330.

[S05] D. Soudry, *On Langlands functoriality from classical groups to* GL_n, Automorphic forms. I, Astérisque No. **298** (2005), 335–390.

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