ON CORRESPONDENCES BETWEEN CERTAIN AUTOMORPHIC FORMS ON $\mathrm{Sp}_{4n}(\mathbb{A})$ AND $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$

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ABSTRACT. In 2005, Ginzburg, Rallis and Soudry constructed, in terms of residues of certain Eisenstein series, and by use of the descent method, families of non-tempered automorphic representations of $\mathrm{Sp}_{4nm}(\mathbb{A})$ and $\widetilde{\mathrm{Sp}}_{2n(2m-1)}(\mathbb{A})$, which generalized the classical work of Piatetski-Shapiro on Saito-Kurokawa liftings. In this paper, we introduce a new framework (Diagrams of Constructions) in order to establish explicit relations among the representations introduced in [GRS05]. In particular, we prove that these constructions yield bijections between a certain set of cuspidal automorphic forms on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ and a certain set of square-integrable automorphic forms of $\mathrm{Sp}_{4n}(\mathbb{A})$. The proofs use new interpretations of composition of two consecutive descents with explicit identities, which we expect to be very useful to further investigation of the automorphic discrete spectrum of classical groups.

1. INTRODUCTION

Let τ be an irreducible, unitary, cuspidal, automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$, where $\mathbb{A}$ is the Adele ring of a number field F. Assume that the partial exterior square L-function $L^S(s,\tau,\wedge^2)$ has a pole at $s=1$ and the partial standard L-function $L^S(s,\tau)$ is nonzero at $s=\frac{1}{2}$. Fix a nontrivial character ψ of $F\backslash\mathbb{A}$. The automorphic descent method ([GRS99b]) constructs a genuine, cuspidal, automorphic representation $\widetilde{\pi}_\psi(\tau)$ of the double cover $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ of $\mathrm{Sp}_{2n}(\mathbb{A})$. This representation is irreducible, lifts weakly to τ with respect to ψ, and is ψ^{-1}-globally generic. (See Sec. 2.2 for the precise definition.) See [S05] for a survey on the descent method. A complete account of this theory appears in the book [GRS11]. The irreducibility of $\widetilde{\pi}_\psi(\tau)$ follows from [JS03]. We note that the conditions on the partial L-functions can be replaced by the corresponding conditions on the complete L-functions. This can be easily checked by using the bounds of the real exponents of the elements in the local unitary dual of GL_{2n} ([Vd86], [Tm86]). See also [Kh99], Prop. 3.4, for a proof that the local exterior square L-function of irreducible unitary generic representations of GL_{2n} is holomorphic at $\mathrm{Re}(s)\geq 1$.

The representation $\widetilde{\pi}_\psi(\tau)$ is in fact obtained as follows. Let $\mathcal{E}_\tau$ be the residual representation of $\mathrm{Sp}_{4n}(\mathbb{A})$ generated by the residues of the Eisenstein series corresponding to

$$\mathrm{Ind}_{P^{4n}_{2n}(\mathbb{A})}^{\mathrm{Sp}_{4n}(\mathbb{A})}(\tau|\det|^{\frac{1}{2}}), \tag{1.1}$$

where P^{4n}_{2n} is the standard Siegel parabolic subgroup. Our assumptions on τ imply that this residue exists. The ψ-descent $\widetilde{\pi}_\psi(\tau)$ is the representation by right translation acting in the space generated by the Fourier-Jacobi coefficients $\mathrm{FJ}^{\psi}_{\phi,n}$ applied to $\mathcal{E}_\tau$, viewed as

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automorphic functions on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. See Sec. 2.2 for the definition. The proof that $\widetilde{\pi}_\psi(\tau)$ is cuspidal depends just on the fact that at one finite place v, where the local v-components τ_v and ψ_v are unramified, τ_v is self dual and has a trivial central character. This fact is also sufficient to determine the Satake parameters, with respect to ψ_v, of each irreducible constituent of $\widetilde{\pi}_\psi(\tau)$, at any such unramified place v. See Section 2 of [GRS99c] for definition and detailed discussion, and in particular, the fact that these Satake parameters correspond to those of τ_v, by means of the local Langlands functorial ψ-lift to $\mathrm{GL}_{2n}(F_v)$.

