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On Some Topics in Automorphic Representations

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Abstract

This paper is extended notes of the author's lecture at the ICCM2007 in Hangzhou, China, which discuss the progress of the author's research after my lecture at the ICCM2001 ([Jng04]). Some parts of the topics has also been discussed in the author's recent survey paper ([Jng07]).

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1 Introduction

Automorphic forms or more classically modular forms have been a very active subject in mathematics in the past two centuries. The classical theta functions in the theory of representing a number by a sum of squares and in the theory of Riemann zeta functions are typical examples. Modular forms related to elliptic functions and elliptic curves are more sophisticated examples. More recently, modular forms have been used to interpret discoveries in mathematical physics (string theory, M-duality, for instance), algebraic geometry (the theory of motives, for instance) and number theory (representations of Galois groups, for instance). The conjectural framework for the theory of automorphic forms and its intrinsic relations to algebraic geometry and number theory is called the Langlands Program. The relations of automorphic forms to mathematics physics is roughly referred as the geometric Langlands program.

The main ingredient in the Langlands Program is the notion of automorphic representations. The Langlands conjectures describes the basic structures of automorphic representations and their implications to algebraic geometry and number

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theory. The basic structures of automorphic forms are now described more precisely by the Arthur conjectures. In the past forty years, the studies of the basic structures of automorphic representations have been conducted via two essential approaches: the trace formula approach and the L-function approach. The trace formula approach provides a general and existence method to understand the basic structures of automorphic forms, especially these belonging to the discrete spectrum. The recent progress on the Fundamental Lemma by Laumon and Ngo, and by Ngo for the function fields, which can be transferred to the cases of p-adic fields by the work of Waldspurger, yields the light for complete understanding of the endoscopy structures of automorphic forms over classical groups. We refer to [A05] for the detailed discussion of this approach. Soudry's paper at ICM 2006 ([Sd06]) contains the fundamental part of the approach via the theory of L-functions. In this paper, I will discuss in some details my joint work with David Ginzburg and David Soudry along this line, in addition to [Sd06].

1.1 Automorphic representations

Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. For simplicity, take G to be a reductive k-split algebraic group, or even take G to be a k-split classical groups. For example, take $G = \mathrm{GL}_m$, the general linear group consisting of all $m \times m$-matrices with nonzero determinant, or $G = \mathrm{SO}_{2n+1}$, the special odd orthogonal group, which is defined by

$$\mathrm{SO}_{2n+1} = \{g \in \mathrm{GL}_{2n+1} \mid {}^t g J_{2n+1} g = J_{2n+1}, \det g = 1\}.$$

Here J_{2n+1} is defined inductively by

$$J_{2n+1} := \begin{pmatrix} & & 1 \\ & J_{2n-1} & \\ 1 & & \end{pmatrix}.$$

It is known that the diagonal embedding of $G(k)$ into $G(\mathbb{A})$ has discrete image in $G(\mathbb{A})$ and the quotient $Z_G(\mathbb{A}) \cdot G(k) \backslash G(\mathbb{A})$ has finite volume with respect to the canonical Haar measure on the quotient space, where Z_G denotes the center of G.

Automorphic functions may be taken as functions in the following L^2-space

$$L^2(G, \omega) = L^2(G(k) \backslash G(\mathbb{A}))_\omega,$$

which consists of all $\mathbb{C}$-valued square integrable functions f on $G(k) \backslash G(\mathbb{A})$, such that

$$f(zg) = \omega(z) f(g),$$

where $z \in Z_G(\mathbb{A})$ and ω is a character of $Z_G(k) \backslash Z_G(\mathbb{A})$, and

$$\int_{Z_G(\mathbb{A}) G(k) \backslash G(\mathbb{A})} |f(g)|^2 dg < \infty.$$

Naturally, the space $L^2(G, \omega)$ has a $G(\mathbb{A})$-module structure given by

$$(g \cdot f)(x) = f(xg)$$

for all $g, x \in G(\mathbb{A})$ and $f \in L^2(G,\omega)$. A function $f \in L^2(G,\omega)$ is called *cuspidal* if the following integral

$$\int_{N(k)\backslash N(\mathbb{A})} f(ng)dn$$

is zero for almost all $g \in G(\mathbb{A})$ and for the unipotent radical N of all standard parabolic subgroup $P = MN$ of G. We denote by B a Borel subgroup of G. When $G = \mathrm{GL}_m$ or SO_{2n+1}, B can be taken to be the subgroup consisting of all upper-triangular matrices in G. Then there is a Levi decomposition

$$B = TU$$

where T is the maximal k-split torus and U is the unipotent radical of B. When $G = \mathrm{GL}_m$ or SO_{2n+1}, T is the diagonal subgroup and U consists of all upper-triangular matrices in G with all diagonal entries 1. A parabolic subgroup $P = MN$ of G is called standard if it contains $B = TU$. When $G = \mathrm{GL}_m$, M is isomorphic to $\mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_r}$ with $m = m_1 + \cdots + m_r$, and N is given by the following unipotent elements of GL_m

$$\begin{pmatrix} I_{m_1} & * & * \\ & \ddots & * \\ & & I_{m_r} \end{pmatrix}.$$

For detailed discussions on algebraic groups we prefer to [Sp98].

If a smooth cuspidal function $f \in L^2(G,\omega)$ generates an irreducible $G(\mathbb{A})$-submodule in $L^2(G,\omega)$, then f is called a cuspidal automorphic form on $G(\mathbb{A})$. We prefer to [BrJ79] or [MW95] for a formal definition of cuspidal automorphic forms. Any irreducible $G(\mathbb{A})$-submodule of $L^2(G,\omega)$ generated by a cuspidal automorphic form is called an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Let (π, V_π) be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Then any function in V_π is cuspidal.

