On the Nonvanishing of the Central Value of the Rankin-Selberg L-functions II

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Abstract. We characterize the nonvanishing of the central value of the Rankin-Selberg L-functions $L(s, \pi \times \pi')$ in terms of the period of the Gross-Prasad type for the case where π is an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2r}(\mathbb{A})$ of symplectic type and π' is an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2l}(\mathbb{A})$ of orthogonal type. The other case was treated in **[GJR03b]**.

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1. Introduction

We started in **[GJR03b]** our study on the nonvanishing of the central value of the Rankin-Selberg tensor product L-functions in terms of periods of automorphic forms. In **[GJR03b]**, we considered the cases where periods are defined over either a symplectic group or metaplectic double cover of a symplectic group, which was referred to as **Case (1)** in **[GJR03b]**. In this paper we will consider **Case (2)**, where the periods are defined over either an odd orthogonal group or an even orthogonal group.

Let π_1 and π_2 be irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ and $\mathrm{GL}_m(\mathbb{A})$, respectively, where $\mathbb{A}$ is the ring of adeles attached to a number field k. In **Case (2)**, we have that (i) $n = 2r$ is even and π_1 is *symplectic* (i.e. the exterior square L-function $L(s, \pi_1, \Lambda^2)$ has a simple pole at $s = 1$), and (ii) $m = 2l$ is even and π_2 is *orthogonal* (i.e. the symmetric square L-function $L(s, \pi_2, \mathrm{Sym}^2)$ has a simple pole at $s = 1$). Our objective is to characterize the non-vanishing of the Rankin-Selberg L-function $L(s, \pi_1 \times \pi_2)$ at $s = \frac{1}{2}$ (the center of symmetry) in terms of periods attached to π_1 and π_2 over orthogonal groups. The precise definition of the period will be given in Section 2. In this case, by the Theorem of Ginzburg-Rallis-Soudry ([**GJR03b**], §1), there exist an irreducible unitary generic cuspidal automorphic representation τ of $\mathrm{SO}_{2r+1}(\mathbb{A})$ which lifts to π_1 (since π_1 is symplectic), and an irreducible unitary generic cuspidal automorphic representation σ of $\mathrm{SO}_{2l}(\mathbb{A})$ which lifts to π_2 (since π_2 is orthogonal).

Conjecture. *The central value of the Rankin-Selberg L-function $L(\frac{1}{2}, \pi_1 \times \pi_2)$ is non-zero if and only if there exist a relevant pair $(\mathrm{SO}'_{2r+1}, \mathrm{SO}'_{2l})$ of inner forms of $(\mathrm{SO}_{2r+1}, \mathrm{SO}_{2l})$ (see Section 2 for definition) and a pair of irreducible cuspidal automorphic representations (τ', σ') in which σ' (resp. τ') is nearly equivalent to σ (resp. τ), such that the period*

$$\mathcal{P}_{2r+1,2l}(\phi_{\tau'}, \phi_{\sigma'})\ (r \geq l) \quad \text{or} \quad \mathcal{P}_{2l,2r+1}(\phi_{\sigma'}, \phi_{\tau'})\ (r \leq l)$$

is non-zero. Here the period $\mathcal{P}_{2r+1,2l}(\phi_{\tau'}, \phi_{\sigma'})$ $(r \geq l)$ is defined by

$$\int_{\mathrm{SO}'_{2l}(k)\backslash \mathrm{SO}'_{2l}(\mathbb{A})} \phi_{\sigma'}(h) \int_{V_{r,l}(k)\backslash V_{r,l}(\mathbb{A})} \phi_{\tau'}(vh)\psi_{V_{r,l}}(v)dvdh$$

and the period $\mathcal{P}_{2l,2r+1}(\phi_{\sigma'}, \phi_{\tau'})$ $(r \leq l)$ is defined in a similar way. The details of the definition will be given in Section 2.

We sketch the relation between this conjecture and the Gross-Prasad conjecture [**GP92**]. By the Langlands functoriality from generic cuspidal automorphic representations of SO to GL ([**CKPSS03**]), one might define

$$L(s, \sigma \times \tau) := L(s, \pi_1 \times \pi_2).$$

Then the assertion on the characterization of the nonvanishing of the central value $L(\frac{1}{2}, \sigma \times \tau)$ in terms of periods attached to either $(\sigma, \tau, \mathrm{SO}_{2l})$ if $r \geq l$ or $(\sigma, \tau, \mathrm{SO}_{2r+1})$ if $l > r$ is a conjecture of Gross and Prasad (see [**GP92**] and [**GP94**] for the global conjecture when $r = l$ or $l - 1$ and for the local conjecture in general).

When $r = 1$ and $l = 1$, the assertion was proved by Waldspurger in [**W85**]. When $l = 2$ and $r = 1$, it is a conjecture of Jacquet on the relation between the nonvanishing of the central value of the triple product L-function and the nonvanishing of the trilinear periods. For the split period case, it was proved in [**Jng98b**] and [**Jng01**]. For general period cases, it was proved completely in [**HK91**] and [**HK**]. When $r = 2$ and $l = 2$, some special cases were studied in [**HK92**] and [**BFSP**]. A special case of our conjecture ($r = 1$ and $l \geq 2$) was studied in [**GJR03a**], but by a different approach.

It is important to note that for an irreducible generic cuspidal automorphic representation of SO, the image of the Langlands functorial transfer to GL is an isobaric sum. It is almost certain that the refinement of the arguments in this paper can be carried over to this generality. We will consider this in our future work. The analogue for unitary groups is our work in progress.

In this paper, we shall use similar ideas as in [**GJR03b**] to study the above Conjecture. The following two Theorems are our main results.

Theorem A. *With the notations as in the Conjecture, if the period*

$$\mathcal{P}_{2r+1,2l}(\phi_{\tau}, \phi_{\sigma})\ (r \geq l) \quad \text{or} \quad \mathcal{P}_{2l,2r+1}(\phi_{\sigma}, \phi_{\tau})\ (r \leq l)$$

is non-zero, then the central value $L(\frac{1}{2}, \pi_1 \times \pi_2)$ of the Rankin-Selberg L-function $L(s, \pi_1 \times \pi_2)$ is non-zero.

The proof of Theorem A is to realize the above period as an 'inner' period of certain period of residue of Eisenstein series obtained from the pair (σ', τ'). This method has been used to study many interesting cases, about which we refer to the introduction of **[GJR03b]** for further discussions.

Theorem B. *With the notations as in the Conjecture, under certain assumptions, if the central value of the Rankin-Selberg L-function $L(\frac{1}{2}, \pi_1 \times \pi_2)$ is non-zero, then the period*

$$\mathcal{P}_{2r+1,2l}(\phi_{\tau'}, \phi_{\sigma'}) \ (r \geq l) \quad or \quad \mathcal{P}_{2l,2r+1}(\phi_{\sigma'}, \phi_{\tau'}) \ (r \leq l)$$

is non-zero for a pair (τ', σ').

We shall only prove Theorem B for a special case ($r \geq l = 1$) without the assumptions. For the general case, we need an assumption on the non-vanishing of certain Fourier coefficient of the residue of Eisenstein series obtained from the pair (τ', σ'), which we present the details for **Case (1)** in Section 6, **[GJR03b]**. We shall not provide the same details here for **Case (2)**. Further study of Fourier coefficients of the residual representations will be the subject matter of our on-going project.

We shall provide the general setting for **Case (2)** in Section 2, which includes the definitions of relevant pairs of orthogonal groups and the periods of automorphic forms, and other notations needed for the rest of the paper. In Section 3, we first recall basic facts of Eisenstein series and their residues, prove nonvanishing results on certain Fourier coefficients of the residues of relevant Eisenstein series, and then establish a basic identity (Theorem 3.8) which relates the 'outer' period to the 'inner' period. We remark that Theorem 3.8 deals with the residual representations with nearly generic cuspidal data. This makes the method used here potentially applicable to the investigation of residual representations without assuming the genericity of cuspidal data. This basic identity is used in Section 4 to prove Theorem A. In Section 5, we prove Theorem B for the special case $r \geq l = 1$. The general case will be considered in our future work.

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2. Periods

In this section, we give a formal definition of period integrals for automorphic forms. The remaining sections are devoted to the study of these periods for automorphic forms occurring in the discrete spectrum.

2.1. Relevant Pairs. We recall from [**GP94**] the notion of relevant pairs of orthogonal groups. Let $(V, <\cdot,\cdot>_V)$ be a quadratic space over a field k. We assume that all quadratic spaces considered in this paper are non-degenerate and of finite dimension. Let $(W, <\cdot,\cdot>_W)$ be another quadratic space, which can be embedded into V. We fix an embedding and define

$$W^\perp := \{v \in V \mid\ <v, w>_V = 0,\ \textit{for all}\ w \in W\}.$$

Then, following Witt's Theorem, one knows that $W^\perp$ is a quadratic space and

$$V = W \oplus W^\perp. \tag{2.1}$$

We say the pair (V, W) is *relevant* if W embeds in V and $W^\perp$ is a split quadratic space with odd dimension over k. For more details about relevant pairs, we refer to Section 3 in [**GP94**].

2.2. Periods of Automorphic Forms. Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. Let (V, W) be a relevant pair of quadratic spaces. By definition,

$$W^\perp = (X + X') \oplus <a>$$

is a k-split quadratic space of dimension $2e+1$ with X and X' dual isotropic subspaces of dimension e. Let $Y = X + X'$, which is k-split and nondegenerate. We write

$$V = Y \oplus Y^\perp.$$

Then it is easy to see that W can be embedded into $Y^\perp$ with codimension one. Let $M_e = \mathrm{GL}(X) \times \mathrm{SO}(Y^\perp)$. It is clear that M_e is a Levi subgroup of $G = \mathrm{SO}(V)$ and it is non-trivial if $e = \dim_k X > 0$. Let $P_e = M_e U_e$ be a standard parabolic subgroup of $G = \mathrm{SO}(V)$. It is proper and maximal if $e > 0$. In this paper, we include in our discussion the case that $e = \dim_k X = 0$. We have

$$0 \to \Lambda^2(X) \to U_e \to U_e^{ab} = X \otimes Y^\perp \to 0,$$

which is trivial if $e = 0$.

When $e = 0$, we consider the diagonal embedding of $H = \mathrm{SO}(W)$ into $G \times H$. When $e \geq 1$, we let $X_1 \subset X$ be a hyperplane and

$$\ell_1 \ : \ X \to \mathbb{G}_a$$

be a non-zero linear functional which vanishes on X_1. Let

$$\ell_W \ : \ Y^\perp \to \mathbb{G}_a$$

be a non-zero linear functional which vanishes on the hyperplane W. Consider the composite map ρ_e of the projection $U_e \to U_e^{ab} = X \otimes Y^\perp$ with

$$\ell_1 \otimes \ell_W \ : \ U_e^{ab} = X \otimes Y^\perp \to \mathbb{G}_a.$$

Since the Levi subgroup M_e normalizes U_e, the stabilizer of ρ_e in M_e is

$$\mathrm{GL}(X)_{\ell_1} \times \mathrm{SO}(W),$$

where $\mathrm{GL}(X)_{\ell_1}$ is the subgroup of $\mathrm{GL}(X)$ which fixes the linear functional ℓ_1. One can check that it is the subgroup of $\mathrm{GL}(X)$ consisting of elements of the following type

$$\begin{pmatrix} h & * \\ 0 & 1 \end{pmatrix}, \quad h \in \mathrm{GL}_{e-1}.$$

Let $P_{1^e} = [\mathrm{GL}_1^e \times \mathrm{SO}(Y^\perp)]V_e$ be a standard parabolic subgroup of G. Then we have

$$V_e = (N_e \times I_{Y^\perp}) \ltimes U_e.$$

Let $\ell_e \ : \ N_e \to \mathbb{G}_a$ be a homomorphism which is non-trivial when restricted to each simple roots of GL_e in N_e. We define

$$\ell \ : \ V_e \to \mathbb{G}_a$$

to be a homomorphism which equals ℓ_e on N_e and equals ρ_e on U_e. It is clear that the stabilizer of ℓ in $\mathrm{SO}(Y^\perp)$ is isomorphic to $H = \mathrm{SO}(W)$. Define

$$R_e := H \ltimes V_e$$

which is an algebraic subgroup of G. We consider the natural diagonal embedding of H into $R_e \times H$.

Fix a non-trivial character ψ of $k \backslash \mathbb{A}$. We define

$$\psi_e(n) = \psi(\ell_e(n)) \tag{2.2}$$

which is a character of $N_e(\mathbb{A})$ and is trivial on the subgroup $N_e(k)$, and define

$$\psi_{\rho_e}(u) = \psi(\rho_e(u)), \tag{2.3}$$

which is a character ψ_{ρ_e} of $U_e(\mathbb{A})$ and is trivial on the subgroup $U_e(k)$. Then we define a character ψ_{V_e} of $V_e(\mathbb{A})$ by

$$\psi_{V_e}(v(n,u)) = \psi_e(n)\psi_{\rho_e}(u), \tag{2.4}$$

which is trivial on $V_e(k)$.

Now we define the period of automorphic forms. Let ϕ be an automorphic forms on $G(\mathbb{A})$ and φ be an automorphic forms on $H(\mathbb{A})$. The period of automorphic forms which we shall study is defined by the following integral

$$\mathcal{P}_{R_e}(\phi,\varphi) := \int_{H(k)\backslash H(\mathbb{A})} \varphi(h) \int_{V_e(k)\backslash V_e(\mathbb{A})} \phi(vh)\psi_{V_e}(v)dvdh. \tag{2.5}$$

It is clear that if one of ϕ and φ is cuspidal, then the period $\mathcal{P}_{R_e}(\phi,\varphi)$ converges absolutely. Otherwise, the integral needs to be defined through regularization.