The fact that $\mathcal{E}_\tau$ is the residual representation from the Eisenstein series with the given cuspidal datum is used in the proof of the nonvanishing of the descent $\widetilde{\pi}_\psi(\tau)$. See [GRS99a], [GRS99b], and [GRS02]. What we just explained shows that if we replace $\mathcal{E}_\tau$ by an irreducible, automorphic representation π, nearly equivalent to (1.1), and apply the same Fourier-Jacobi coefficients, then we get an automorphic representation $\widetilde{\pi}$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which is cuspidal. Furthermore, each irreducible subrepresentation of $\widetilde{\pi}$ lifts weakly, with respect to ψ, to τ. In order that $\widetilde{\pi}$ be nonzero, we simply require that the coefficients $\mathrm{FJ}^{\psi}_{\phi,n}$ are not identically zero on the space of π. Thus, we can repeat the descent construction for such π, as we did for $\mathcal{E}_\tau$. We denote $\widetilde{\pi} = \widetilde{\mathcal{D}}^{4n}_{2n,\psi}(\pi)$. We also call this representation the ψ-descent of π from $\mathrm{Sp}_{4n}(\mathbb{A})$ to $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$.

When we consider an irreducible, cuspidal automorphic representation $\widetilde{\pi}$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which admits a ψ-weak lift to $\mathrm{GL}_{2n}(\mathbb{A})$, to an irreducible, unitary, cuspidal automorphic representation τ, then, clearly, τ must be self dual and have trivial central character. One expects that the exterior square L-function of τ has a pole at $s = 1$. (This is proved in [GJRS09] when $\widetilde{\pi}$ is globally generic; see also [GJRS09], Theorem 5.2, for a generalization.) So, we will keep this condition in our assumptions. However, we relax the condition on the nonvanishing of the central value of the standard L-function of τ, and demand that there exists a twist of τ by a quadratic character, such that the central value of the twisted standard L-function of τ is nonzero. See (3.1).

Assume then that τ is such a representation. In Sec. 3, we consider two sets of irreducible, automorphic representations associated to τ and ψ. The first set

$$\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$$

is the set of irreducible, genuine, cuspidal automorphic representations of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which have a weak lift with respect to ψ to $\mathrm{GL}_{2n}(\mathbb{A})$, with image τ. The second set

$$\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$$

is the set of all irreducible, automorphic representations π of $\mathrm{Sp}_{4n}(\mathbb{A})$, which occur in the discrete automorphic spectrum, are nearly equivalent to (1.1), and are such that the coefficients $\mathrm{FJ}^{\psi}_{\phi,n}$ are not identically zero on the space of π. In both these sets the representations are counted with their multiplicities. Note that by [A05], these multiplicities are expected to be identically one. However, we do not use the multiplicity one property in this paper. Instead, we prove a weak version of multiplicity one for $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, as a consequence of our proofs (Theorem 4.6).

Consider the set $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$, which consists of the cuspidal members of $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$ and also the residual representation $\mathcal{E}_\tau$, when it exists. We expect that the sets $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$ and $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$ are equal (as a consequence of [A05]). We prove in Sec. 3 that the sets $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$ and $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$ are not empty. Our first main theorem is

Theorem 1.1. *Let $\pi \in \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$. Then the ψ-descent of π from $\mathrm{Sp}_{4n}(\mathbb{A})$ to $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$*

$$\Psi'(\pi) = \widetilde{\mathcal{D}}^{4n}_{2n,\psi}(\pi),$$

is an irreducible representation, which lies in $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$.

This is Theorem 4.1 and it is proved in Sec. 6. This theorem generalizes the special case when $\pi = \mathcal{E}_\tau$, mentioned in the beginning, and provides another proof for the irreducibility of $\widetilde{\pi}_\psi(\tau)$ ([JS03]).