1.2 Discrete spectrum

We denote by $L^2_d(G,\omega)$ the Hilbert sum of all irreducible $G(\mathbb{A})$-submodules in $L^2(G,\omega)$, which is called the discrete spectrum of G. Let $L^2_c(G,\omega)$ be the submodule in $L^2(G,\omega)$ generated by all irreducible cuspidal $G(\mathbb{A})$-submodules in $L^2(G,\omega)$. It is clear that $L^2_c(G,\omega)$ is a $G(\mathbb{A})$-submodules in $L^2_d(G,\omega)$, which is called the cuspidal spectrum of G. Following the Langlands theory of Eisenstein series ([L76] and [MW95]), the non-cuspidal discrete spectrum of G is realized by the residues of Eisenstein series, which is often called the residual spectrum of G. Hence one has

$$L^2_d(G,\omega) = L^2({}_cG,\omega) \bigoplus L^2_r(G,\omega).$$

One of the main problems concerning the modern theory of automorphic forms is

Problem 1.1. *Understand the decomposition of $L^2_d(G,\omega)$ into the irreducible ones.*

In other words, it is to understand the multiplicity, which is denoted by $m_d(\pi)$, of an irreducible admissible representation π of $G(\mathbb{A})$ occurring in $L^2_d(G,\omega)$. More precisely, one may consider the multiplicity $m_c(\pi)$ of an irreducible admissible representation π of $G(\mathbb{A})$ occurring in $L^2_c(G,\omega)$, and the multiplicity $m_r(\pi)$ of an irreducible admissible representation π of $G(\mathbb{A})$ occurring in $L^2_r(G,\omega)$. Hence one has, for an irreducible admissible representation π of $G(\mathbb{A})$

$$m_d(\pi) = m_c(\pi) + m_r(\pi).$$

In general, by a theorem of Gelfand and Piatetski-Shapiro, the cuspidal multiplicity, $m_c(\pi)$, is finite for any reductive algebraic group G over k. On the other hand, for $G = GL(m)$, it is a theorem of J. Shalika and of I. Piatetski-Shapiro, independently, that the cuspidal multiplicity is one. Then by a theorem of Jacquet and Shalika and the work of Moeglin and Waldspurger, the discrete multiplicity for $G = GL(m)$ is also one. For classical groups, $G = \mathrm{SO}_m$ or Sp_{2n}, the Arthur conjecture asserts that

$$m_d(\pi) \leqslant \begin{cases} 1, & \text{if } G = \mathrm{SO}_{2n+1} \text{ or } \mathrm{Sp}_{2n} \\ 2, & \text{if } G = \mathrm{SO}_{2n}. \end{cases}$$

Some special cases have been investigated by various people.

- $G = \mathrm{SL}_2$: Based on the work of Langlands and Lebaase, D. Ramakrishnan proves that the cuspidal multiplicity is at most one ([Rm00]).
- $G = \mathrm{SL}_n$ for $n \geqslant 3$: D. Blasius finds a family of cuspidal automorphic representations with higher multiplicity, i.e. $m_c(\pi) > 1$ ([Bl94]). See also the work of E. Lapid ([Lp99]).
- $G = U(3)$: J. Rogawski shows that the discrete multiplicity is at most one ([Rg92]).
- $G = G_2$: Gan, Gurevich and Jiang show that there exists a family of cuspidal automorphic representations of $G_2(\mathbb{A})$, whose cuspidal multiplicity can be as high as possible ([GnGJ02]), and see [Gn05] for more complete result. It turns out that such a result can be expected for other exceptional groups, although it does not happen for classical groups according to Arthur's conjecture.
- $G = \mathrm{GSp}(4)$: Jiang and Soudry shows that for irreducible generic cuspidal automorphic representations of GSp(4), the cuspidal multiplicity is at most one ([JngS07c]).

1.3 Local components of automorphic representations

We recall first the structure of the $\mathbb{A}$-rational point of G. Let S be a finite set of local places of k, including the archimedean local places of k. We set

$$\mathbb{A}(S) = (\prod_{v\in S} k_v) \times (\prod_{v\notin S} \mathcal{O}_v)$$

where $\mathcal{O}_v$ for a finite local place v is the ring of v-integers in the local field v. Note that $\mathcal{O}_v$ is compact. Hence $\mathbb{A}(S)$ is locally compact. Then we have

$$\mathbb{A} = \varinjlim_{S} \mathbb{A}(S).$$

It follows that

$$G(\mathbb{A}(S)) = (\prod_{v \in S} G(k_v)) \times (\prod_{v \notin S} G(\mathcal{O}_v))$$

and

$$G(\mathbb{A}) = \varinjlim_{S} G(\mathbb{A}(S)).$$

It is a theorem of Harish-Chandra and Bernstein that the local groups $G(k_v)$ are tame ([Cl06]), i.e. of type I in the sense of C^*-algebras. It follows that any irreducible unitary representation of $G(\mathbb{A}$ is a restricted tensor product

$$\pi = \otimes_v \pi_v$$

where π_v is an irreducible admissible unitary representation of $G(k_v)$ and π_v is spherical or unramified or of type I for almost all local places v of k. Note that π_v is spherical or unramified if π_v has nonzero K_v-fixed vectors for some hyperspecial maximal compact subgroup K_v of $G(k_v)$, i.e.$V_{\pi_v}^{K_v} \neq 0$, where

$$V_{\pi_v}^{K_v} = \{u \in V_{\pi_v} \ : \ \pi_v(h)(u) = u, \text{for all } h \in K_v\}.$$

When G is k-split, one may take $K_v = G(\mathcal{O}_v)$.

From Satake's theory of p-adic spherical functions ([St63]), we have the following properties,

(1) $\dim \pi_v^{K_v} \leqslant 1$.
(2) If $\pi_v^{K_v} \neq 0$ (π_v is spherical), there is a unramified character χ_v of $T(\mathbb{Q}_v)$ s.t. π_v is the irreducible spherical constituent of $\mathrm{Ind}_{B(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\chi_v)$, where $B = TU$ is a Borel subgroup of G.
(3) The K_v-invariant vector of π_v is characterized by a semi-simple conjugacy class t_{π_v} in the Langlands dual group ${}^L G$ (which will be defined below), which is called the Satake parameter attached to π_v.

When π_v is spherical, the dimension of $V_{\pi_v}^{K_v}$ is one. We choose a nonzero vector u_v° in V_{π_v}. Take all finite subset S of local places in Ω_k, such that S contains all archimedean local places of k and for any local place v, which is not contained in S, the local component π_v is spherical. Consider all the factorizable vectors of the following form

$$(\otimes_{v \in S} u_v) \otimes (\otimes_{v \notin S} u_v^\circ).$$

Then the set of all factorizable vectors generates a dense subspace of the irreducible unitary representation (π, V_π) with $\pi = \otimes_v \pi_v$.