From the definition of relevant pairs, the parity of $\dim_k V$ and $\dim_k W$ are different. When $\dim_k V$ is even, we denote $\dim_k V = 2l$ and $\dim_k W = 2r+1$. This implies that $r < l$. When $\dim_k V$ is odd, we denote $\dim_k V = 2r+1$ and $\dim_k W = 2l$. This implies that $r \geq l$. Hence we have $e = r - l$ if $r \geq l$; and $e = l - r - 1$ if $r < l$. We also denote the period by

$$\mathcal{P}_{R_e}(\phi,\varphi) = \begin{cases} \mathcal{P}_{2r+1,2l}(\phi,\varphi) & \textit{if } r \geq l \\ \mathcal{P}_{2l,2r+1}(\phi,\varphi) & \textit{if } r < l. \end{cases} \tag{2.6}$$

3. Periods of Residues of Eisenstein Series

We shall consider period $\mathcal{P}_{R_e}(\phi,\varphi)$ when both ϕ and φ are residues of Eisenstein series.

3.1. Residues of Eisenstein series. We consider orthogonal groups SO_{4r+q} with $q = 2r+1$ or $2l$. The k-split rank of SO_{4r+q} is assumed to be at least $2r$. Let $P_{2r,q}$ be the standard maximal parabolic subgroup of SO_{4r+q} with the Levi decomposition

$$P_{2r,q} = M_{2r,q} \ltimes U_{2r,q} = (\mathrm{GL}_{2r} \times \mathrm{SO}_q) \ltimes U_{2r,q}.$$

We denote by $w_{2r,q}$ the Weyl element which conjugates $P_{2r,q}$ to its opposite $P_{2r,q}^-$. The elements of $M_{2r,q}$ will be often written as

$$m = m(a,b) \in \mathrm{GL}_{2r} \times \mathrm{SO}_q. \tag{3.1}$$

Let $\pi \otimes \sigma_q$ be an irreducible generic cuspidal automorphic representation of $\mathrm{GL}_{2r} \times \mathrm{SO}_q$. We assume that the central character of π is trivial. This assumption is natural as long as we consider the existence of the residual representations that we are going to define below.

To the given generic cuspidal datum $(P_{2r,q}, \pi \otimes \sigma_q)$, one may attach an Eisenstein series $E(g, s, \phi_{\pi\otimes\sigma_q})$ on $\mathrm{SO}_{4r+q}(\mathbb{A})$. More precisely, the cuspidal automorphic representation $\pi \otimes \sigma_q$ can be realized in the space of square integrable automorphic functions $\mathrm{L}^2(Z_{M_{2r,q}}(\mathbb{A})M_{2r,q}(k)\backslash M_{2r,q}(\mathbb{A}))$, where $Z_{M_{2r,q}}$ is the center of $M_{2r,q}$. Let K be the maximal compact subgroup of SO_{4r+q} such that

$$\mathrm{SO}_{4r+q}(\mathbb{A}) = P_{2r,q}(\mathbb{A})\mathrm{K}$$

is the Iwasawa decomposition. Let $\phi_{\pi\otimes\sigma_q}$ be a $\mathrm{K}\cap M_{2r,q}(\mathbb{A})$-finite automorphic form in $\pi \otimes \sigma_q$. It can be extended as a function of $\mathrm{SO}_{4r+q}(\mathbb{A})$, so that for $g = um\mathrm{k} \in \mathrm{SO}_{4r+q}(\mathbb{A})$

$$\phi_{\pi\otimes\sigma_q}(g) = \phi_{\pi\otimes\sigma_q}(mk)$$

and for any fixed $\mathrm{k} \in \mathrm{K}$, the function

$$m \mapsto \phi_{\pi\otimes\sigma_q}(mk)$$

is a $\mathrm{K} \cap M_{2r,q}(\mathbb{A})$-finite automorphic function in $\pi \otimes \sigma_q$. We define

$$\Phi(g, s, \phi_{\pi\otimes\sigma_q}) := \phi_{\pi\otimes\sigma_q}(g) \exp < s + \rho_{P_{2r,q}}, H_{P_{2r,q}}(g) > \quad (3.2)$$

for $g \in \mathrm{SO}_{4r+q}(\mathbb{A})$. Here the parameter s is normalized as in **[GSh88]**. In our case we have

$$\exp < s + \rho_{P_{2r,q}}, H_{P_{2r,q}}(g) >= |\det a|^{s+\frac{2r+q-1}{2}} \quad (3.3)$$

where we write $g = um(a, b)\mathrm{k} \in \mathrm{SO}_{4r+q}(\mathbb{A})$ and $m(a, b) \in \mathrm{GL}_{2r} \times \mathrm{SO}_q$. Then the Eisenstein series is given by

$$E(g, s, \phi_{\pi\otimes\sigma_q}) = \sum_{\gamma\in P_{2r,q}(k)\backslash \mathrm{SO}_{4r+q}(k)} \Phi(\gamma g, s, \phi_{\pi\otimes\sigma_q}). \quad (3.4)$$

The constant term of the Eisenstein series $E(g, s, \phi_{\pi\otimes\sigma_q})$ along a standard parabolic subgroup P is always zero unless $P = P_{2r,q}$. In this case, one has

$$\begin{aligned} E_{P_{2r,q}}(g, s, \phi_{\pi\otimes\sigma_q}) &= \int_{[U_{2r,q}]} E(ug, s, \phi_{\pi\otimes\sigma_q})du \quad (3.5) \\ &= \Phi(g, s, \phi_{\pi\otimes\sigma_q}) + \mathcal{M}(w_{2r,q}, s)(\Phi(\cdot, s, \phi_{\pi\otimes\sigma_q}))(g) \end{aligned}$$

where $[U_{2r,q}] = U_{2r,q}(k)\backslash U_{2r,q}(\mathbb{A})$. We denote here by $\mathcal{M}(w_{2r,q}, s)$ the standard intertwining operator attached to the Weyl element $w_{2r,q}$, which maps the (normalized) induced representation

$$\mathrm{I}(s, \pi \otimes \sigma_q) = \mathrm{Ind}_{P_{2r,q}(\mathbb{A})}^{\mathrm{SO}_{4r+q}(\mathbb{A})}(\pi \otimes \sigma_q \otimes \exp < s, H_{P_{2r,q}}(\cdot) >) \quad (3.6)$$

to $\mathrm{I}(-s, w_{2r,q}(\pi \otimes \sigma_q))$. It follows from the general theory of Eisenstein series of Langlands that the Eisenstein series $E(g, s, \phi_{\pi\otimes\sigma_q})$ has a pole at $s = s_0 > 0$ if and only if the term $\mathcal{M}(w_{2r,q}, s)(\Phi(\cdot, s, \phi_{\pi\otimes\sigma_q}))$ has a pole at $s = s_0$ for some holomorphic (or standard) section $\Phi(g, s, \phi_{\pi\otimes\sigma_q})$ in $\mathrm{I}(s, \pi\otimes\sigma_q)$. Since factorizable sections generate a dense subspace in $\mathrm{I}(s, \pi\otimes\sigma_q)$, it suffices to consider the factorizable sections for the existence of poles of the Eisenstein series $E(g, s, \phi_{\pi\otimes\sigma_q})$, or for the existence of poles of $\mathcal{M}(w_{2r,q}, s)(\Phi(\cdot, s, \phi_{\pi\otimes\sigma_q}))$. When the section $\Phi(\cdot, s, \phi_{\pi_2\otimes\sigma_q})$ is factorizable, i.e.

$$\Phi(\cdot, s, \phi_{\pi\otimes\sigma_q}) = \otimes_v \Phi_v(\cdot, s, \phi_{\pi_v\otimes\sigma_{q,v}}), \tag{3.7}$$

where $\Phi_v(\cdot, s, \phi_{\pi_v\otimes\sigma_{q,v}})$ is a section in $\mathrm{I}(s, \pi_v\otimes\sigma_{q,v})$ and is unramified at almost all finite local places v, the term $\mathcal{M}(w_{2r,q}, s)(\Phi(\cdot, s, \phi_{\pi\otimes\sigma_q}))$ can be expressed as an infinite product

$$\mathcal{M}(w_{2r,q}, s)(\Phi(\cdot, s, \phi_{\pi\otimes\sigma_q})) = \prod_v \mathcal{M}_v(w_{2r,q}, s)(\Phi_v(\cdot, s, \phi_{\pi_v\otimes\sigma_{q,v}})). \tag{3.8}$$

By the Langlands-Shahidi theory **[L71]** and **[Sh88]**, we have

$$\mathcal{M}(w_{2r,q}, s) = \frac{L(s, \pi\times\sigma_q)L(2s, \pi, \mathrm{r}_2)}{L(s+1, \pi\times\sigma_q)L(2s+1, \pi, \mathrm{r}_2)} \cdot \prod_v \mathcal{N}_v(w_{2r,q}, s), \tag{3.9}$$

where r_2 is the symmetric square representation $\mathrm{Sym}^2(\mathbb{C}^{2r})$ of $\mathrm{GL}_{2r}(\mathbb{C})$ if $q = 2r+1$, and the exterior square representation $\Lambda^2(\mathbb{C}^{2r})$ of $\mathrm{GL}_{2r}(\mathbb{C})$ if $q = 2l$. Here we denote by $\mathcal{N}_v(w_{2r,q}, s)$ the normalized intertwining operator

$$\mathcal{N}_v(w_{2r,q}, s) = \frac{L(s+1, \pi\times\sigma_q)L(2s+1, \pi, \mathrm{r}_2)}{L(s, \pi\times\sigma_q)L(2s, \pi, \mathrm{r}_2)} \cdot \mathcal{M}_v(w_{2r,q}, s)$$

which defines a mapping from $\mathrm{I}(s, \pi_v\otimes\sigma_{q,v})$ to $\mathrm{I}(-s, w_{2r,q}(\pi_v\otimes\sigma_{q,v}))$.

Proposition 3.1 (Theorem 4.11 [K]). The normalized local intertwining operator $\mathcal{N}_v(w_{2r,q}, s)$ is holomorphic and nonzero for the real part of s greater than or equal to $\frac{1}{2}$, i.e. for any holomorphic section $\Phi_v(\cdot, s, \phi_{\pi_v\otimes\sigma_{q,v}})$ in $\mathrm{I}(s, \pi_v\otimes\sigma_{q,v})$, as a function in s, $\mathcal{N}_v(w_{2r,q}, s)(\Phi_v(\cdot, s, \phi_{\pi_v\otimes\sigma_{q,v}}))$ is holomorphic and nonzero for the real part of s greater than or equal to $\frac{1}{2}$.

By Proposition 3.1, one deduces

Proposition 3.2. The Eisenstein series $E(g, s, \phi_{\pi\otimes\sigma_q})$ can possibly have a simple pole at $s = \frac{1}{2}$ or $s = 1$. The existence of the pole at $s = \frac{1}{2}$ or $s = 1$ of $E(g, s, \phi_{\pi\otimes\sigma_q})$ is equivalent to the existence of the pole at $s = \frac{1}{2}$ or $s = 1$ of the product of L-functions

$$L(s, \pi\times\sigma_q)L(2s, \pi, \mathrm{r}_2)$$

respectively.

Proof. By the Langlands theory of constant terms of Eisenstein series, the Eisenstein series has a pole at $s = s_0$ if and only if the constant terms of the Eisenstein series has a pole at $s = s_0$. By (3.5), this is equivalent to say that the global intertwining operator $\mathcal{M}(w_{2r,q}, s)$ has a pole at $s = s_0 \geq \frac{1}{2}$. From identity (3.9), if the global intertwining operator $\mathcal{M}(w_{2r,l}, s)$ has a pole at $s = s_0$, then the quotient

$$\frac{L(s, \pi \times \sigma_q)L(2s, \pi, \mathrm{r}_2)}{L(s+1, \pi \times \sigma_q)L(2s+1, \pi, \mathrm{r}_2)}$$

must have a pole at $s = s_0$ since, by Proposition 3.1, the product

$$\prod_v \mathcal{N}_v(w_{2r,q}, s)$$

does not vanish for $s = s_0 \geq \frac{1}{2}$. Now both L-functions $L(s, \pi \times \sigma_q)$ and $L(s, \pi, \mathrm{r}_2)$ are non-zero for the real part of s greater than one. It follows that the product

$$L(s, \pi \times \sigma_q)L(2s, \pi, \mathrm{r}_2)$$

must have a pole at $s = s_0 \geq \frac{1}{2}$ if $\mathcal{M}(w_{2r,l}, s)$ has a pole at $s = s_0 \geq \frac{1}{2}$. Conversely, if the product of L-functions

$$L(s, \pi \times \sigma_q)L(2s, \pi, \mathrm{r}_2)$$

has a pole at $s = s_0$, then the global intertwining operator $\mathcal{M}(w_{2r,q}, s)$ has a pole at $s = s_0$, because in (3.9), we can always choose a particular factorizable section $\Phi(\cdot, s, \phi_{\pi\otimes\sigma_q})$ as in (3.7), so that the product

$$\prod_v \mathcal{N}_v(w_{2r,q}, s)(\Phi_v(\cdot, s, \phi_{\pi_v\otimes\sigma_{q,v}}))$$

is holomorphic and non-zero at $s = s_0$. Note that in this part, we do not need Proposition 3.1.