We remark that in previous work on automorphic descents ([GRS99a], [GRS99b], [GRS02], [GRS05], and [G08]), the descent constructions produced only distinguished members in near equivalence classes of automorphic representations. However, in Theorem 1.1 and the other main results in this paper, we obtain more general (if not all) members in the near equivalence classes. Thus, the automorphic descent Ψ' defines a map

$$\Psi' : \ \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi) \mapsto \ \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi). \tag{1.2}$$

Our second main theorem is

Theorem 1.2. *The map Ψ' is surjective.*

This theorem follows from Theorem 4.2, which is proved in Sec. 5. In order to prove the surjectivity of Ψ', we define a sort of inverse map Φ from $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$. For $\widetilde{\pi} \in \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$, $\Phi(\widetilde{\pi})$ is a direct sum of elements in $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$. By definition,

$$\Phi(\widetilde{\pi}) = \mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}), \tag{1.3}$$

the ψ^{-1}-descent, from $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$ to $\mathrm{Sp}_{4n}(\mathbb{A})$, of the residual representation $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ corresponding to the ψ-parabolic induction from $\tau|\det|^1 \otimes \widetilde{\pi}$ (on the Levi subgroup $\mathrm{GL}_{2n} \times \mathrm{Sp}_{2n}$ of Sp_{6n}) to $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$ (see Sec. 2.3 for the details). This construction can be visualized by the following diagram:

$$\begin{array}{ccccc}
\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}} & \widetilde{\mathrm{Sp}}_{6n} & & & \\
 & & \searrow \mathcal{D}^{6n}_{4n,\psi^{-1}} & & \\
 & \mathrm{Res} \uparrow & & \mathrm{Sp}_{4n} & \pi \in \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi) \\
 & & {}^{\Phi} \nearrow \swarrow {}_{\Psi' = \widetilde{\mathcal{D}}^{4n}_{2n,\psi}} & & \\
\widetilde{\pi} \in \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi) & \widetilde{\mathrm{Sp}}_{2n} & & &
\end{array} \tag{1.4}$$

where Res denotes the mapping which takes $\widetilde{\pi}$ on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ to the residual representation $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ of $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$. In Theorem 4.2, we prove that

$$\Psi'(\Phi(\tilde{\pi})) = \tilde{\pi}.$$

Moreover, Ψ' is nontrivial on each irreducible subrepresentation of $\Phi(\tilde{\pi})$. This means that for each irreducible subrepresentation π of $\Phi(\tilde{\pi})$, we have

$$\Psi'(\pi) = \widetilde{\pi}.$$

Our third main theorem is

Theorem 1.3. *For each* $\pi \in \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau, \psi)$*, we have*

$$\pi \subset \Phi(\Psi'(\pi)),$$

that is, π *is a subrepresentation of* $\Phi(\Psi'(\pi))$*.*

This is Theorem 4.3 and is proved in Sec. 6. If we add the assumption (Assumption (A) in §4) that the residual representations $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ are irreducible, then we can prove that

$$\pi = \Phi(\Psi'(\pi))$$

in Theorem 1.3. This is Theorem 4.4, which is proved in Sec. 6. Note that Assumption (A) is true when $\widetilde{\pi}$ is in addition generic. With Assumption (A), Φ defines a mapping

$$\Phi : \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi) \mapsto \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau, \psi), \tag{1.5}$$

and the theorems above can be summarized together by

Theorem 1.4. *Suppose that* Assumption (A) *in §4 holds. Then* Φ *is the inverse mapping to* Ψ'*, and hence Diagram* (1.4) *is commutative.*

Thus, the set $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$ parameterizes the set $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau, \psi)$, by means of the ψ-descent mapping from $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$ to $\mathrm{Sp}_{4n}(\mathbb{A})$, composed with the mapping which associates the residual representation $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ to $\widetilde{\pi} \in \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$. Note that in the case where $L^S(\frac{1}{2}, \tau) = 0$, the set $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau, \psi)$ (is nonempty and) consists of cuspidal representations. These are CAP representations with respect to (1.1), and are straightforward generalizations of the Saito-Kurokawa representations of $\mathrm{Sp}_4(\mathbb{A})$, constructed by Piatetski-Shapiro. See [PS83].