1.4 Complex dual groups

For k-split reductive algebraic groups G, take T to be a maximal k-split torus of G. Let $R(T,G)$ be the set of roots of G with respect to T and $R^\vee(T,G)$ be the set of coroots of G with respect to T. Let X be the $\mathbb{R}$-vector space generated by $R(T,G)$ and $X^\vee$ be the $\mathbb{R}$-vector space generated by $R^\vee(T,G)$. Finally, let Δ be the set of simple root in $R(T,G)$ with respect to a given Borel subgroup $B=TU$, and $\Delta^\vee$ be the dual of Δ in $R^\vee(T,G)$. Then $(X,\Delta;X^\vee,\Delta^\vee)$ is the root datum attached to (G,B,T). It follows from a standard theorem in the theory of linear algebraic groups ([Sp98]) that G is determined over k, up to isogeny, by a combinatorical datum, called the root datum attached to G.

The complex dual group of G is the complex algebraic group $G^\vee$ determined, uniquely up to isogeny, by the root datum dual to the one of G. The relations are given by the following diagram

$$\begin{array}{ccc} G & \Longleftrightarrow & (X,\Delta;X^\vee,\Delta^\vee) \\ \updownarrow & & \updownarrow \\ {}^LG & \Longleftrightarrow & (X^\vee,\Delta^\vee;X,\Delta) \end{array}$$

For example, we have the following table

G	$G^\vee$
$\mathrm{GL}(m)$	$\mathrm{GL}(m,\mathbb{C})$
$\mathrm{SL}(m)$	$\mathrm{PGL}(m,\mathbb{C})$
$\mathrm{SO}(2n+1)$	$\mathrm{Sp}(2n,\mathbb{C})$
$\mathrm{Sp}(2n)$	$\mathrm{SO}(2n+1,\mathbb{C})$
$\mathrm{SO}(2n)$	$\mathrm{SO}(2n,\mathbb{C})$
G_2	$G_2(\mathbb{C})$

The Langlands dual group ${}^LG$ is defined to be the semiproduct of the complex dual group ${}^LG$ and the absolute Galois group $\Gamma_k=\mathrm{Gal}(\bar{k}/k)$. When G is k-split, then the semiproduct is a direct product. Hence we may take the complex dual group as the Langlands dual group.

1.5 Nearly equivalence and L-functions

Let S be a finite set of local places, which includes the archimedean local places of k. For each local place v, we denote by c_v a semisimple conjugacy class in $G^\vee(\mathbb{C})$ (assuming that G is k-split). We set

$$c(S):=\{c_v \mid v\notin S\}.$$

For any two sets S and S', we say that $c(S)$ and $c'(S')$ are equivalent if there is a set S'' such that $c(S'')=c'(S'')$ as conjugacy classes in $G^\vee(\mathbb{C})$. We denote by $\mathcal{C}(G)$ the equivalence classes of all such sets $c(S)$.

Let $\pi=\otimes_v\pi_v$ be an irreducible automorphic representation of $G(\mathbb{A})$. Then there is a finite set S_π of local places, which includes the archimedean local places of k, such that for any local place $v \notin S_\pi$, π_v is unramified. Let $c(\pi_v)$ be the semisimple conjugacy class in $G^\vee(\mathbb{C})$ corresponding to the unramified π_v. We denote by $c(\pi)$ the collection $\{c(\pi_v) \mid v \notin S_\pi\}$. In other words, we have

$$c(\pi) = c(S_\pi).$$

We denote by $\mathcal{A}(G)$ the equivalence classes of all irreducible admissible representations of $G(\mathbb{A})$. Then we have the following mapping

$$c \; : \; \mathcal{A}(G) \to \mathcal{C}(G), \qquad \pi \mapsto c(\pi).$$

The fiber of this mapping is called a nearly equivalence class of irreducible admissible representations of $G(\mathbb{A})$. In other words, π and π' to be nearly equivalent if $c(\pi)$ and $c(\pi')$ are equivalent. The fiber of π is denoted by Π_π.

Problem 1.2. *Determine the structures of π in terms of $c(\pi)$ for irreducible cuspidal automorphic representations π of $G(\mathbb{A})$.*

This is one of the major problems in the modern theory of automorphic forms. For $G = \mathrm{GL}(m)$, Jacquet and Shalika proved the following theorem.

Theorem 1.3. (Jacquet-Shalika [JS81]) *For irreducible cuspidal automorphic representations π_1 and π_2 of $\mathrm{GL}_m(\mathbb{A})$, π_1 and π_2 are isomorphic if and only if $c(\pi_1) = c(\pi_2)$.*

This theorem has been extended to $G = \mathrm{SO}_{2n+1}$ as follows.

Theorem 1.4. (Jiang-Soudry [JngS03]) *For irreducible generic cuspidal automorphic representations π_1 and π_2 of $\mathrm{SO}_{2n+1}(\mathbb{A})$, π_1 and π_2 are isomorphic if and only if $c(\pi_1) = c(\pi_2)$.*

It was proved by Soudry that Theorem 1.4 holds for GSp(4) ([Sd87]).

1.6 Tensor product L-functions

Let $\pi = \otimes_v \pi_v$ be an irreducible unitary automorphic representation of $G(\mathbb{A})$ and $\tau = \otimes_v \tau_v$ be an irreducible unitary automorphic representation of $\mathrm{GL}_m(\mathbb{A})$. The partial tensor product L-functions associated to *pi* and τ is defined by the following eulerian product

$$L^S(s, \pi \times \tau) := \prod_{v \notin S} (\det(I - c(\pi_v) \otimes c(\tau_v) q_v^{-s})^{-1},$$

where $S = S_{\pi,\tau}$ is a finite set of local places of k, including all archimedean local places, such that for $v \notin S$, both π_v and τ_v are unramified. Note here that G is a k-split classical group.

It is known from the Langlands theory of Eisenstein series that the partial tensor product L-functions $L^S(s, \pi \times \tau)$ converges absolutely for the real part of s large, and has meromorphic continuation to the whole complex plane. One can show that these L-functions satisfy a functional equation relating s to $1 - s$, by the Langlands-Shahidi method (when π is generic) or the Rankin-Selberg method (when π is general).