Finally, by Theorem 7.2 in **[CKPSS03]**, the irreducible generic cuspidal automorphic representation σ_q of $\mathrm{SO}_q(\mathbb{A})$ has a Langlands functorial lift $\pi(\sigma)$, which is an irreducible unitary automorphic representation of $\mathrm{GL}_q(\mathbb{A})$ (if $q = 2l$) or $\mathrm{GL}_{q-1}(\mathbb{A})$ (if $q = 2r+1$), is uniquely determined by σ_q, and is of isobaric type. Hence one has

$$L(s, \pi \times \sigma_q) = L(s, \pi \times \pi(\sigma_q)).$$

It follows that the L-function $L(s, \pi \times \sigma_q)$ is holomorphic and non-vanishing at a real point $s_0 > 1$, and $L(s, \pi \times \sigma_q)$ has a pole at $s = 1$ if and only if π is isomorphic to one of the isobaric summands of $\pi(\sigma)$. By Theorem 3.1 in **[K00]**, the complete L-function $L(s, \pi, \mathrm{r}_2)$ is holomorphic and non-vanishing at any real point $s_0 > 1$. By Theorem A in **[S]**, one knows that $L(s, \pi, \mathrm{r}_2)$ has a pole at $s = 1$ if and only if π is the image of an irreducible unitary generic cuspidal automorphic representation τ of $\mathrm{SO}_{2r}(\mathbb{A})$ (if r_2 is the symmetric square) or $\mathrm{SO}_{2r+1}(\mathbb{A})$ (if r_2 is the exterior square), under the Langlands functorial lifting from SO_{2r} and SO_{2r+1} to GL_{2r}, respectively. Hence one knows that for the real value $s \geq \frac{1}{2}$, both L-functions $L(s, \pi \times \sigma_q)$ and $L(s, \pi, \mathrm{r}_2)$ can have possible poles only at $s = 1$. The Proposition follows. $\square$

In the following we are going to study some special type of Eisenstein series. Let τ be an irreducible generic unitary cuspidal automorphic representation of $\mathrm{SO}_{2r+1}(\mathbb{A})$ which lifts to an irreducible unitary cuspidal automorphic representation $\pi(\tau)$ of $\mathrm{GL}_{2r}(\mathbb{A})$. Let σ be an irreducible generic unitary cuspidal automorphic representation of $\mathrm{SO}_{2l}(\mathbb{A})$ which lifts to an irreducible unitary cuspidal automorphic representation $\pi(\sigma)$ of $\mathrm{GL}_{2l}(\mathbb{A})$. By the automorphic descent established in **[GRS01]** and **[S]**, such τ and σ exist.

Let τ' be an irreducible unitary cuspidal automorphic representation of $\mathrm{SO}'_{2r+1}(\mathbb{A})$ (an inner form of SO_{2r+1}) nearly equivalent to τ, and σ' be an irreducible unitary cuspidal automorphic representation of $\mathrm{SO}'_{2l}(\mathbb{A})$ (an inner form of SO_{2l}) nearly equivalent to σ. If SO'_{2l} is an inner form of SO_{2l} and SO'_{2r+1} is an inner form of SO_{2r+1}, then SO'_{6l} is the inner form of SO_{6l} such that $\mathrm{GL}_{2l} \times \mathrm{SO}'_{2l}$ is a Levi subgroup of SO'_{6l}, and SO'_{6r+1} is the inner form of SO_{6r+1} such that $\mathrm{GL}_{2r} \times \mathrm{SO}'_{2r+1}$ is a Levi subgroup of SO'_{6r+1}.

As in Proposition 3.2, we expect that both Eisenstein series $E(g, s, \phi_{\pi(\tau)\otimes\sigma'})$ and $E(g, s, \phi_{\pi(\sigma)\otimes\tau'})$ have a possible pole only at $s = \frac{1}{2}$ and the nonvanishing of the residue at $s = \frac{1}{2}$ is characterized by the nonvanishing of the value at $s = \frac{1}{2}$ of the L-function $L(s, \pi(\tau) \times \sigma')$ and $L(s, \pi(\sigma) \otimes \tau')$, respectively. We denote the residue by

$$\mathcal{E}_{\pi(\tau)\times\sigma'}(g) := \mathrm{Res}_{s=\frac{1}{2}} E(g, s, \phi_{\pi(\tau)\times\sigma'}) \tag{3.10}$$

$$\mathcal{E}_{\pi(\sigma)\otimes\tau'}(g) := \mathrm{Res}_{s=\frac{1}{2}} E(g, s, \phi_{\pi(\sigma)\otimes\tau'}). \tag{3.11}$$

One of the main results of this paper is to characterize the nonvanishing of these residues in terms of periods defined over the cuspidal support of the residues. To this end we have to introduce Eisenstein series $E(g, s, \phi_{\pi(\tau)\otimes\tau'})$ and $E(g, s, \phi_{\pi(\sigma)\otimes\sigma'})$. We show that both Eisenstein series have a non-zero residue at $s = 1$.

Lemma 3.3. With the notations as above, the following statements hold.

(1) The residue at $s = 1$ of the Eisenstein series $E(g, s, \phi_{\pi(\tau)\otimes\tau'})$ on $\mathrm{SO}'_{6r+1}(\mathbb{A})$ is non-zero.

(2) The residue at $s = 1$ of the Eisenstein series $E(g, s, \phi_{\pi(\sigma)\otimes\sigma'})$ on $\mathrm{SO}'_{6l}(\mathbb{A})$ is nonzero.

Proof. We shall prove Part (1), while Part(2) follows by a similar argument. By the general argument in Section 3.1, we have

$$\mathcal{M}(w_{2r,2r+1}, s) = \frac{L(s, \pi(\tau) \times \tau')L(2s, \pi(\tau), \mathrm{Sym}^2)}{L(s+1, \pi(\tau) \times \tau')L(2s+1, \pi(\tau), \mathrm{Sym}^2)} \cdot \prod_v \mathcal{N}_v(w_{2r,2r+1}, s).$$

We can choose the datum such that $\mathcal{N}_v(w_{2r,2r+1}, s)$ is non-zero at $s = 1$. Since τ' is nearly equivalent to τ and the local L-factor of $L(s, \pi(\tau) \times \tau')$ does not vanish at $s = 1$, we deduce that the L-function $L(s, \pi(\tau) \times \tau')$ has a pole at $s = 1$. Since the L-function $L(s, \pi(\tau), \mathrm{Sym}^2)$ does not vanish at $s = 2$, we know that the global intertwining operator $\mathcal{M}(w_{2r,2r+1}, s)$ has a pole at $s = 1$ with some choice of data. This implies that the Eisenstein series $E(g, s, \phi_{\pi(\tau)\otimes\tau'})$ has a pole at $s = 1$. □

We denote the residues by

$$\mathcal{E}_{\pi(\tau)\otimes\tau'}(g) = \mathrm{Res}_{s=1} E(g, s, \phi_{\pi(\tau)\otimes\tau'}) \tag{3.12}$$

$$\mathcal{E}_{\pi(\sigma)\otimes\sigma'}(g) = \mathrm{Res}_{s=1} E(g, s, \phi_{\pi(\sigma)\otimes\sigma'}). \tag{3.13}$$

3.2. Certain Fourier Coefficients of the Residues. We shall study the vanishing property of certain Fourier coefficients of the residue $\mathcal{E}_{\pi(\tau)\otimes\tau'}(g)$ and $\mathcal{E}_{\pi(\sigma)\otimes\sigma'}(g)$. The proof is based on the structure of the local unramified components of the residues.

Let ν be a local finite place of k and $k_\nu = F$. Let τ_ν be the local ν-component of τ and σ_ν be the local ν-component of σ. We may make assume that ν is such a local finite place such that

(1) both τ_ν and σ_ν are unramified representations of F-split SO_{2r+1} and SO_{2l}, respectively, and

(2) both τ_ν and σ_ν are generic.

We write

$$\tau_\nu = \mathrm{Ind}_{B_{\mathrm{SO}_{2r+1}}(F)}^{\mathrm{SO}_{2r+1}(F)}(\chi_\nu) \tag{3.14}$$

$$\sigma_\nu = \mathrm{Ind}_{B_{\mathrm{SO}_{2l}}(F)}^{\mathrm{SO}_{2l}(F)}(\mu_\nu) \tag{3.15}$$

where both χ_ν and μ_ν are unramified characters. Here we consider the normalized (or unitary) induced representations. Let $T_{\mathrm{SO}_{2r+1}}$ denote the maximal torus of SO_{2r+1} whose elements are written as

$$t = \mathrm{diag}(a_1, \cdots, a_r, 1, a_r^{-1}, \cdots, a_1^{-1}).$$

Then we write $\chi_\nu(t) := \prod_{i=1}^r \chi_i(a_i)$. Similarly, we write $\mu_\nu := \prod_{j=1}^l \mu_j$.

Let θ_{τ_ν} be the irreducible unramified constituent of $\mathrm{I}(1, \pi(\tau_\nu) \otimes \tau_\nu)$ and θ_{σ_ν} be the irreducible unramified constituent of $\mathrm{I}(1, \pi(\sigma_\nu) \otimes \sigma_\nu)$. Let P_{3^r} be the standard parabolic subgroup of SO_{6r+1} with Levi subgroup $(\mathrm{GL}_3)^r$ and Q_{3^l} be the standard parabolic subgroup of SO_{6l} with Levi subgroup $(\mathrm{GL}_3)^l$. We have

Lemma 3.4. Let det be the determinant of GL_3.

(1) The irreducible unramified representation θ_{τ_ν} is a constituent of the spherical unitarily induced representation

$$\mathrm{Ind}_{P_{3r}(F)}^{\mathrm{SO}_{6r+1}(F)}(\chi_1 \circ \det \otimes \cdots \otimes \chi_r \circ \det).$$

(2) The irreducible unramified representation θ_{σ_ν} is a constituent of the spherical unitarily induced representation

$$\mathrm{Ind}_{Q_{3^l}(F)}^{\mathrm{SO}_{6l}(F)}(\mu_1 \circ \det \otimes \cdots \otimes \mu_l \circ \det).$$

Proof. We prove Part (1), while Part (2) is similar.

The representation θ_{τ_ν} is the unramified subquotient of $\mathrm{I}(1, \pi(\tau_\nu) \otimes \tau_\nu)$. and $\tau_\nu = \mathrm{Ind}_{B_{\mathrm{SO}_{2r+1}}(F)}^{\mathrm{SO}_{2r+1}(F)}(\chi_\nu)$. The unramified local Langlands functorial lift of τ_ν to $\mathrm{GL}_{2r}(F)$ is $\pi(\tau_\nu) = \mathrm{Ind}_{B_{\mathrm{GL}_{2r}}(F)}^{\mathrm{GL}_{2r}(F)}(\chi_\nu)$. Here χ_ν is the character of the maximal torus $T_{\mathrm{GL}_{2r}}$ of GL_{2r} defined by

$$\chi_\nu(\mathrm{diag}(u_1, \cdots, u_r, u_{r+1}, \cdots, u_{2r})) = \prod_{i=1}^{r} \chi_i(u_i u_{2r-i+1}^{-1}).$$

Then the induced representation $\mathrm{I}(1, \pi(\tau_\nu) \otimes \tau_\nu)$ can be written as

$$\mathrm{Ind}_{B_{\mathrm{SO}_{6r+1}}(F)}^{\mathrm{SO}_{6r+1}(F)}(\chi' \cdot \chi_0)$$

where

$$\begin{aligned} \chi'(t) &= \prod_{i=1}^{r} \chi_i(t_i t_{2r-i+1}^{-1}) \prod_{j=2r+1}^{3r} \chi_{j-2r}(t_j) \\ \chi_0(t) &= |t_1 \cdots t_{2r}| \end{aligned}$$

for $t = \mathrm{diag}(t_1, \cdots, t_{3r}, 1, t_{3r}^{-1}, \cdots, t_1^{-1})$ in the maximal torus $T_{\mathrm{SO}_{6r+1}}$ of SO_{6r+1}. It is not difficult to find a Weyl element w such that

$$\mathrm{Ind}_{B_{\mathrm{SO}_{6r+1}}(F)}^{\mathrm{SO}_{6r+1}(F)}(\chi' \cdot \chi_0) \cong \mathrm{Ind}_{B_{\mathrm{SO}_{6r+1}}(F)}^{\mathrm{SO}_{6r+1}(F)}(\widetilde{\chi})$$

where the character $\widetilde{\chi}$ of $T_{\mathrm{SO}_{6r+1}}$ is given by

$$\widetilde{\chi}(t) = \prod_{i=1}^{r} \chi_i(a_{i1} a_{i2} a_{i3}) |a_{i1} a_{i3}^{-1}|$$

for $t = \mathrm{diag}(a_{11}, a_{12}, a_{13}, \cdots, a_{r1}, a_{r2}, a_{r3}, 1, a_{r3}^{-1}, a_{r2}^{-1}, a_{r1}^{-1}, \cdots, a_{13}^{-1}, a_{12}^{-1}, a_{11}^{-1})$ in $T_{\mathrm{SO}_{6r+1}}$. By induction in stages, Part (1) of the Lemma follows. □

Lemma 3.4 determines the structure of the local unramified component of the residues we want to study. In the following we shall show that such an unramified local component does not support certain local Fourier functional, which is related to the Fourier coefficient of the residues.