The mapping Φ was considered in a more general setup in [GRS05], where a construction of certain CAP representations was suggested (assuming the vanishing of certain Fourier coefficients on the construction data (Theorems A, B, C in [GRS05]). In particular, the existence of the CAP representations of $\mathrm{Sp}_{4n}(\mathbb{A})$ just mentioned was obtained there. The new idea in this paper is to consider the explicit relations between the map Φ, which was introduced in [GRS05] and the automorphic descent Ψ' introduced in [GRS99a]. In particular, from Theorems 1.1 - 1.3, we deduce the irreducibility and the surjectivity properties of the map Ψ', and the analogs for Φ. We picture this in the following diagram of constructions, which is an extended diagram of Diagram (1.4):

$$
\begin{array}{ccccc}
 & & & \mathrm{Sp}_{8n} & \mathcal{E}_{\tau,\pi} \\
 & & \swarrow {\scriptstyle \widetilde{\mathcal{D}}^{8n}_{6n,\psi}} & & \\
\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}} & \widetilde{\mathrm{Sp}}_{6n} & & \mathrm{FC} \downarrow\uparrow \mathrm{Res} & \\
 & & \searrow {\scriptstyle \mathcal{D}^{6n}_{4n,\psi^{-1}}} & & \\
 & \mathrm{Res} \uparrow & & \mathrm{Sp}_{4n} & \pi \in \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi) \\
 & & {\scriptstyle \Phi} \nearrow\swarrow {\scriptstyle \Psi'=\widetilde{\mathcal{D}}^{4n}_{2n,\psi}} & & \\
\widetilde{\pi} \in \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi) & \widetilde{\mathrm{Sp}}_{2n} & & &
\end{array}
\tag{1.6}
$$

In Diagram (1.6), we want to show that the composition of the descent $\widetilde{\mathcal{D}}^{8n}_{6n,\psi}$ with the descent $\mathcal{D}^{6n}_{4n,\psi^{-1}}$ on the residual representation $\mathcal{E}_{\tau,\pi}$, with cuspidal datum

$$(\mathrm{GL}_{2n} \times \mathrm{Sp}_{4n}, \tau|\det|^{\frac{3}{2}} \otimes \pi),$$

essentially recovers π on $\mathrm{Sp}_{4n}(\mathbb{A})$, where $\mathrm{GL}_{2n} \times \mathrm{Sp}_{4n}$ is the Levi part of the standard maximal parabolic subgroup P^{8n}_{2n} of Sp_{8n}. More precisely, the $\mathrm{Sp}_{4n}(\mathbb{A})$-module $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{D}}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi}))$ is equal to the $\mathrm{Sp}_{4n}(\mathbb{A})$-module obtained by first applying to $\mathcal{E}_{\tau,\pi}$ the constant term along P^{8n}_{2n}, which results with the $\mathrm{GL}_{2n}(\mathbb{A}) \times \mathrm{Sp}_{4n}(\mathbb{A})$-module $\delta^{\frac{1}{2}}_{P^{8n}_{2n}}|\det\cdot|^{-\frac{3}{2}}\tau \otimes \pi$, and then taking the Whittaker coefficient on the first factor (that is $\delta^{\frac{1}{2}}_{P^{8n}_{2n}}|\det\cdot|^{-\frac{3}{2}}\tau$). Clearly, the $\mathrm{Sp}_{4n}(\mathbb{A})$-module thus obtained is π. The composition of the last two maps (constant term and Whittaker coefficient) is marked by the vertical arrow pointing down FC. The more precise relations among all the mappings in Diagram (1.6) are stated as two identities in Theorems 5.1 and 5.4. These identities are new and crucial to the proofs of the main results of this paper. They can be viewed as generalizations of the identity (5.27) in [GRS99b], which computes the ψ^{-1}-Whittaker coefficient of the ψ^{-1}-descent of $\mathcal{E}_\tau$ (assuming it exists). The idea of the proofs are very similar, but the arguments are more technical.