Problem 1.5. *Determine the location of the poles of* $L^S(s, \pi \times \tau)$ *for* $s \geqslant \frac{1}{2}$.

When π is generic, it can be proved by either the Langlands-Shahidi method or the Rankin-Selberg method that $L^S(s, \pi \times \tau)$ is holomorphic for the real part of s greater than one. In general, it is an open problem. By using Arthur's conjecture in a later section, the poles can be explicitly determined.

2 Langlands functoriality

We discuss some preliminary versions of the Langlands functoriality conjecture, which is the central problem in the Langlands Program.

2.1 Langlands functorial transfers

We state here the weak version of the Langlands functorial transfers, which is denoted by WLT_ρ.

Conjecture 2.1. (The Weak Langlands Transfer) *Let* G *and* H *be* k*-split reductive algebraic groups and let* ρ *be any group homomorphism*

$$\rho \ : \ H^\vee(\mathbb{C}) \to G^\vee(\mathbb{C})$$

from the complex dual group $H^\vee$ *to the complex dual group* $G^\vee$. *For any irreducible admissible automorphic representation* σ *of* $H(\mathbb{A})$, *there is an irreducible admissible automorphic representation* π *of* $G(\mathbb{A})$ *such that*

$$c(\rho(\sigma)) = c(\pi)$$

as conjugacy classes in $G^\vee(\mathbb{C})$, *where* $c(\rho(\sigma)) = \{\rho(c(\sigma_v))\}$.

The (strong) Langlands functorial transfer can be formulated in terms of the complete L-functions, which is denoted by LT_ρ

Conjecture 2.2. (The Langlands Functorial Transfer) *Let* G *and* H *be* k*-split reductive algebraic groups and let* ρ *be any group homomorphism*

$$\rho \ : \ H^\vee \to G^\vee$$

from the complex dual group $H^\vee$ *to the complex dual group* $G^\vee$. *For any irreducible admissible automorphic representation* σ *of* $H(\mathbb{A})$, *there is an irreducible admissible automorphic representation* π *of* $G(\mathbb{A})$ *such that*

$$L(s, \sigma \times \tau) = L(s, \pi \times \tau), \qquad \epsilon(s, \sigma \times \tau) = \epsilon(s, \pi \times \tau),$$

for all irreducible unitary cuspidal automorphic representations τ *of* $\mathrm{GL}_m(\mathbb{A})$ *with* m *being all positive integers.*

The Langlands functorial transfer conjecture is one of the fundamental conjectures in the Langlands Program. Some of the known cases were proved either by Arthur-Selberg trace formula method, by the converse theorem and L-function method, or by various types of theta correspondence method.

- Let D be a division algebra over k of index n. The generalized Jacquet-Langlands correspondence between $\mathrm{GL}_m(D)$ and $\mathrm{GL}_{mn}(k)$ (Jacquet-Langlands for $m = 1$ and $n = 2$, and Arthur-Clozel for $m = 1$ and general n ([AC89]). Based on the method of Arthur-Clozel, Badulescu recently ([BG07]) established the case for general m and n, without technical assumption on the archimedean part of D.
- Let G_n be one of the k-split classical groups: SO_{2n+1}, Sp_{2n}, and SO_{2n}. The complex dual group $G_n^\vee$ of G_n is $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{SO}_{2n+1}(\mathbb{C})$, or $\mathrm{SO}_{2n}(\mathbb{C})$, respectively. The natural embedding ι_n of $G_n^\vee$ to a general linear group is given by
$$\iota_n(G_n^\vee) \subset \mathrm{GL}_{2n}(\mathbb{C})$$
if G_n is SO_{2n+1} or SO_{2n}, and by
$$\iota_n(G_n^\vee) \subset \mathrm{GL}_{2n+1}(\mathbb{C})$$
if G_n is Sp_{2n}. This is proved in [CKPSS04] (see [CKPSS01] for the case that $G_n = \mathrm{SO}_{2n+1}$), by using the converse theorem and L-function method.
- Let G_n be either k-quasisplit unitary group $U(n,n)$ or $U(n+1,n)$. To define the group G_n, we need a quadratic extension F/k. Then the Langlands dual group ${}^LG_n$ when $G_n = U(n,n)$ is a semi-direct product $\mathrm{GL}_{2n}(\mathbb{C}) \rtimes \mathrm{Gal}(F/k)$ of the complex group $\mathrm{GL}_{2n}(\mathbb{C})$ and the Galois group $\mathrm{Gal}(F/k)$. The target group for $G_n = U(n,n)$ is $\mathrm{Res}_{F/k}(\mathrm{GL}_{2n})$, the Langlands dual group of which is
$$(\mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C})) \rtimes \mathrm{Gal}(F/k).$$
The Langlands functorial transfer for both cases were proved by H. Kim and M. Krishnamurthy in [KK04] and [KK05], by using the converse theorem and L-function method. We refer to [KK04] and [KK05] for details.
- In [AS06a], M. Asgari and F. Shahidi established the weak Langlands functorial transfer from general spin groups GSpin_m to the general linear group for irreducible generic cuspidal automorphic representations. This completes the weak Langlands functorial transfers for the list of reductive k-split algebraic groups whose Langlands dual groups have classical derived groups, by using the converse theorem and L-function method. A particular case of this work provides the Langlands functorial transfer from GSp_4 to GL_4, which has been long expected. We refer to [AS06b] for more explicit results related to this Langlands transfer.
- Some lower rank, very interesting cases: (i) The symmetric square transfer of GL_2 was prove by Gelbart-Jacquet, the symmetric cube transfer of GL_2 was proved by Kim-Shahidi ([KSh02]), and the symmetric fourth power transfer of GL_2 was proved by Kim ([K03]). (ii) The tensor product transfer of $\mathrm{GL}_2 \times \mathrm{GL}_2$ was proved by D. Ramakrishnan ([Rm00]), and that of $\mathrm{GL}_2 \times \mathrm{GL}_3$ was proved by Kim-Shahidi ([KSh02]). (iii) The exterior square transfer from GL_4 to GL_6 was proved by Kim ([K03]). These cases were proved by the converse theorem and L-function method. (iv) The non-endoscopy transfer from G_2 to GSp_6 for generic cuspidal automorphic representations was proved by Ginzburg and Jiang by using exceptional theta correspondence ([GJng01]).