Let G be the F-split SO_{6r+1}. As in Section 2.2, we let $e \geq r+1$ and define the unipotent subgroup $V_e = V_{6r+1,e}$ of SO_{6r+1}. The elements of V_e can be written as

$$v(n,x,z) = \begin{pmatrix} n & x & z \\ & I_{6r-2e+1} & x^* \\ & & n^* \end{pmatrix} \in \mathrm{SO}_{6r+1}, \tag{3.16}$$

where n is in N_e, the standard maximal unipotent subgroup of GL_e, $x \in \mathrm{Mat}_{e\times(6r-2e+1)}$ and $z \in \{Z \in \mathrm{Mat}_{e\times e} \ : \ {}^tZJ_e + J_eZ = 0\}$, where we define inductively

$$J_e = \begin{pmatrix} & & 1 \\ & J_{e-2} & \\ 1 & & \end{pmatrix}.$$

Given a vector $\zeta \in F^{6r-2e+1}$ of non-zero length, we define a character $\psi^{\zeta}_{V_e}$ of the group $V_{6r+1,e}$ by

$$\psi^{\zeta}_{V_e}(v) = \psi(n_{1,2} + \cdots + n_{e-1,e} + x_e \cdot \zeta)$$

where x_e is the last row of x.

In a similar way, for $e \geq l$, we define the unipotent subgroup $V_{6l,e}$ to be the unipotent radical of the parabolic subgroup of the split orthogonal group SO_{6l} whose Levi part is $\mathrm{GL}_1^e \times \mathrm{SO}_{6l-2e}$. Given a vector $\zeta \in F^{6l-2e}$ with nonzero length we define the character $\psi^{\zeta}_{V_{6l,e}}$ of $V_{6l,e}$.

Given a representation ϵ of $\mathrm{SO}_{6r+1}(F)$ we define a functional $L_{e,\zeta}$ on the space of ϵ with quasi-invariance property that

$$L_{e,\zeta}(v\eta) = \psi^{\zeta}_{V_{6r+1,e}}(v)L_{e,\zeta}(\eta)$$

for all $v \in V_{6r+1,e}$ and all η in the space of ϵ. Similarly, we define a functional $L_{e,\zeta}$ of $V_{6l,e}$ on a representation ϵ of SO_{6l}. We now prove

Lemma 3.5. With the notations defined as above, the following statements hold.

(1) For all $e \geq r+1$, the representation θ_{τ_ν} of SO_{6r+1} has no nonzero functional $L_{e,\zeta}$ for all $\zeta \in F^{6r-2e+1}$ with nonzero length.

(2) For all $e \geq l$, the representation θ_{σ_ν} of SO_{6l} has no nonzero functional $L_{e,\zeta}$ for all $\zeta \in F^{6l-2e}$ with nonzero length.

Proof. The proof is the same in both cases and is done as in the proof of Lemma 2 in [**GRS03**]. We just indicate the steps needed for the proof. From Lemma 3.4 it is enough to show that

$$\mathrm{Ind}_{P_{3^r}(F)}^{\mathrm{SO}_{6r+1}(F)}(\chi_1 \circ \det \otimes \cdots \otimes \chi_r \circ \det)$$

has no nonzero functional $L_{e,\zeta}$. Using Mackey theory it is enough to prove that if $\gamma \in P_{3^r}\backslash \mathrm{SO}_{6r+1}/V_{6r+1,e}$ then there exists a $v \in V_{6r+1,e}$ such that $\gamma v\gamma^{-1} \in P_{3^r}$ and $\psi^{\zeta}_{V_{6r+1,e}}(v) \neq 1$. By the Bruhat theory every such γ has the form $\gamma = wu_w$ where w is a Weyl element of SO_{6r+1} and u_w is a certain upper unipotent matrix. At first we show that we can get "rid" of u_w and consider only the cases where $\gamma = w$. Then as in the proof of Lemma 2 in **[GRS03]** we show that given any Weyl element w we can find a one-dimensional unipotent subgroup generated by $v \in V_{6r+1,e}$ such that $wvw^{-1} \in P_{3^r}$ and $\psi^{\zeta}_{V_{6r+1,e}}(v) \neq 1$. This finishes the proof. □

With the preparation of the local results above, we return to the global case. Let ϵ be an automorphic representation of $\mathrm{SO}'_{6r+1}(\mathbb{A})$. We define the Fourier coefficient

$$\int_{V_{6r+1,e}(k)\backslash V_{6r+1,e}(\mathbb{A})} \phi_\epsilon(vg)\psi^{\zeta}_{V_{6r+1,e}}(v)dv. \tag{3.17}$$

Similarly if ϵ is an automorphic representation on SO'_{6l} we define the Fourier coefficient

$$\int_{V_{6l,e}(k)\backslash V_{6l,e}(\mathbb{A})} \phi_\epsilon(vg)\psi^{\zeta}_{V_{6l,e}}(v)dv. \tag{3.18}$$

As a consequence of Lemma 3.5, we prove

Proposition 3.6. With the notations defined as above the following statements hold.

(1) For all $e \geq r+1$ the residual representation $\mathcal{E}_{\pi(\tau)\otimes\tau'}(g)$ of $\mathrm{SO}'_{6r+1}(\mathbb{A})$ has no nontrivial Fourier coefficient as defined by (3.17) for all $\zeta \in F^{6r-2e+1}$ with nonzero length.

(2) For all $e \geq l$ the residual representation $\mathcal{E}_{\pi(\sigma)\otimes\sigma'}(h)$ of $\mathrm{SO}'_{6l}(\mathbb{A})$ has no nontrivial Fourier coefficient as defined by (3.18) for all $\zeta \in F^{6l-2e}$ with nonzero length. Here $\mathcal{E}_{\pi(\sigma)\otimes\sigma'}(h)$ is the residue at $s = 1$ of the Eisenstein series $E(h, s, \phi_{\pi(\sigma)\otimes\sigma'})$ of $\mathrm{SO}'_{6l}(\mathbb{A})$.

3.3. Regularization of Periods. We study the periods of the residues of Eisenstein series defined below. If $r \geq l$, we consider

$$\begin{aligned} &\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \mathcal{E}_{\pi(\tau)\otimes\sigma'}) \qquad (3.19)\\ := &\int_{[\mathrm{SO}'_{4r+2l}]} \mathcal{E}_{\pi(\tau)\otimes\sigma'}(h) \int_{[V_{6r+1,r-l}]} \mathcal{E}_{\pi(\tau)\otimes\tau'}(vh)\psi_{V_{6r+1,r-l}}(v)dvdh \end{aligned}$$

where $[H] := H(k)\backslash H(\mathbb{A})$ for any algebraic k-group H. If $r < l$, we consider

$$\begin{aligned} &\mathcal{P}_{4r+2l,6r+1}(\mathcal{E}_{\pi(\tau)\otimes\sigma'},\mathcal{E}_{\pi(\tau)\otimes\tau'}) \qquad (3.20)\\ := &\int_{[\mathrm{SO}'_{6r+1}]}\mathcal{E}_{\pi(\tau)\otimes\tau'}(g)\int_{[V_{4r+2l,l-r-1}]}\mathcal{E}_{\pi(\tau)\otimes\sigma'}(vg)\psi_{V_{4r+2l,l-r-1}}(v)dvdg. \end{aligned}$$

The convergence of these periods needs to be justified through Arthur's truncation method. We shall study the case $r \geq l$ in complete detail, while the study for case $l > r$ is similar, and the details will be omitted.

We recall that elements of $V_{6r+1,r-l}$ can be written as

$$v(n,x,z) = \begin{pmatrix} n & x & z \\ & I_{4r+2l+1} & x^* \\ & & n^* \end{pmatrix} \in \mathrm{SO}_{6r+1}. \qquad (3.21)$$

Here $n \in N_{r-l}$, the standard maximal unipotent subgroup of GL_{r-l}, and x is a $(r-l)\times(4r+2l+1)$-matrix. Take $\zeta = (0_{2r+l}, a, 0_{2r+l}) \in k^{4r+2l+1}$ with $a \neq 0$. The character $\psi^{\zeta}_{V_{6r+1,r-l}}$ can be expressed as

$$\psi^{\zeta}_{V_{6r+1,r-l}}(v(n,x,z)) = \psi(n_{1,2}+\cdots+n_{r-l-1,r-l}+ax_{r-l,2r+l+1}). \qquad (3.22)$$

The period $\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'},\mathcal{E}_{\pi(\tau)\otimes\sigma'})$ in (3.19) is

$$\int_{[\mathrm{SO}'_{4r+2l}]}\mathcal{E}_{\pi(\tau)\otimes\sigma'}(h)\int_{[V_{6r+1,r-l}]}\mathcal{E}_{\pi(\tau)\otimes\tau'}(vh)\psi^{\zeta}_{V_{6r+1,r-l}}(v)dvdh. \qquad (3.23)$$

As in Section 4.1 of **[GJR03b]**, we shall regularize the above period by truncating the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma'}(h)$.

In the following we denote by $P = MU = P_{2r,2l} = M_{2r,2l}U_{2r,2l}$, the maximal parabolic subgroup of $G := \mathrm{SO}'_{4r+2l}$. We identify $\mathfrak{a}_P$ with $\mathbb{R}$. Then a regular $T \in \mathfrak{a}_P$ will correspond to a real number $c \in \mathbb{R}_{>1}$, where we denote by $\mathbb{R}_{>c}$ the set of all real numbers greater than c. We denote

$$H(g) := \exp < 1, H_P(g) >= |\det m(g)| \qquad (3.24)$$

for $g = um(g)\mathrm{k} \in \mathrm{SO}_{4r+2l}(\mathbb{A})$ (Iwasawa decomposition given in §3.1). Let τ_c ($c \in \mathbb{R}_{>1}$) be the characteristic function of the subset $\mathbb{R}_{>c}$.

Following **[Arthur78]** and **[Arthur80]**, the truncation of the Eisenstein series $E(g,s,\phi)$ (where $\phi = \phi_{\pi(\tau)\otimes\sigma'}$) is defined as follows:

$$\Lambda_c E(g,s,\phi) = E(g,s,\phi) - \sum_{\gamma\in P(k)\backslash G(k)} E_P(\gamma g,s,\phi)\tau_c(H(\gamma g)). \qquad (3.25)$$

The constant term $E_P(g,s,\phi)$ of the Eisenstein series $E(g,s,\phi)$ along P can be expressed as (see (3.5) for definition)

$$E_P(g,s,\phi) = \Phi(g,s,\phi) + \mathcal{M}(w,s)(\Phi(\cdot,s,\phi))(g)$$

where $w = w_{2r,2l}$ and $\mathcal{M}(w,s)$ is the intertwining operator as described in §3.1. We remark that the summation in (3.25) has only finitely many terms and converges absolutely (Corollary 5.2, **[Arthur78]**). The truncated Eisenstein series can then be rewritten as follows

$$\begin{aligned}\Lambda_c E(g,s,\phi) &= \sum_{\gamma\in P(k)\backslash G(k)} \Phi(\gamma g,s,\phi)(1-\tau_c(H(\gamma g))) \\ &\quad - \sum_{\gamma\in P(k)\backslash G(k)} \mathcal{M}(w,s)(\Phi(\cdot,s,\phi))(\gamma g)\tau_c(H(\gamma g)) \\ &:= \mathcal{E}_1(g)-\mathcal{E}_2(g). \qquad (3.26)\end{aligned}$$

Let s_0 be a positive real number. Assume that the Eisenstein series $E(g,s,\phi)$ has a simple pole at $s=s_0$. We denote by $E_{s_0}(g,\phi)$ the non-zero residue of $E(g,s,\phi)$. Then we have

$$\begin{aligned}\Lambda_c E_{s_0}(g,\phi) &= E_{s_0}(g,\phi) - \sum_{\gamma\in P(k)\backslash G(k)} \mathcal{M}(w,s)(\Phi(\cdot,s,\phi))_{s_0}(\gamma g)\tau_c(H(\gamma g)) \\ &:= E_{s_0}(g,\phi)-\mathcal{E}_3(g). \qquad (3.27)\end{aligned}$$

Following the argument in **[GJR03b]**, §4.1, we obtain from (3.26) and (3.27) the formula for the period $\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, E_{s_0}(g,\phi))$:

$$\begin{aligned}&\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, E_{s_0}(\cdot,\phi)) \\ =\ &\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \mathcal{E}_3) + \mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \Lambda_c E_{s_0}(\cdot,\phi)) \\ =\ &\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \mathcal{E}_3) \\ &+[\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \mathcal{E}_1) - \mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \mathcal{E}_2)]_{s_0}.\end{aligned}$$

The following Proposition will be proved in the next subsection.

Proposition 3.7. For $i=1,2$, the following periods

$$\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \mathcal{E}_i)$$

converge absolutely for $Re(s)$ large and have meromorphic continuation to the whole complex plane.

Since both $\Lambda_c E(g,s,\phi)$ and $\Lambda_c E_{s_0}(g,\phi)$ rapidly decay in the usual sense, the periods

$$\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \Lambda_c E(\cdot,s,\phi))$$

and

$$\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \Lambda_c E_{s_0}(\cdot,\phi))$$

converge absolutely. By Proposition 3.7, both periods $\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \mathcal{E}_1)$ and $\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \mathcal{E}_2)$ converge absolutely for $Re(s)$ large and have meromorphic continuation to the complex plane. Hence we have

$$\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'},\mathcal{E}_2)_{s_0}=\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'},\mathcal{E}_3).$$

It follows that

$$\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'},E_{s_0}(\cdot,\phi))=\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'},\mathcal{E}_1)_{s_0}. \tag{3.28}$$

3.4. Period Identity. We shall prove Proposition 3.7 through the following explicit calculation.

First, the convergence for the real part of s large follows as in the proof of Proposition 3.1, **[GJR01]**. The meromorphic continuation of both periods will follow from the explicit calculation.