(v) The endoscopy transfer from $\mathrm{GL}_2 \times \mathrm{GL}_2$ to G_2 was proved by Ginzburg by using the combination of (iv) with refined argument of automorphic descent construction of Ginzburg-Rallis-Soudry ([G05]).

2.2 The image of the Langlands functorial transfer

One of the refinements for the Langlands functorial transfer conjecture is to determine and to characterize the image of the Langlands functorial transfers. The key ingredient to the current known cases is from the Rankin-Selberg method. This is the work of Ginzburg, Rallis, and Soudry, generalizing the earlier work of Gelbart and Piatetski-Shapiro ([GPSR97], and [GlPSR87]. The following are the theorem for SO_{2n+1}.

Theorem 2.3. ([GRS01]) *Let π be an irreducible generic cuspidal automorphic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$, and let τ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$. Assume that the partial L-function $L^S(s, \pi \times \tau)$ has a pole at $s = 1$. Then m is even, τ is self-dual, and the partial exterior square L-function of τ, $L^S(s, \tau, \Lambda^2)$ has a pole at $s = 1$.*

From this theorem, we obtain the extra information for τ from the existence of the pole at $s = 1$ of the tensor product L-function $L^S(s, \pi \times \tau)$. The following theorem indicates the significance of this extra information.

Theorem 2.4. *Assume that the partial exterior square L-function of τ, $L^S(s, \tau, \Lambda^2)$ has a pole at $s = 1$. Then the following hold.*

1. *τ is self-dual, and m must be even, say, $m = 2r$.*
2. *There is a unique irreducible generic cuspidal automorphic representation σ of $\mathrm{SO}_{2r+1}(\mathbb{A})$, such that τ is the Langlands functorial transfer from σ.*
3. *Write $\tau = \otimes_v \tau_v$. Each local component τ_v is a local Langlands functorial transfer from $\mathrm{SO}_{2r+1}(k_v)$.*

Part one was proved in [JS90] and [K00]. The existence of σ in part two was proved in [GRS01] by the automorphic descent method, and the uniqueness of σ in part two was proved in [JngS03] by the local converse theorem (see [Jng06b] for detailed discussion on the general version of the local converse theorem.). Part three was proved in [JngS04] by the local descent method and the local converse theorem. We refer to [Jng04] for more detailed discussion of the local theory.

It is expected that this theorem holds for other classical groups with suitable modification.

Based on these results, certain properties of the image of the Langlands functorial transfer can be determined as follows. We state below the theorem for SO_{2n+1} and refer to [CKPSS04] and [Sd05] for the statements for other classical groups.

Theorem 2.5. ([GRS01], [CKPSS01], [CKPSS04], [JngS03], [JngS04], and [Sd05]) *There is a unique one-to-one correspondence between the set $\mathcal{B}_n$ and the set $\mathcal{A}_n$, which is the Langlands functorial transfer from $\mathrm{SO}_{2n+1}(\mathbb{A})$ to $\mathrm{GL}_{2n}(\mathbb{A})$, where*

$\mathcal{B}_n$ is the set of the equivalence classes of irreducible generic cuspidal automorphic representations σ of $\mathrm{SO}_{2n+1}(\mathbb{A})$, and $\mathcal{A}_n$ is the set of equivalence classes of irreducible self-dual unitary automorphic representations τ of $\mathrm{GL}_{2n}(\mathbb{A})$ with the following properties:

- *There is a partition $n = \sum_{i=1}^r n_i$ and for each i, there is an irreducible unitary self-dual cuspidal automorphic representation τ_i of $\mathrm{GL}_{2n_i}(\mathbb{A})$ such that*
$$\tau = \tau_1 \boxplus \cdots \boxplus \tau_r;$$
- *if $i \neq j$, then $\tau_i \ncong \tau_j$;*
- *for all i, $L^S(s, \tau_i, \Lambda^2)$ has a pole at $s = 1$.*

Remark 2.1. *For SO_{2n} or for Sp_{2n}, the results are not as precise as in Theorem 4.3 for SO_{2n+1}, since the results in [JngS03] and [JngS04] for SO_{2n+1} have not been completely established for either SO_{2n} or for Sp_{2n}. Also for GSpin_m, the automorphic descent has not been carried over. The analogue of Theorem 4.2 is not valid yet.*

It is very interesting to mention that Theorem 4.3 has applications to the Inverse Galois Problem recently by C. Khare, M. Larsen, and G. Savin ([KLS06]).

3 The Arthur theorem

In [A05], Arthur states his theorem assuming the various types of the Fundamental Lemmas, which gives the explicit description of the discrete spectrum of all classical groups in terms of the discrete spectrum of the general linear group. By Arthur's theorem, the weak Langlands transfer from classical groups to the general linear group holds. We describe below the Arthur theorem for SO_{2n+1}.

Let $\mathcal{A}^{uc}(\mathrm{GL}_m)$ be the set of irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A})$, modulo equivalence. For $\tau \in \mathcal{A}^{uc}(\mathrm{GL}_m)$, the Rankin product L-function $L^S(s, \tau \times \tau)$ has a pole at $s = 1$ if and only if τ is self-dual. Assume that τ is self-dual. Then by

$$L^S(s, \tau \times \tau) = L^S(s, \tau, S^2) L^S(s, \tau, \Lambda^2),$$

it follows that one and only one of the symmetric square L-function $L^S(s, \tau, S^2)$ and the exterior square L-function $L^S(s, \tau, \Lambda^2)$ has a simple pole at $s = 1$, since both $L^S(s, \tau, S^2)$ and $L^S(s, \tau, \Lambda^2)$ do not vanish at $s = 1$ by a theorem of Shahidi. We say τ is of symplectic type if $L^S(s, \tau, \Lambda^2)$ has a pole at $S = 1$; otherwise we say τ is of orthogonal type.

Let $n = \sum_{i=1}^r n_i$ be a partition of n. We write $2n_i = m_i a_i$ for $i = 1, 2, \cdots, r$. Take a self-dual τ_i in $\mathcal{A}^{uc}(\mathrm{GL}_{m_i})$. Assume that if a_i is even, τ_i is of symplectic type; and if a_i is odd, τ_i is of orthogonal type. For each i, we introduce a symbol $\psi_i = (\tau_i, a_i)$. By a theorem of Moeglin and Waldspurger on the discrete spectrum of the general linear group, the symbol $\psi_i = (\tau_i, a_i)$ is in one-to-one correspondence with the Speh residual representation $\Delta(\tau_i, a_i)$ attached to the normalized induced representation

$$\mathrm{Ind}_{P_{m_i^{a_i}}(\mathbb{A})}^{\mathrm{GL}_{2n_i}(\mathbb{A})}(\tau_i |\det|^{\frac{a_i - 1}{2}} \otimes \cdots \otimes \tau_r |\det|^{\frac{1 - a_i}{2}}).$$

Define the set of square integrable Arthur parameters

$$\Psi_2(\mathrm{SO}_{2n+1}) = \{\psi = \boxplus_{i=1}^r \psi_i \mid \psi_i = (\tau_i, a_i) \text{ as above } \psi_i \not\cong \psi_j \text{ if } i \neq j\}.$$

Arthur states the following theorem in Chapter 30, [A05].

Theorem 3.1. (Arthur) *The discrete spectrum of* SO_{2n+1} *decomposes into a direct sum in terms of the square integrable Arthur parameters:*

$$L^2_d(\mathrm{SO}_{2n+1}) \cong \oplus_{\psi\in\Psi_2(\mathrm{SO}_{2n+1})} m_\psi(\oplus_{\pi\in\Pi_\psi, m_d(\pi)\neq 0}\pi),$$

where $m_d(\pi)$ *is the discrete multiplicity of* π*, and* m_ψ *is the multiplicity of the members in the global Arthur packet* Π_ψ*, which depends only on* ψ*.*

To define the global Arthur packet Π_ψ, one has to introduce the corresponding local Arthur packets. Let $\psi = \boxplus_{i=1}^r \psi_i$ be an Arthur parameter. For each i, we have $\psi_i = (\tau_i, a_i)$ as above. We write $\tau_i = \otimes_v \tau_{i,v}$. For each local place v of k, by the Generalized Ramanujan conjecture, $\tau_{i,v}$ is tempered. Hence by the local Langlands conjecture for the general linear group (by Langlands for archimedean fields, and by Harris-Taylor and by Henniart for p-adic local fields), $\tau_{i,v}$ is in one-to-one correspondence with a local Langlands parameter $(\rho_{i,v}, b_i)$, which is an m_i-dimensional representation of $W_{k_v} \times \mathrm{SL}_2(\mathbb{C})$, where W_{k_v} is the Weil group of k_v. Hence one obtains the local Arthur parameter $(\rho_{i,v}, b_i, a_i)$, which is a $2n_i$-dimensional representation of $W_{k_v} \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$. By Arthur's conjecture, for each local Arthur parameter

$$\psi_{i,v} = (\rho_{i,v}, b_i, a_i),$$

and more generally,

$$\psi_v = \boxplus_{i=1}^r \psi_{i,v},$$

there is a finite sets of irreducible admissible representations of $\mathrm{SO}_{2n_i+1}(k_v)$ and of $\mathrm{SO}_{2n+1}(k_v)$, which are denoted by $\Pi_{\psi_{i,v}}$ and Π_{ψ_v}, respectively. Note that for almost all finite local places v, the local Arthur packet Π_{ψ_v} contains an irreducible unramified representation, which is denoted by π_v°. Finally the global Arthur packet Π_ψ consists of all

$$\pi = \otimes_v \pi_v$$

with the property that for all v, $\pi_v \in \Pi_{\psi_v}$, and for almost all finite local places v, $\pi_v = \pi_v^\circ$.

It is easy to see that by Arthur's theorem each member π in Π_ψ has weak Langlands transfer from SO_{2n+1} to GL_{2n}, whose image is nearly equivalent to the following automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ associated to the normalized induced representation

$$\mathrm{Ind}_{P_{2n_1,\cdots,2n_r}(\mathbb{A})}^{\mathrm{GL}_{2n}(\mathbb{A})}(\Delta(\tau_1, a_1)\otimes\cdots\otimes\Delta(\tau_r, a_r)).$$

We remark that with the recent progress on the Fundamental Lemmas by Laumon and Ngo, and by B.-C. Ngo, and the work of Waldspurger, it is expected that the complete proof of Arthur's theorem will appear soon.

4 Generic cases

We reformulate the results on generic cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ by Cogdell, Kim, Piatetski-Shapiro, and Shahidi, using the converse theorem and L-function method; by Ginzburg, Rallis, and Soudry using automorphic descent constructions; and by Jiang and Soudry using combination of local and global descents with local and global theta correspondences, in the framework of Arthur's theorem. In order to simplify the statement, we assume that for SO_{2n+1}, there is no irreducible cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ which are isomorphic to a residual representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$. This follows from the Arthur multiplicity theorem for discrete spectrum of SO_{2n+1}. However, it seems that one does not know any alternative proof.

Theorem 4.1. *For $G = \mathrm{SO}_{2n+1}$, let $\psi = \boxplus_{i=1}^r \psi_i = \boxplus_{i=1}^r (\tau_i, a_i)$ as before, and let Π_ψ be the global Arthur packet attached to ψ.*

1. *Π_ψ contains at most one generic member.*
2. *Π_ψ contains a generic member if and only if $a_i = 1$ for $i = 1, 2, \cdots, r$.*
3. *Assume that Π_ψ contains a generic member. For $\pi \in \Pi_\psi$, the second fundamental L-function $L^S(s, \pi, \rho_2)$ has a pole at $s = 1$ with order $r - 1$ if and only if there is a partition $n = \sum_{i=1}^r n_i$ such that π is an endoscopy transfer from the elliptic endoscopy group*

$$\mathrm{SO}_{2n_1+1} \times \cdots \times \mathrm{SO}_{2n_r+1}.$$

We remark that part 1. is a reformulation of the work of Jiang and Soudry ([JngS03]), part 2. is a reformulation of the work of Cogdell, Kim, Piatetski-Shapiro and Shahidi ([CKPSS01]), and part 3. is reformulation of my work ([Jng06a]).