We shall investigate the period $\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'},\mathcal{E}_1)$ in detail, while the conclusion for the period $\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'},\mathcal{E}_2)$ follows from the same argument. By (3.23) and (3.28), the period $\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'},\mathcal{E}_1)$ is given by the following integral

$$\int_{[\mathrm{SO}'_{4r+2l}]}\mathcal{E}_1(g)\int_{[V_{6r+1,r-l}]}\mathcal{E}_{\pi(\tau)\otimes\tau'}(vg)\psi_{V_{6r+1,r-l}}(v)dvdg. \tag{3.29}$$

where

$$\mathcal{E}_1(g)=\sum_{\gamma\in P(k)\backslash G(k)}\Phi(\gamma g,s,\phi)(1-\tau_c(H(\gamma g))).$$

In the following we set

$$\Phi^c(g,s):=\Phi(g,s,\phi)(1-\tau_c(H(g))).$$

From the above, we know that (3.29) is absolutely convergent when the real part of s is large. Unfolding the Eisenstein series we obtain

$$\int_{P(k)\backslash\mathrm{SO}'_{4r+2l}(\mathbb{A})}\Phi^c(g)\int_{[V_{6r+1,r-l}]}\mathcal{E}_{\pi(\tau)\otimes\tau'}(vg)\psi_{V_{6r+1,r-l}}(v)dvdg. \tag{3.30}$$

Recall that U is the unipotent radical of the parabolic subgroup P. Factoring the measure $U(k)\backslash U(\mathbb{A})$, we obtain the following integral as an inner integration in (3.30)

$$\int_{[U]}\int_{[V_{6r+1,r-l}]}\mathcal{E}_{\pi(\tau)\otimes\tau'}(vug)\psi_{V_{6r+1,r-l}}(v)dvdu. \tag{3.31}$$

Define a Weyl element w of SO'_{6r+1} by

$$w = \begin{pmatrix} 0 & I_r & & & \\ I_{r-l} & 0 & & & \\ & & I_{2r+2l+1} & & \\ & & & 0 & I_{r-l} \\ & & & I_r & 0 \end{pmatrix}. \tag{3.32}$$

By the embedding of SO'_{4r+2l} into SO'_{6r+1}, we have

$$U = \{u = \begin{pmatrix} I_{r-l} & & & & \\ & I_{2r} & u_1 & u_2 & \\ & & I_{2l+1} & u_1^* & \\ & & & I_{2r} & \\ & & & & I_{r-l} \end{pmatrix} \in \mathrm{SO}'_{6r+1}\}. \tag{3.33}$$

Since U is the unipotent radical of the parabolic subgroup $P_{2r,2l}$ (of SO'_{4r+2l}), when embedded in SO'_{6r+1}, we write $u_1 \in \mathrm{Mat}_{2r\times 2l}$ as a matrix in $\mathrm{Mat}_{(2r)\times(2l+1)}$ according to the standard embedding of SO'_{2l} and SO'_{2l+1}. Here $u_2 \in \{X \in \mathrm{Mat}_{2r\times 2r} : {}^tXJ_{2r} + J_{2r}X = 0\}$. Also, we write elements in the unipotent subgroup $V_{6r+1,r-l}$ as

$$v = v(n,p,x_2,x_3,z) := \begin{pmatrix} n & p & x_2 & x_3 & z \\ & I_{2r} & 0 & 0 & x_3^* \\ & & I_{2l+1} & 0 & x_2^* \\ & & & I_{2r} & p^* \\ & & & & n^* \end{pmatrix}. \tag{3.34}$$

where $n \in N_{r-l}$, the unipotent radical consisting of all upper unipotent matrices in GL_{r-l}. Also $p, x_3 \in \mathrm{Mat}_{(r-l)\times 2r}, x_2 \in \mathrm{Mat}_{(r-l)\times(2l+1)}$ and $z \in \{X \in \mathrm{Mat}_{(r-l)\times(r-l)} : {}^tXJ_{r-l} + J_{r-l}X = 0\}$. We write

$$wv(n,p,x_2,x_3,z)u(u_1,u_2)w^{-1} = \ell_1(u_1,x_3,u_2)v'\hat{p} \tag{3.35}$$

where

$$\ell_1(u_1,x_3,u_2) := \begin{pmatrix} I_{2r} & 0 & u_1 & x_3 & u_2 \\ & I_{r-l} & 0 & 0 & x_3^* \\ & & I_{2l+1} & 0 & u_1^* \\ & & & I_{r-l} & 0 \\ & & & & I_{2r} \end{pmatrix}, \tag{3.36}$$

$v' \in V_{2r+1,r-l}$, the embedding of which in SO'_{6r+1} is

$$v' = \begin{pmatrix} I_{2r} & 0 & 0 & 0 & 0 \\ & n & x_2 & z & 0 \\ & & I_{2l+1} & x_2^* & 0 \\ & & & n^* & 0 \\ & & & & I_{2r} \end{pmatrix} \tag{3.37}$$

and

$$\hat{p} = \begin{pmatrix} I_{2r} & & & & \\ p & I_{r-l} & & & \\ & & I_{2l+1} & & \\ & & & I_{r-l} & \\ & & & p^* & I_{2r} \end{pmatrix}. \tag{3.38}$$

After conjugating in the above integral by w from left to right, we obtain

$$\int_{[U]} \int_{[V_{6r+1,r-l}]} \mathcal{E}_{\pi(\tau)\otimes\tau'}(\ell_1(u_1, x_3, u_2)v'\hat{p}wg)\psi^{\zeta}_{V_{6r+1,r-l}}(v)dvdu. \tag{3.39}$$

Notice that from the definition of ζ, then after the conjugation by w the character $\psi^{\zeta}_{V_{6r+1,r-l}}$ agrees with $\psi^{\zeta}_{V_{2r+1,r-l}}$.

Let L_1 denote the subgroup of SO'_{6r+1} which consists of the matrices $\ell_1(u_1, x_3, u_2)$ as defined in (3.36), where, as explained above, $u_1 \in \mathrm{Mat}_{2r\times 2l}$. Let

$$L_2 = \{\ell_2(x_4, x_3, u_2) = \begin{pmatrix} I_{2r} & 0 & x_4 & x_3 & u_2 \\ & I_{r-l} & 0 & 0 & x_3^* \\ & & I_{2l+1} & 0 & x_4^* \\ & & & I_{r-l} & 0 \\ & & & & I_{2r} \end{pmatrix} \in \mathrm{SO}'_{6r+1}\}$$

where $x_4 \in \mathrm{Mat}_{2r\times(2l+1)}$ and x_3 and u_2 are as above. Then L_2/L_1 can be identified with a k-vector space of column vectors of size $2r$. In the above integral we expand along this column vector L_2/L_1 with points in $k\backslash\mathbb{A}$. We obtain

$$\sum_{\epsilon_i \in k} \int \mathcal{E}_{\pi(\tau)\otimes\tau'}(\ell_2((t, u_1), x_3, u_2)v'\hat{p}wg)\psi^{\zeta}_{V_{2r+1,r-l}}(v')\psi(\sum_{i=1}^{2r} \epsilon_i t_i)d\ell_2 dv' d\hat{p}$$

where t is a column vector whose transpose is $(t_1, \cdots, t_{2r})$ and by (t, u_1) we indicate that we added the integration of t over $(k\backslash\mathbb{A})^{2r}$ to the integration over variable u_1. The integration over all other variables are the same as before.

Let α be a row vector of size $2r$ embedded in SO'_{6r+1} as the last row of p in $\hat{p}$. Then one can find such a row $\alpha \in k^{2n}$, depending on ϵ_i and ξ such that after we conjugate it from the left to right in the above integral, change variables in v and collapse the summation over $\epsilon_i \in k$ with the adelic integration over

$(k\backslash\mathbb{A})^{2r}$ which consists of the last row of p, we obtain

$$\int_{\hat{p}_1,\hat{p}_2}\int_{[L_2]\times[V_{2r+1,r-l}]}\mathcal{E}_{\pi(\tau)\otimes\tau'}(\ell_2(x_4,x_3,u_2)v'\hat{p}_1\hat{p}_2wg)\psi^{\zeta}_{V_{2r+1,r-l}}(v')d\ell_2dv'd\hat{p}_1d\hat{p}_2, \tag{3.40}$$

where variable x_4 is in $\mathrm{Mat}_{2r\times(2l+1)}(k)\backslash\mathrm{Mat}_{2r\times(2l+1)}(\mathbb{A})$. Note that $\hat{p}$ is as defined in (3.38). In (3.40), variable p_2 is the last row of p integrated over $\mathbb{A}^{2r}$, and variable p_1 consists of all matrices in p with bottom row being zero and is integrated over points in $\mathrm{Mat}_{(r-l-1)\times(2r)}(k)\backslash\mathrm{Mat}_{(r-l-1)\times(2r)}(\mathbb{A})$. Let

$$L_3=\{\ell_3(y)=\begin{pmatrix}I_{2r}&y&&&\\&I_{r-l}&&&\\&&I_{2l+1}&&\\&&&I_{r-l}&y^*\\&&&&I_{2r}\end{pmatrix}\in\mathrm{SO}'_{6r+1}\} \tag{3.41}$$

where $y\in\mathrm{Mat}_{2r\times(r-l)}$ such that its first column is zero. Taking the Fourier expansion over $L_3(k)\backslash L_3(\mathbb{A})$ of integral (3.40) and using the integration over $\hat{p}_1$ as was done in $\hat{p}_2$, we deduce that integral (3.40) equals the following integral

$$\int_{\hat{p}}\int_{[L_3\times L_2\times V_{2r+1,r-l}]}\mathcal{E}_{\pi(\tau)\otimes\tau'}(\ell_3(y)\ell_2(x_4,x_3,u_2)v'\hat{p}wg)\psi^{\zeta}_{V_{2r+1,r-l}}(v')dyd\ell_2dv'd\hat{p}, \tag{3.42}$$

where $\hat{p}$ is integrated over $\mathrm{Mat}_{(r-l)\times 2r}(\mathbb{A})$.

Let

$$L_4=\{\ell_4(y)=\begin{pmatrix}I_{2r}&y&&&\\&I_{r-l}&&&\\&&I_{2l+1}&&\\&&&I_{r-l}&y^*\\&&&&I_{2r}\end{pmatrix}\ :\ y\in\mathrm{Mat}_{2r\times(r-l)}\}. \tag{3.43}$$

Then L_4/L_3 is a column vector which can be identified as the first column of the above matrix y. We take Fourier expansion of integral (3.42) along $(L_4/L_3)(k)\backslash(L_4/L_3)(\mathbb{A})$. Let the group GL_{2r} act on this column (L_4/L_3) via the adjoint action, which has two orbits. Here GL_{2r} is embedded in SO'_{6r+1} via the Levi subgroup $\mathrm{GL}_{2r}\times\mathrm{SO}'_{2r+1}$.

The trivial orbit will contribute

$$\int_{\hat{p}}\int_{[U_{2r,2r+1}\times V_{2r+1,r-l}]}\mathcal{E}_{\pi(\tau)\otimes\tau'}(uv'\hat{p}wg)\psi^{\zeta}_{V_{2r+1,r-l}}(v')dudv'd\hat{p}, \tag{3.44}$$

where $U_{2r,2r+1}$ is the unipotent radical of the maximal parabolic subgroup $P_{2r,2r+1}$ of SO'_{6r+1} with Levi subgroup $\mathrm{GL}_{2r}\times\mathrm{SO}'_{2r+1}$. We may write elements in $\mathrm{GL}_{2r} \times \mathrm{SO}'_{2r+1}$ as $m(a,b)$ and elements in $U_{2r,2r+1}$ as

$$u = u(y, z') = \begin{pmatrix} I_{2r} & y & z' \\ & I_{2r+1} & y^* \\ & & I_{2r} \end{pmatrix}. \tag{3.45}$$

The nontrivial orbit will contribute

$$\sum_{\gamma \in P^0_{2r}(k)\backslash \mathrm{GL}_{2r}(k)} \int_{\hat{p}} \int_{(u,v')} \mathcal{E}_{\pi(\tau)\otimes\tau'}(uv'\hat{\gamma}\hat{p}wg)\psi^{\zeta}_{V_{2r+1,r-l}}(v')\psi(y_{2r,2r+1})dudv'd\hat{p}, \tag{3.46}$$

where (u, v') is in $[U_{2r,2r+1} \times V_{2r+1,r-l}]$, and $\hat{\gamma} = m(\gamma, I_{2r+1})$ is in $\mathrm{GL}_{2r}(k) \times \mathrm{SO}'_{2r+1}(k)$. We denote by P^0_{2r} is the subgroup of GL_{2r} which fixes the character $\psi(y_{2r,2r+1})$.

From (3.37) and (3.45) an element in $U_{2r,2r+1} \times V_{2r+1,r-l}$ can be written as

$$uv' = \begin{pmatrix} I_{2r} & y_1 & y_2 & y_3 & z' \\ & n & y_4 & z & y_3^* \\ & & I_{2l+1} & y_4^* & y_2^* \\ & & & n^* & y_1^* \\ & & & & I_{2r} \end{pmatrix}. \tag{3.47}$$

The product of characters, $\psi^{\zeta}_{V_{2r+1,r-l}}(v')\psi(y_{2r,2r+1})$ equals

$$\psi((y_1)_{2r,1} + n_{1,2} + \cdots + n_{r-l-1,r-l} + a(y_4)_{r-l,l+1}), \tag{3.48}$$

according to (3.22). We are going to show that (3.46) is identically zero. To this end, we consider the Fourier expansion of (3.46) along $N_{2r}(k)\backslash N_{2r}(\mathbb{A})$, where N_{2r} is the maximal unipotent subgroup of GL_{2r} in the Levi subgroup $\mathrm{GL}_{2r} \times \mathrm{SO}'_{2r+1}$. Note that the product of N_{2r} and $U_{2r,2r+1} \times V_{2r+1,r-l}$ is equal to $V_{6r+1,3r-l}$ (see (3.16)), whose elements has form

$$v = \begin{pmatrix} n & x & z \\ & I_{2l+1} & x^* \\ & & n^* \end{pmatrix} \in \mathrm{SO}'_{6r+1}. \tag{3.49}$$

We consider the Fourier expansion, column by column, and obtain

$$\sum_{\gamma \in N_{2r}(k)\backslash \mathrm{GL}_{2r}(k)} \int_{[V_{6r+1,3r-l}]} \mathcal{E}_{\pi(\tau)\otimes\tau'}(v\hat{\gamma}\hat{p}wg)\psi^{\zeta}_{V_{6r+1,3r-l},\epsilon_1,\cdots,\epsilon_{2r-1}}(v)dvd\hat{p}, \tag{3.50}$$

where the character $\psi^{\zeta}_{V_{6r+1,3r-l},\epsilon_1,\cdots,\epsilon_{2r-1}}(v)$ is given by

$$\psi(\epsilon_1 n_{1,2} + \cdots + \epsilon_{2r-1}n_{2r-1,2r} + n_{2r,2r+1} + \cdots + n_{3r-l-1,3r-l} + ax_{3r-l,l+1}),$$

with $\epsilon_i \in \{0, 1\}$ and $a \in k^{\times}$. This is compatible with (3.22).

If there is an i such that $\epsilon_i = 0$ then the integral in (3.50) has an inner integration, which represents the constant term of $\mathcal{E}_{\pi(\tau)\otimes\tau'}$ with respect to the unipotent radical of a standard maximal parabolic subgroup not equal to $P_{2r,2r+1}$. By the cuspidal support of $\mathcal{E}_{\pi(\tau)\otimes\tau'}$, such a constant term must be zero and hence the integral in (3.50) with $\epsilon_i = 0$ must vanish. We are left with the integral in (3.50) having all $\epsilon_i = 1$. This integral represents the Fourier coefficient of $\mathcal{E}_{\pi(\tau)\otimes\tau'}$ as given in (3.17) with $e = 3r - l > r + 1$. By Part (1) of Proposition 3.6, this last integral in (3.50) must also vanish. Therefore, we prove that (3.50) is identically zero. In other words, we show that (3.31) is equal to (3.44).

In (3.44) the integration over $U_{2r,2r+1}(k)\backslash U_{2r,2r+1}(\mathbb{A})$ is the constant term of $\mathcal{E}_{\pi(\tau)\otimes\tau'}$. It follows that

$$\int_{[U_{2r,2r+1}]} \mathcal{E}_{\pi(\tau)\otimes\tau'}(ug)du = \phi_{\pi(\tau)\otimes\tau'}(g) \exp < -1 + \rho_{P_{2r,2r+1}}, H_{P_{2r,2r+1}}(g) > .$$

It is easy to check that

$$\exp < -1 + \rho_{P_{2r,2r+1}}, H_{P_{2r,2r+1}}(g) >= |\det a|^{2r-1}$$

if we write $g = um(a,b)\mathrm{k}$ using the Iwasawa decomposition

$$P_{2r,2r+1}(\mathbb{A})\mathrm{K}.$$

Hence (3.44) is equal to

$$\int_{\hat{p}} \int_{[V_{2r+1,r-l}]} |\det a(wg)|^{2r-1} \phi_{\pi(\tau)\otimes\tau'}(v'\hat{p}wg)\psi^{\zeta}_{V_{2r+1,r-l}}(v')dv'd\hat{p}, \qquad (3.51)$$

We now return to to the explicit calculation of $\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_1)$, the period defined in (3.29). In (3.30) together with (3.31), the integration over the variable g can factor through the Iwasawa decomposition, i.e.

$$\int_{M_{2r,2l}(k)U_{2r,2l}(\mathbb{A})\backslash \mathrm{SO}'_{4r+2l}(\mathbb{A})} = \int_{\mathrm{K}} \int_{[\mathrm{GL}_{2r}]\times[\mathrm{SO}'_{2r+1}]} .$$

By plugging this and (3.44) in (3.30) and changing variable in $\hat{p}$, we obtain

$$\int_{\mathrm{K}} \int_{\hat{p}} \int_{[\mathrm{GL}_{2r}\times \mathrm{SO}'_{2l}]} \Phi^c(m(a,b)\mathrm{k})|\det a|^{-(r+l)} \qquad (3.52)$$
$$\int_{[V_{2r+1,r-l}]} \phi_{\pi(\tau)\otimes\tau'}(v'm(a,b)\hat{p}w\mathrm{k})\psi^{\zeta}_{V_{2r+1,r-l}}(v')dv'd\hat{p}dmd\mathrm{k}.$$

Recall that

$$\begin{aligned}\Phi^c(m(a,b)\mathrm{k}, s) &= \Phi(m(a,b)\mathrm{k}, s, \phi)(1 - \tau_c(H(a))) \\ &= \phi_{\pi(\tau)\otimes\sigma'}(m(a,b)\mathrm{k})|\det a|^{s+\frac{2r+2l-1}{2}}(1 - \tau_c(H(a))).\end{aligned}$$

Then (3.52) can be re-written as

$$\int_{\mathrm{K}}\int_{\hat{p}}\int_{[\mathrm{GL}_{2r}]}|\det a|^{s-\frac{1}{2}}(1-\tau_c(H(a)))\int_{[\mathrm{SO}'_{2l}]}\phi_{\pi(\tau)\otimes\sigma'}(m(a,b)\mathrm{k}) \quad (3.53)$$
$$\int_{[V_{2r+1,r-l}]}\phi_{\pi(\tau)\otimes\tau'}(v'm(a,b)\hat{p}w\mathrm{k})\psi^{\zeta}_{V_{2r+1,r-l}}(v')dv'dmd\hat{p}d\mathrm{k}.$$

In (3.53) we set

$$\mathcal{F}^{\psi^{\zeta}}(\phi_{\pi(\tau)\otimes\tau'})(g) := \int_{[V_{2r+1,r-l}]}\phi_{\pi(\tau)\otimes\tau'}(v'g)\psi^{\zeta}_{V_{2r+1,r-l}}(v')dv'. \quad (3.54)$$

We consider the Langlands decomposition for $\mathrm{GL}_{2r}(\mathbb{A})$

$$\mathrm{GL}_{2r}(\mathbb{A}) = \mathrm{GL}_{2r}(\mathbb{A})^1\cdot A^+,$$

and set $\mathrm{PM}_{2r,2l} = (Z_{\mathrm{GL}_{2r}}(\mathbb{A})\mathrm{GL}_{2r}(k)\backslash\mathrm{GL}_{2r}(\mathbb{A}))\times(\mathrm{SO}'_{2l}(k)\backslash\mathrm{SO}'_{2l}(\mathbb{A}))$. Let d be the degree of the number field k over $\mathbb{Q}$. Then (3.53) equals

$$\mathrm{vol}(\mathbb{A}^1/k^{\times})\int_{\mathbb{R}^+}|t|^{2rd(s-\frac{1}{2})}(1-\tau_c(t))dt^{\times}\int_{\mathrm{K}}\int_{\hat{p}} \quad (3.55)$$
$$\int_{\mathrm{PM}_{2r,2l}}\phi_{\pi(\tau)\otimes\sigma'}(m\mathrm{k})\mathcal{F}^{\psi^{\zeta}}(\phi_{\pi(\tau)\otimes\tau'})(m\hat{p}w\mathrm{k})dmd\hat{p}d\mathrm{k}.$$

It is easy to see that

$$\int_{\mathbb{R}^+}|t|^{2rd(s-\frac{1}{2})}(1-\tau_c(t))dt^{\times} = \frac{c^{2rd(s-\frac{1}{2})}}{2rd(s-\frac{1}{2})}$$

which has a simple pole at $s=\frac{1}{2}$. In (3.54) we are left with

$$\int_{\mathrm{K}\times\mathrm{Mat}_{r-l,2r}(\mathbb{A})\times\mathrm{PM}_{2r,2l}}\phi_{\pi(\tau)\otimes\sigma'}(m\mathrm{k})\mathcal{F}^{\psi^{\zeta}}(\phi_{\pi(\tau)\otimes\tau'})(m\hat{p}w\mathrm{k})dmd\hat{p}d\mathrm{k} \quad (3.56)$$

which is holomorphic in s. Hence we obtain from the above explicit calculation and (3.28) the main identity which relates the 'outer' period to the 'inner' period.

Theorem 3.8. When $s\neq\frac{1}{2}$, the period

$$\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, E_s(\cdot,\phi_{\pi(\tau)\otimes\sigma'}))$$

is identically zero. When $s=\frac{1}{2}$, the period

$$\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \mathcal{E}_{\pi(\tau)\otimes\sigma'})$$

is equal to

$$\frac{\mathrm{vol}(\mathbb{A}^1/k^{\times})}{2rd}\int_{\mathrm{K}\times\mathrm{Mat}_{r-l,2r}(\mathbb{A})\times\mathrm{PM}_{2r,2l}}\phi_{\pi(\tau)\otimes\sigma'}(m\mathrm{k})\mathcal{F}^{\psi^{\zeta}}(\phi_{\pi(\tau)\otimes\tau'})(m\hat{p}w\mathrm{k})dmd\hat{p}d\mathrm{k}$$

where $\mathrm{PM}_{2r,2l} = (Z_{\mathrm{GL}_{2r}}(\mathbb{A})\mathrm{GL}_{2r}(k)\backslash\mathrm{GL}_{2r}(\mathbb{A})) \times (\mathrm{SO}'_{2l}(k)\backslash\mathrm{SO}'_{2l}(\mathbb{A}))$.

Remark 3.9. From formulas (3.55) and (3.56), the period

$$\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \mathcal{E}_1)$$

has meromorphic continuation to the whole complex plane.

This completes the proof of Proposition 3.7 for this period. The statement for the period

$$\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau'}, \mathcal{E}_2)$$

follows from the same argument (as in **[GJR01]**). Hence the proof for Proposition 3.7 is now completed.

Remark 3.10. We can use the same argument to prove Theorem 3.8 for the case of $l > r$. In this case, we can also use the following 'outer' period

$$\int_{[\mathrm{SO}'_{4l+2r+1}]} \mathcal{E}_{\pi(\sigma)\otimes\tau'}(g) \int_{[V_{6l,l-r-1}]} \mathcal{E}_{\pi(\sigma)\otimes\sigma'}(vg)\psi_{V_{6l,,l-r-1}}(v)dvdg$$

where $\pi(\sigma)$ is the Langlands transfer of σ to GL_{2l}. We omit the details.

4. The Proof of Theorem A

In this section we prove Theorem A which gives a sufficient condition for $L(\frac{1}{2}, \pi_1 \times \pi_2)$ to be non-zero.

Let π_1 be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{2r}(\mathbb{A})$ such that the exterior square L-function $L(s, \pi_1, \Lambda^2)$ has a pole at $s = 1$, and let τ be an irreducible unitary generic cuspidal automorphic representation of $\mathrm{SO}_{2r+1}(\mathbb{A})$ which has a weak lift to π_1. Let π_2 be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{2l}(\mathbb{A})$ with the properties that the symmetric square L-function $L(s, \pi_2, \mathrm{Sym}^2)$ has a pole at $s = 1$ and let σ be an irreducible unitary generic cuspidal automorphic representation of $\mathrm{SO}_{2l}(\mathbb{A})$ which has a weak lift to π_2. Note that the existence of τ and σ is established by the automorphic descent method ([**GRS99c**], [**GRS01**], and [**S**]). By the work of [**CKPSS03**], [**JS03a**] and [**JS03b**], such weak lifts are Langlands functorial.

We first discuss the cases more general than what Theorem A asserts. Let τ' be an irreducible unitary generic cuspidal automorphic representation of $\mathrm{SO}'_{2r+1}(\mathbb{A})$ (an inner form of SO_{2r+1}), which is nearly equivalent to τ, and σ' be an irreducible unitary generic cuspidal automorphic representation of $\mathrm{SO}'_{2l}(\mathbb{A})$ (an inner form of SO_{2l}) which is nearly equivalent to σ.

Proposition 4.1. If there exists a pair (τ, σ') as above such that the period

$$\mathcal{P}_{2r+1,2l}(\phi_\tau, \phi_{\sigma'}) \ (r \geq l) \quad \textit{or} \quad \mathcal{P}_{2l,2r+1}(\phi_{\sigma'}, \phi_\tau) \ (r < l)$$

does not vanish for some given $\phi_{\sigma'} \in \sigma'$ and $\phi_\tau \in \tau$, then the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma'}$ at $s = \frac{1}{2}$ of the Eisenstein series $E(g, s, \phi_{\pi(\tau)\otimes\sigma'})$ does not vanish.