5 Ginzburg-Rallis-Soudry descents

The automorphic descent construction was first discovered by Ginzburg, Rallis and Soudry based on the Rankin-Selberg constructions of the tensor product L-functions for classical groups. It provides special cases of endoscopy descents, which is the inverse map of endoscopy transfer. We only consider the case of the Ginzburg-Rallis-Soudry descent from GL_{2n} to SO_{2n+1}, which is the inverse of the Langlands transfer from SO_{2n+1} to GL_{2n}.

Let τ be an irreducible unitary self-dual cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$. If τ has a descent to SO_{2n+1}, then τ is an image of the Langlands transfer from SO_{2n+1}. Hence it is natural to assume that the exterior square L-function $L^S(s, \tau, \Lambda^2)$ has a pole at $s = 1$.

With the given datum, we can build an Eisenstein series $E(g, \Phi_\tau, s)$ on $\mathrm{SO}_{4n}(\mathbb{A})$ associated to the normalized induced representation

$$\mathrm{I}(s, \tau) = \mathrm{Ind}_{P_{2n}(\mathbb{A})}^{\mathrm{SO}_{4n}(\mathbb{A})}(\tau \otimes |\det|^{\frac{s}{2}}).$$

It is easy to show that this Eisenstein series $E(g, \Phi_\tau, s)$ with the given datum has a simple pole at $s = 1$, whose residue is denoted by $\mathcal{E}_1(g, \Phi_\tau)$. By a theorem of Jacquet and Shalika, τ has a nonzero Shalika period of GL_{2n}. By a theorem of Jiang and Qin, the residue $\mathcal{E}_1(g, \Phi_\tau)$ has a nonzero generalized Shalika period of SO_{4n}. See [JngQ07] for details.

The main idea of the Ginzburg-Rallis-Soudry descents is to analyze the residue $\mathcal{E}_1(g, \Phi_\tau)$ in terms of a family of the generalized Gelfand-Graev periods ([GRS99]). We give some details about the generalized Gelfand-Graev periods below.

Let $(V_{4n}, (\cdot,\cdot))$ be a nondegenerate quadratic vector space over k of dimension $4n$ with Witt index $2n$. The symmetric bilinear form is given by

$$J_{4n} = \begin{pmatrix} & & 1 \\ & J_{4n-2} & \\ 1 & & \end{pmatrix} \tag{5.1}$$

inductively. We may choose a basis

$$\{e_1, \cdots, e_{2n}; e_{-2n}, \cdots, e_{-1}\} \tag{5.2}$$

such that

$$(e_i, e_j) = \begin{cases} 1 & \text{if } j = -i, \\ 0 & \text{if } j \neq -i. \end{cases}$$

For each $r \in \{0, 1, 2, \cdots, 2n-1\}$, we have the following partial polarization

$$V_{4n} = X_r \oplus V_{2(2n-r)} \oplus X_r^* \tag{5.3}$$

where X_r is a totally isotropic subspace of dimension r and X_r^* is the dual of X_r with respect to the non-degenerate bilinear form $(\cdot,\cdot)$, and the subspace $V_{2(2n-r)}$ is the orthogonal complement of $X_r \oplus X_r^*$. Without loss of generality, we may assume that X_r is generated by $e_1, \cdots, e_r$ and X_r^* is generated by $e_{-r}, \cdots, e_{-1}$.

It is clear that $\mathrm{GL}(X_r)$ is isomorphic to GL_r. Let U_r be the standard maximal unipotent subgroup of GL_r. Let $N^r = N^r_{2n}$ be the standard unipotent subgroup of SO_{4n} consisting of elements of type

$$n = \begin{pmatrix} u & x & z \\ & I_{2(2n-r)} & x^* \\ & & u^* \end{pmatrix} \in \mathrm{SO}_{4n}$$

where $u \in U_r$. Let ψ be a nontrivial additive character of $\mathbb{A}$ which is trivial on k. We define a character ψ_r of $N^r(\mathbb{A})$ by

$$\psi_r(n) := \psi(u_{1,2} + \cdots + u_{r-1,r})\psi(x_{r,2n-r} + x_{r,2n-r+1}). \tag{5.4}$$

Let φ be an automorphic form on $\mathrm{SO}_{4n}(\mathbb{A})$. We define

$$\mathcal{F}^{\psi_r}(g; \varphi) := \int_{N^r(k)\backslash N^r(\mathbb{A})} \varphi(ng)\psi_r^{-1}(n)dn \tag{5.5}$$

If the integral is not identically zero, we say that the automorphic form φ has a nonzero ψ_r-Fourier coefficient. If $r = 2n-1$, then the ψ_{2n}-Fourier coefficient $\mathcal{F}^{\psi_{2n}}(g;\varphi)$ is the usual Whittaker-Fourier coefficient.

It is clear that the connected component of the stabilizer of ψ_r is $\mathrm{SO}_{2(2n-r)}$ is $\mathrm{SO}_{2(2n-r)-1}$. Hence the ψ_r-Fourier coefficient $\mathcal{F}^{\psi_r}(g;\varphi)$ is an automorphic form when restricted to $\mathrm{SO}_{2(2n-r)-1}(\mathbb{A})$. When $r=0$, we just restrict φ from $\mathrm{SO}_{4n}(\mathbb{A})$ to $\mathrm{SO}_{4n-1}(\mathbb{A})$.

The Ginzburg-Rallis-Soudry descent of τ from GL_{2n} to SO_{2n+1} is to investigate the ψ_r-Fourier coefficient of the residue $\mathcal{E}_1(g,\Phi_\tau)$.