We shall prove this Theorem by using Theorem 3.8 for the case $r \geq l$. The case $r < l$ follows by a similar argument. Recall from (2.5) that the period $\mathcal{P}_{2r+1,2l}(\phi_\tau, \phi_{\sigma'})$ is defined by

$$\mathcal{P}_{2r+1,2l}(\phi_\tau, \phi_{\sigma'}) = \int_{[\mathrm{SO}'_{2l}]} \phi_{\sigma'}(g) \int_{[V_{2r+1,r-l}]} \phi_\tau(vg) \psi^{\zeta}_{V_{2r+1,r-l}}(v) dv dg. \quad (4.1)$$

We consider the cuspidal data $(P_{2r,2l}, \pi(\tau) \otimes \sigma')$ and $(P_{2r,2r+1}, \pi(\tau) \otimes \tau)$ of $\mathrm{SO}_{4r+2l}(\mathbb{A})$ and $\mathrm{SO}_{6r+1}(\mathbb{A})$, respectively. Note that σ' may not be generic. We may assume that

$$\phi_{\pi(\tau)\otimes\sigma'} = \phi_{\pi(\tau)} \otimes \phi_{\sigma'} \quad (4.2)$$

$$\phi_{\pi(\tau)\otimes\tau} = \overline{\phi}_{\pi(\tau)} \otimes \phi_\tau \quad (4.3)$$

where $\overline{\phi}_{\pi(\tau)}$ is the complex conjugate of $\phi_{\pi(\tau)}$. Recall from (3.2) that

$$\begin{aligned} \Phi(g, s, \phi_{\pi(\tau)\otimes\sigma'}) &= \phi_{\pi(\tau)\otimes\sigma'}(g) \exp < s + \rho_{P_{2r,2l}}, H_{P_{2r,2l}}(g) > \\ \Phi(g, s, \phi_{\pi(\tau)\otimes\tau}) &= \phi_{\pi(\tau)\otimes\tau}(g) \exp < s + \rho_{P_{2r,2r+1}}, H_{P_{2r,2r+1}}(g) > . \end{aligned}$$

We need the following Proposition to finish the proof of Proposition 4.1.

Proposition 4.2. If the period

$$\mathcal{P}_{2r+1,2l}(\phi_\tau, \phi_{\sigma'})$$

does not vanish for some given $\phi_{\sigma'} \in \sigma'$ and $\phi_\tau \in \tau$, then integral (3.56)

$$\int_{\mathrm{K} \times \mathrm{Mat}_{r-l,2r}(\mathbb{A}) \times \mathrm{PM}_{2r,2l}} \phi_{\pi(\tau)\otimes\sigma'}(m\mathrm{k}) \mathcal{F}^{\psi^\zeta}(\phi_{\pi(\tau)\otimes\tau'})(m\hat{p}w\mathrm{k}) dm d\hat{p} d\mathrm{k}$$

does not vanish for the corresponding data defined in (4.2) and (4.3), and for $\phi_{\pi(\tau)} \in \pi(\tau)$, where $\mathcal{F}^{\psi^\zeta}(\phi_{\pi(\tau)\otimes\tau'})$ is defined as in (3.54).

Proof. The proof is the same as that of Proposition 5.3 in **[GJR03b]**. We omit the details here. □

By Proposition 4.2 and Theorem 3.8, if the period

$$\mathcal{P}_{2r+1,2l}(\phi_\tau, \phi_{\sigma'})$$

does not vanish for some given $\phi_{\sigma'} \in \sigma'$, $\phi_\tau \in \tau$, and $\phi_{\pi(\tau)} \in \pi(\tau)$, then the period

$$\mathcal{P}_{6r+1,4r+2l}(\mathcal{E}_{\pi(\tau)\otimes\tau}, \mathcal{E}_{\pi(\tau)\otimes\sigma'})$$

does not vanish for the corresponding data. In particular, this implies that the residue

$$\mathcal{E}_{\pi(\tau)\otimes\sigma'}$$

does not vanish for the given data. This proves Proposition 4.1.

It is worthwhile noting that Proposition 4.1 provides a characterization of the existence of residue in terms of a period of the cuspidal datum without assuming the genericity of the cuspidal support of the residue.

To link the existence of the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma'}$ with the nonvanishing of the central value $L(\frac{1}{2}, \pi(\tau) \times \sigma')$, we have to use Formula (3.9). The key point here is Proposition 3.2, which has been proved only when σ' is generic.

In the following we assume that $\sigma' = \sigma$ is generic. Then from Proposition 3.2 the product of L-functions

$$L(s, \pi(\tau) \times \sigma)L(2s, \pi(\tau), \Lambda^2)$$

has a simple pole at $s = \frac{1}{2}$. Since the exterior square L-function

$$L(s, \pi(\tau), \Lambda^2)$$

has a simple pole at $s = 1$, the product L-function

$$L(s, \pi(\tau) \times \sigma)$$

does not vanish at $s = \frac{1}{2}$. Since $\pi_2 = \pi(\sigma)$ and $\pi_1 = \pi(\tau)$, we obtain that the tensor product L-function

$$L(s, \pi_1 \times \pi_2)$$

does not vanish at $s = \frac{1}{2}$.

Since in Theorem A, both τ and σ are generic, the above argument gives a complete proof of Theorem A.

5. The Proof of Theorem B

In this section we prove Theorem B for the case $r \geq l = 1$. Let π_1 be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{2r}(\mathbb{A})$ such that the exterior square L-function $L(s, \pi_1, \Lambda^2)$ has a pole at $s = 1$ and let π_2 be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ such that the symmetric square L-function $L(s, \pi_2, \mathrm{Sym}^2)$ has a pole at $s = 1$. We assume that $\pi_1 = \pi(\tau)$ is a lift from an irreducible unitary generic cuspidal automorphic representation of $\mathrm{SO}_{2r+1}(\mathbb{A})$ and $\pi_2 = \pi(\sigma)$ is a lift from an irreducible unitary automorphic representation of $\mathrm{SO}_2(\mathbb{A})$ (here σ is a character $\mathrm{SO}_2(k)\backslash\mathrm{SO}_2(\mathbb{A})$). Let SO'_{2r+1} be an inner form of SO_{2r+1} and SO'_2 be an inner form of SO_2. Assume that $(\mathrm{SO}'_{2r+1}, \mathrm{SO}'_2)$ is a relevant pair.

Theorem 5.1 **($r \geq l = 1$).** If the central value of the Rankin-Selberg L-function $L(\frac{1}{2}, \pi_1 \times \pi_2)$ is non-zero, then there is a pair (τ', σ') of $(\mathrm{SO}'_{2r+1}, \mathrm{SO}'_2)$ which is nearly equivalent to (τ, σ) such that the period $\mathcal{P}_{2r+1,2}(\phi_{\sigma'}, \phi_{\tau'})$ is not identically zero.

To prove this Theorem, we have to consider two cases. First is the case when SO_2 is the split orthogonal group. In this case $\mathrm{SO}_2 \cong \mathrm{GL}_1$ and the only relevant pair is $(\mathrm{SO}_{2r+1}, \mathrm{GL}_1)$ where SO_{2r+1} is the split orthogonal group. In this case the period $\mathcal{P}_{2r+1,2}(\phi_{\sigma'}, \phi_{\tau'})$ is just the Norodvorsky integral [**N75**] evaluated at $s = 1/2$. Since the Norodvorsky integral represents the L-function $L(s, \pi_1 \times \pi_2)$. The Theorem follows in this case.

The second case is when SO_2 is anisotropic. For this case we consider the Eisenstein series $E(g, s, \phi_{\pi(\tau)\otimes\sigma})$ on SO'_{4r+2} as studied in Section 3.1. Note that the quasisplit orthogonal group SO'_{4r+2} has a Levi subgroup isomorphic to $\mathrm{GL}_{2r} \times \mathrm{SO}_2$. By Proposition 3.2, we know that the residue at $s = \frac{1}{2}$ of $E(g, s, \phi_{\pi(\tau)\otimes\sigma})$ does not vanish if the L-function

$$L(s, \pi_1 \times \pi_2) = L(s, \pi(\tau) \times \sigma)$$

is non-zero at $s = \frac{1}{2}$. As before, we denote the residue by $\mathcal{E}_{\pi(\tau)\otimes\sigma}$. The proof of the Theorem in this case is based on the study of certain Bessel model of the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma}$. We need the information on the structure of the local unramified component of the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma}$ before we investigate the global Bessel coefficients of $\mathcal{E}_{\pi(\tau)\otimes\sigma}$.

Fix a local finite place ν of k and set $k_\nu = F$. Assume that the local ν-components of τ and σ are unramified, the relevant pair $(\mathrm{SO}_{2r+1}, \mathrm{SO}_2)$ is split over F. Since τ_ν is generic, we may write

$$\tau_\nu = \mathrm{Ind}_{B_{\mathrm{SO}_{2r+1}}(F)}^{\mathrm{SO}_{2r+1}(F)}(\chi)$$

and then the lift to $\mathrm{GL}_{2r}(F)$ of τ_ν is

$$\pi(\tau_\nu) = \mathrm{Ind}_{B_{\mathrm{GL}_{2r}}(F)}^{\mathrm{GL}_{2r}(F)}(\chi)$$

where the character χ is defined as in the proof of Lemma 3.4. Let μ denote an unramified character of $\mathrm{SO}_2(F) = \mathrm{GL}_1(F)$. We denote by $P_{2^r,1}$ the parabolic subgroup of SO_{4r+2} whose Levi part is $\mathrm{GL}_2^r \times \mathrm{GL}_1$. Define a character $\mu \otimes \chi$ of $\mathrm{GL}_2^r(F) \times \mathrm{GL}_1(F)$ to be

$$\mu \otimes \chi(h_1, \cdots, h_r, a) = \prod_{i=1}^{r} \chi_i(\det h_i)\mu(a),$$

where $h_i \in \mathrm{GL}_2(F)$ and $a \in \mathrm{GL}_1(F)$. We have

Lemma 5.2. The local unramified component of the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma}$ is the unique unramified quotient of $\mathrm{Ind}_{P_{2^r,1}(F)}^{\mathrm{SO}_{4r+2}(F)}(\mu \otimes \chi)$.

Proof. This is proved exactly as in Proposition 1, [**GRS99b**]. □

We now return to the global situation and consider the Bessel coefficients of the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma}$. Let U denote the upper unipotent subgroup of SO'_{4r+2}, the elements of which can be written as

$$u = \begin{pmatrix} n & x & z \\ & I_2 & x^* \\ & & n^* \end{pmatrix} \in \mathrm{SO}'_{4r+2}$$

were $n \in N_{2r}$, the maximal upper unipotent subgroup of GL_{2r}, and $z \in \{X \in \mathrm{Mat}_{2r\times 2r} \ : \ {}^tXJ_r + J_rX = 0\}$. Assume that SO'_{4r+2} is defined in terms of the symmetric matrix

$$\begin{pmatrix} & & J_{2r} \\ & \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} & \\ J_{2r} & & \end{pmatrix}$$

where $J_{2r} = \begin{pmatrix} & & 1 \\ & J_{2r-2} & \\ 1 & & \end{pmatrix}$ and $\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ defines SO_2. Then if the two columns of x are $\alpha\ell_1$ and $\beta\ell_2$, then the rows of x^* are $-(J_r\ell_1)^t$ and $-(J_r\ell_2)^t$.

For $1 \le e \le 2r$ let V_e be the subgroup of U which consists of all matrices of the form

$$v(v_1, v_2, v_3) = \begin{pmatrix} v_1 & v_2 & v_3 \\ & I_{4r-2e+2} & v_2^* \\ & & v_1^* \end{pmatrix} \in U,$$

where $v_1, v_3 \in \mathrm{Mat}_{e\times e}, v_2 \in \mathrm{Mat}_{e\times(4r-2e+2)}$. Notice that $U = V_{2r}$. Let ψ be an additive character of $F\backslash\mathbb{A}$. Let $\xi \in k^{4r-2e+2}$ be a vector with nonzero length. We define a character ψ_e^ξ of V_e by

$$\psi_e^\xi(v(v_1, v_2, v_3)) = \psi((v_1)_{1,2} + \cdots + (v_1)_{e-1,e} + (v_2)_e \cdot \xi)$$

where $(v_1)_{i,j}$ is the (i,j)- entry of v_1, $(v_2)_e$ is the e-th row of v_2 and $(v_2)_e \cdot \xi$ is the inner product of $(v_2)_e$ with ξ.

Let ϵ be an automorphic representation on $\mathrm{SO}'_{4r+2}(\mathbb{A})$. For all $1 \le e \le 2r$ we define

$$\mathcal{B}_e(\epsilon, \xi)(g) \ = \int_{V_e(F)\backslash V_e(\mathbb{A})} \phi_\epsilon(vg)\psi_e^\xi(v)dv$$

for $\phi_\epsilon \in \epsilon$, and refer to it as the e-th Bessel coefficient of ϕ_ϵ. We have

Proposition 5.3. Suppose that $r < e \le 2r$. Then the the e-th Bessel coefficient of the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma}$, i.e. the integral $\mathcal{B}_e(\mathcal{E}_{\pi(\tau)\otimes\sigma}, \xi)$ vanishes for all choice of data and all $\xi \in k^{4r-2e+2}$ which has nonzero length.

Proof. The proof is based on the structure of the local unramified component of the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma}$. In other words, it is enough to show that at one local place the local component of the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma}$ has no such functionals. From Lemma 5.2 we can find a local unramified place ν such that $(\mathcal{E}_{\pi(\tau)\otimes\sigma})_\nu$ is the unramified quotient of $\mathrm{Ind}_{P_{2^r,1}(F)}^{\mathrm{SO}_{4r+2}(F)}(\mu\otimes\chi)$ $(F=k_\nu)$. It suffices to show that $\mathrm{Ind}_{P_{2^r,1}(F)}^{\mathrm{SO}_{4r+2}(F)}(\mu\otimes\chi)$ has no such functionals. This is done in the same way as in Lemma 3.5 and the references cited there. Namely, we consider the double cosets $P_{2^r,1}\backslash\mathrm{SO}_{4r+2}/V_e$ and show that if γ is any representative of a double coset, then there exists $v\in V_e$ such that $\gamma v\gamma^{-1}\in P_{2^r,1}$ and $\psi_e^\xi(v)\neq 1$. This proves the proposition. □

Next we consider the r-th Bessel coefficient of the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma}$, i.e. $\mathcal{B}_e(\mathcal{E}_{\pi(\tau)\otimes\sigma},\xi)(g)$. Depending on ξ this coefficient is an automorphic function on $\mathrm{SO}(r+1,r)(\mathbb{A})$ or $\mathrm{SO}(r+2,r-1)(\mathbb{A})$. We have

Proposition 5.4. As an automorphic function in g, the r-th Bessel coefficient $\mathcal{B}_r(\mathcal{E}_{\pi(\tau)\otimes\sigma},\xi)(g)$ is cuspidal.

Proof. This is done in a similar way as in Theorem 8 p. 839–844, [**GRS95**]. Indeed, when one computes the constant terms of $\mathcal{B}_e(\mathcal{E}_{\pi(\tau)\otimes\sigma},\xi)(g)$ along various maximal unipotent radicals of SO_{2r+1} one obtains integrations which either contain constant terms of the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma}$ or contain as inner integration the e-th Bessel coefficients of the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma}$ for $r<e\le 2r$. By Proposition 5.3 these e-th Bessel coefficients are zero. As for the constant terms obtained, one can check that none of them are along the unipotent radical of $P_{2r,2}$. Hence, by cuspidality of $\pi(\tau)$, they are all zero. □

The following nonvanishing property is the key to prove Theorem 5.1.

Proposition 5.5. There exists a $\xi_0\in k^{2r+2}$ with nonzero length such that the r-th Bessel coefficient $\mathcal{B}_r(\mathcal{E}_{\pi(\tau)\otimes\sigma},\xi_0)(g)$ of the residue $\mathcal{E}_{\pi(\tau)\otimes\sigma}$ is nonzero for some choice of data.

Proof. Let $Q_{2^r,2}$ be the standard parabolic subgroup of SO'_{4r+2} whose Levi part is $\mathrm{GL}_2^r\times\mathrm{SO}_2$. Let $U_{2^r,2}$ denote its unipotent radical, the elements of which can be expressed as

$$u = \begin{pmatrix} I_2 & x_1 & & * & & & & \\ & I_2 & \ddots & & & & * & \\ & & \ddots & x_{r-1} & & & & \\ & & & I_2 & y & z & & \\ & & & & I_2 & y^* & & \\ & & & & & I_2 & & * \\ & & & & & & \ddots & \\ & & & & & & & I_2 \end{pmatrix} \in \mathrm{SO}'_{4r+2}, \tag{5.1}$$

where $x_i, y \in \mathrm{Mat}_{2\times 2}$ and $z = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$. Define below a character $\psi_{U_{2^r,2}}$ of $U_{2^r,2}(\mathbb{A})$, which is trivial on $U_{2^r,2}(k)$

$$\psi_{U_{2^r,2}}(u) = \psi(\mathrm{tr}(x_1) + \cdots + \mathrm{tr}(x_{n-1})).$$

The subgroup of $U_{2^r,2}$ which consists of the (y, z) variables is a Heisenberg group H_5 with five variables. It is also clear that there exists a canonical group homomorphism from $U_{2^r,2}$ to H_5, which we denote by ℓ.

Let $g \in \mathrm{SL}_2$ be embedded in GL_2^r diagonally. Then it is easy to see that SL_2 preserves the character $\psi_{U_{2^r,2}}$ and hence we can define

$$f(g) := \int_{U_{2^r,2}(k)\backslash U_{2^r,2}(\mathbb{A})} \mathcal{E}_{\pi(\tau)\otimes\sigma}(ug)\widetilde{\theta}_\varphi^\psi(\ell(u)g)\psi_{U_{2^r,2}}(u)du$$

where $\widetilde{\theta}_\varphi^\psi$ is the theta function on $\widetilde{\mathrm{Sp}}_4(\mathbb{A})$ and $g \in \mathrm{SL}_2(\mathbb{A})$. Since the copy of SL_2 is part of the dual pair $(\mathrm{SL}_2, \mathrm{SO}_2)$ in Sp_4 it splits under the cover.

Clearly $f(w_0 g) = f(g)$ where w_0 is the Weyl element of $\mathrm{SL}_2(k)$. To prove the Theorem we will prove that under the assumption that $\mathcal{B}_r(\mathcal{E}_{\pi(\tau)\otimes\sigma}, \xi)$ is zero for all choice of data,

(1) $f(g)$ is not identically zero.

(2) $f(\begin{pmatrix} b & \\ & b^{-1} \end{pmatrix} g) = |b|^2 f(g)$ for all $b \in \mathbb{A}^\times$.

This is clearly a contradiction. In fact, by (2) and the automorphicity of $f(g)$, we have

$$|b|^2 f(g) = f(\begin{pmatrix} b & \\ & b^{-1} \end{pmatrix} g) = f(w_0 \begin{pmatrix} b & \\ & b^{-1} \end{pmatrix} g) = f(\begin{pmatrix} b^{-1} & \\ & b \end{pmatrix} w_0 g) = |b|^{-2} f(g).$$

If $f(g)$ is not identically zero, we derive a contradiction. This will prove the Proposition.

In order to show that the function $f(g)$ has properties (1) and (2), we unfold the theta function in the above integral and we obtain

$$\int_{\mathbb{A}^2}\int_{U^0_{2^r,2}(k)\backslash U^0_{2^r,2}(\mathbb{A})} \mathcal{E}_{\pi(\tau)\otimes\sigma}(u(x_1,x_2)^\wedge g)(\omega_\psi(g)\phi)(x_1,x_2)\psi_{U^0_{2^r,2}}(u)dudx_i$$

where $U^0_{2^r,2}$ is the subgroup of $U_{2^r,2}$ where the bottom row of y, as denoted in (5.1), is zero. By $(x_1,x_2)^\wedge$ we denote the matrix u as defined in (5.1) where (x_1,x_2) is the bottom row of y, and all other variables in u are zero. We integrate x_i over $\mathbb{A}^2$. Finally

$$\psi_{U^0_{2^r,2}}(u) = \psi_{U_{2^r,2}}(u)\psi(t)$$

where $z = \begin{pmatrix} t & 0 \\ 0 & -t\end{pmatrix}$ as denoted in (5.1).

We now concentrate on the integral

$$\int_{U^0_{2^r,2}(k)\backslash U^0_{2^r,2}(\mathbb{A})} \mathcal{E}_{\pi(\tau)\otimes\sigma}(um)\psi_{U^0_{2^r,2}}(u)du.$$

Let w be the Weyl element of SO'_{4r+2} defined by $w[i,2i-1] = w[r+i,2r+2i-1] = w[2r+1,2r+1] = w[2r+2,2r+2] = 1$. This determines w uniquely. Conjugating the above integral by w from left to right we obtain

$$\int \mathcal{E}_{\pi(\tau)\otimes\sigma}(\begin{pmatrix} v_1 & v_2 & v_3 \\ & I_2 & v_2^* \\ & & v_1^*\end{pmatrix}\begin{pmatrix} I_{2n} & & \\ p & I_2 & \\ q & p^* & I_{2n}\end{pmatrix} wm)\psi_r(v_1)dv_idpdq$$

where v_1 is integrated over $N_{2r}(k)\backslash N_{2r}(\mathbb{A})$ (N_{2r} is the maximal upper unipotent subgroup of GL_{2r}) and

$$\psi_r(v_1) = \psi((v_1)_{1,2} + \cdots + (v_1)_{2r-1,2r}).$$

The variable v_2 is integrated over $\mathrm{Mat}_{2r\times 2}(k)\backslash\mathrm{Mat}_{2r\times 2}(\mathbb{A})$ such that the last r rows are zero. The variables v_3 and q are integrated over all matrices in

$$\{X \in \mathrm{Mat}_{2r\times 2r} \ : \ {}^tXJ_{2r} + J_{2r}X = 0 \ \textit{and}\ X_{i,j} = 0 \ \textit{for all}\ i \geq j\}.$$

Finally, p is integrated over all matrices in $\mathrm{Mat}_{2\times 2r}$ such that $p_{i,j} = 0$ for all $j \leq r+1$. These variables are integrated over the $\mathbb{A}$-rational points modulo the k-rational points of the corresponding varieties. Using the assumption that $\mathcal{B}_r(\mathcal{E}_{\pi(\tau)\otimes\sigma},\xi)$ is zero for all nonzero length vectors $\xi \in k^{2r+2}$ and apply similar Fourier expansions as in the proof of Proposition 7.1 (**[GJR03b]**) starting at integral (7.6), we deduce that the above integral equals

$$\int \mathcal{E}^{U_{2r,2}}_{\pi(\tau)\otimes\sigma}(\begin{pmatrix} v_1 & & \\ & I_2 & \\ & & v_1^*\end{pmatrix}\begin{pmatrix} I_{2r} & & \\ p & I_2 & \\ q & p^* & I_{2r}\end{pmatrix} wm)\psi_r(v_1)dv_1dpdq.$$

Here $\mathcal{E}^{U_{2r,2}}_{\pi(\tau)\otimes\sigma}$ is the constant term along $U_{2r,2}$ which is the unipotent radical of $Q_{2r,2}$. The matrices v_1, p and q are defined as before but p and q are integrated over the adelic points. Combining this with the Weil representation we thus obtain

$$f(g) = \int \mathcal{E}^{U_{2r,2}}_{\pi(\tau)\otimes\sigma}(\begin{pmatrix} v_1 & & \\ & I_2 & \\ & & v_1^* \end{pmatrix} \begin{pmatrix} I_{2r} & & \\ p & I_2 & \\ q & p^* & I_{2r} \end{pmatrix} w(x_1,x_2)^\wedge g) (\omega_\psi(g)\phi)(x_1,x_2)\psi_r(v_1)d(\ldots).$$

The torus of SL_2 is embedded in SO'_{4r+2} as

$$\mathrm{diag}(b, b^{-1}, \cdots, b, b^{-1}, 1, 1, b, b^{-1}, \cdots, b, b^{-1}).$$

When conjugating by w we obtain $\mathrm{diag}(bI_{2r}, I_2, b^{-1}I_{2r})$ which is the center of the Levi part of $Q_{2r,2}$. Hence $f(g)$ has a left transformation property under this torus. When we conjugate across the $(x_1,x_2)^\wedge$, p and q integration we obtain from a change of variables, a factor of $|b|^{-2r^2}$. From the Weil representation we get a factor of $|b|^2$ and from $\mathcal{E}^{U_{2r,2}}_{\pi(\tau)\otimes\sigma}$ we get $|b|^{2r^2}$. Overall we are left with $|b|^2$ and hence (2) holds for $f(g)$. As for the nonvanishing of $f(g)$, we argue as in Lemmas 1 and 2 on p.895–896, **[GRS99a]** and deduce that $f(g)$ is not zero for some choice of data. □

We now finish the proof of Theorem 5.1. From Propositions 5.4 and 5.5 we deduce that there is a $\xi_0 \in k^{2r+2}$ with nonzero length such the r-th Bessel coefficient $\mathcal{B}_r(\mathcal{E}_{\pi(\tau)\otimes\sigma}, \xi_0)(g)$ is a nonzero cuspidal automorphic function on $\mathrm{SO}'_{2r+1}(\mathbb{A})$. Let τ' be any irreducible summand of the cuspidal automorphic representation of $\mathrm{SO}'_{2r+1}(\mathbb{A})$ generated by all r-th Bessel coefficients $\mathcal{B}_r(\mathcal{E}_{\pi(\tau)\otimes\sigma}, \xi_0)(g)$. It then follows that the integral

$$\int_{\mathrm{SO}'_{2r+1}(k)\backslash \mathrm{SO}'_{2r+1}(\mathbb{A})} \int_{V_r(k)\backslash V_r(\mathbb{A})} \phi_{\tau'}(g)\mathcal{E}_{\pi(\tau)\otimes\sigma}(vg)\psi_r^{\xi_0}(v)dvdg$$

is nonzero for some choice of data. Replacing the residue by the Eisenstein series itself we deduce that the integral

$$\int_{\mathrm{SO}'_{2r+1}(k)\backslash \mathrm{SO}'_{2r+1}(\mathbb{A})} \int_{V_r(k)\backslash V_r(\mathbb{A})} \phi_{\tau'}(g)\mathcal{E}_{\pi(\tau)\otimes\sigma}(vg,s)\psi_r^{\xi_0}(v)dvdg$$

is not zero for the real part of s large.

Unfolding this integral as in **[GPSR97]** we deduce that $\mathcal{P}_{2r+1,2l}(\phi_\sigma, \phi_{\tau'})$ is not zero for some choice of data.

Finally, arguing as in Proposition 5 **[GRS99a]** we deduce that τ' is nearly equivalent to τ. This finishes the proof of Theorem 5.1 and hence the proof of Theorem B.

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