Theorem 5.1. (Ginzburg-Rallis-Soudry) *When $r \geqslant n$, the ψ_r-Fourier coefficient of $\mathcal{E}_1(g,\Phi_\tau)$ is zero. When $r \leqslant n-1$, the ψ_r-Fourier coefficient of $\mathcal{E}_1(g,\Phi_\tau)$ is not identically zero. Moreover, when $r = n-1$, the ψ_{n-1}-Fourier coefficient of $\mathcal{E}_1(g,\Phi_tau)$ is cuspidal. In this case, as representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$, the space σ generated by all $\mathcal{F}^{\psi_{n-1}}(\mathcal{E}_1(g,\Phi_\tau))$ can be written as a direct sum of irreducible generic cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$:*

$$\sigma = \sigma_1 \oplus \sigma_2 \oplus \cdots \oplus .$$

Then σ_i's are nearly equivalent and whose Langlands functorial transfers to $\mathrm{GL}_{2n}(\mathbb{A})$ are equal to τ.

In a joint work with Soudry [JngS03], we prove that σ is in fact irreducible, which is called the automorphic descent of τ, or the Ginzburg-Rallis-Soudry descent of τ.

We remark that when τ is of orthogonal type, the situation is slightly different, we refer the survey paper of Soudry ([Sd05]) for details. There is a local analogue of such descents. We refer to [GRS05] and [JngNQ] for more details.

6 Beyond the genericity

The existence of non-generic irreducible cuspidal automorphic representation for reductive groups which are not of A_n-type was first discovered by R. Howe and I. Piatetski-Shapiro [HPS79]. They provide the first examples of irreducible cuspidal automorphic representations whose local components are nontempered at almost all local places, i.e. the counter-examples of the generalized Ramanujan conjecture. It turns out that this is a general phenomenon. These cuspidal automorphic representations are called in [PS83] CAP automorphic representations, i.e. cuspidal automorphic representations associated to a certain parabolic subgroup. The reason for this is that these cuspidal automorphic representations locally look like the local components of a residual automorphic representation at almost all local places. The basic structure of the discrete spectrum becomes much more complicate when the group is not of A_n-type, because of the existence of the CAP automorphic representations.

We formulate the following most general conjecture for CAP representations.

Let G be a k-quasisplit reductive algebraic group, and G' be an k-inner form of G. Hence at almost all local places v of k, $G(k_v)$ and $G'(k_v)$ are isomorphic over

k_v. It is known that both G and G' share the same Langlands dual group. By the Langlands conjecture, there exists an Langlands transfer from irreducible automorphic representations π' of $G'(\mathbb{A})$ to irreducible automorphic representations π of $G(\mathbb{A})$.

Conjecture 6.1. (The CAP Conjecture) *Assume that G is k-quasisplit reductive group and G' be a k-inner form of G. For any irreducible cuspidal automorphic representation π' of $G'(\mathbb{A})$, there exist a standard parabolic subgroup $P = MN$ of G, an irreducible generic unitary cuspidal automorphic representation σ of $M(\mathbb{A})$, and an unramified character χ of $M(\mathbb{A})^1\backslash M(\mathbb{A})$, such that π' is nearly equivalent to an irreducible constituent of the unitarily induced representation*

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma \otimes \chi).$$

This conjecture is first called the CAP conjecture in [JngS07a] for the case $G = G'$, although it has been long expected. It is expect that Arhtur's theorem should imply the CAP conjecture, although there are technical details to be carried out. For a given π', the parabolic subgroup P is proper, we called π' a CAP automorphic representation.

In the following we discuss some special cases when the CAP conjecture is known.

By the strong multiplicity one theorem for automorphic representations of $\mathrm{GL}(n)$ of H. Jacquet and J. Shalika ([JS81]), the CAP conjecture holds for irreducible cuspidal automorphic representations for $\mathrm{GL}(n)$, since every cuspidal automorphic representation of $\mathrm{GL}(n)$ is generic. In general, let D be a central division algebra over k of index d. The $G' = \mathrm{GL}_m(D)$ is a k-inner form of $G = \mathrm{GL}_{md}(k)$. When $m = 1$, every irreducible automorphic representation $G'(\mathbb{A})$ is cuspidal. In particular, a one-dimensional automorphic representation is cuspidal, whose local components, however, are the local components of the residue of Eisenstein series on $\mathrm{GL}_d(\mathbb{A})$. Hence any one-dimensional automorphic representation of $D^\times(\mathbb{A})$ is a CAP. In Proposition 5.5, [BG07], A. Badulescu proves the CAP conjecture for $\mathrm{GL}_m(D)$ with assumption that D splits at all archimedean local places.

After the pioneer work of Piatetski-Shapiro on the Saito-Kurokawa lift ([PS83]), CAP automorphic representations have attracted a lot of attentions in the investigation of the basic structures of the discrete spectrum of automorphic forms, relating to the Arthur conjectures. Many more examples have been constructed by means of the theory of theta functions and more recently by other new methods.

We list below some references for the known CAP automorphic representations for each group.

- For $G = \mathrm{GSp}(4)$, [HPS79], [PS83], [PSS87], [Sd90], [Sch05], [Pt06].
- For $G = G_2$, the k-split exceptional group of type G_2, [RS89], [GRS97b], [GJng01], [GnGJ02], [GnS03], [Gn05], [GnG05a], [GnG05b], [G05], [GnG06].
- For $G = \mathrm{SO}_{2n}$, k-split special even orthogonal group, [GJngR02].
- For $G = \mathrm{SO}_{2n+1}$, k-split special old orthogonal group, [JngS07a], [JngS07b].
- For $G = \mathrm{Sp}_{2n}$, k-split symplectic group, [Ik01], [GRS05].
- For $G = U_m$, k-quasisplit unitary group, [GlRS97], [Ik].

We make the following remarks.

Remark 6.1. *It is important to point out that all the know CAP automorphic representations confirms the CAP conjecture.*

Since the CAP conjecture requires that the cuspidal datum in the conjecture is generic, it reduces the Langlands functorial transfer for general cuspidal automorphic representations to the case of irreducible generic cuspidal automorphic representations. For irreducible generic cuspidal automorphic representations, the Langlands functorial transfer for various groups have been or will be established by the Converse Theorem and L-function method, as discussed in §4.

On the other hand, if one proves the existence of the Langlands functorial transfer for all irreducible cuspidal automorphic representations of classical groups to the general linear groups, then the CAP conjecture follows from the Ginzburg-Rallis-Soudry descents and their refinements. Hence the CAP conjecture plus the location of poles of tensor product L-functions in general imply the Arthur Theorem, but we will omit the details here.

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