It is clear that Diagram (1.6) can be extended to include the groups $\widetilde{\mathrm{Sp}}_{4mn+2n}$ and Sp_{4mn} for all possible integers m, and we may study the relations among all the descent constructions in various sub-diagrams. At this point, we do not have a good general formulation for such explicit relations and, also, we still face some delicate problems in proving such relations. In this paper we study Diagram (1.6). Further properties of Diagram (1.6) and its analogues for other classical groups have been considered by the authors, and will result in explicit constructions of automorphic representations of endoscopic type, including the completion of proofs, regarding the endoscopic transfers for generic cuspidal representations of classical groups, as started in [G08]. These will appear in forthcoming works of ours.

Finally, as a consequence, we obtain the following interesting applications.

Theorem 1.5. *Assume that $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$ contains a ψ'-generic representation $\widetilde{\pi}$. Then the multiplicity of $\widetilde{\pi}$ in the subspace of ψ'-generic cusp forms on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ is one.*

This is Theorem 4.6, and is proved in Sec. 4.

Although in this paper we restrict attention to irreducible, unitary, cuspidal automorphic representations τ of $\mathrm{GL}_{2n}(\mathbb{A})$, we can repeat our results with representations τ, which are isobaric sums of mutually different, irreducible, unitary, cuspidal automorphic representations

$$\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r,$$

such that the exterior square L-functions $L^S(s, \tau_i, \wedge^2)$ has a pole at $s = 1$, for every $i = 1, 2, \ldots, r$, and there is a quadratic character χ, so that $L^S(\frac{1}{2}, \tau_i \otimes \chi) \neq 0$, for all $i = 1, 2, \ldots, r$. Note that in general, it is a hard and important question to show that the central values $L^S(\frac{1}{2}, \tau_i \otimes \chi)$ for $i = 1, 2, \cdots, r$ are simultaneously nonzero for some quadratic character χ. This could be interpreted in terms of global theta correspondences below. Now, let us define $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$ similarly. We know that every genuine, irreducible, cuspidal automorphic representation $\widetilde{\pi}$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which is globally generic with respect to some Whittaker character, lifts weakly to an automorphic representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$. Using the theta correspondence to $\mathrm{SO}_{2n+1}(\mathbb{A})$ (e.g. [F95], [JS07]), this follows from [CKPSS04]. If $\widetilde{\pi}$ does not come from the split orthogonal group in $2n-1$ variables in the theta correspondence, then τ is of the form above. Now we can repeat Theorem 1.5. This will prove

Theorem 1.6. *Let $\widetilde{\pi}$ be a genuine, irreducible, ψ-generic, cuspidal automorphic representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. Assume that $\widetilde{\pi}$ does not appear in the ψ-theta correspondence from (split) $\mathrm{SO}_{2n-1}(\mathbb{A})$. Then the multiplicity of $\widetilde{\pi}$ is the subspace of ψ-generic cusp forms on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ is one.*

We note that $\widetilde{\pi}$ in the theorem can not appear in the ψ'-theta correspondence to $\mathrm{SO}_{2n-1}(\mathbb{A})$, for any other ψ'. We will address the details of the last theorem in the future.

To end this introduction, we would like to thank the referee for his careful reading and his useful comments.

2. Fourier coefficients

Let F be an algebraic number field and let $\mathbb{A}$ be the ring of Adeles of F. Let Sp_{2k} be the symplectic group of rank k, regarded as an algebraic group over F. It is realized in matrices as follows:

$$\mathrm{Sp}_{2k} := \{g \in \mathrm{GL}_{2k} \mid {}^t g \cdot J_{2k}^- \cdot g = J_{2k}^-\}$$

Here J_{2k}^- is defined by

$$J_{2k}^- := \begin{pmatrix} 0 & J_k \\ -J_k & 0 \end{pmatrix},$$

where J_k is the $k \times k$-matrix defined inductively by $J_k := \begin{pmatrix} 0 & 1 \\ J_{k-1} & 0 \end{pmatrix}$ and $J_1 := (1)$.

Next, we recall the Fourier coefficients of automorphic forms of $\mathrm{Sp}_{2k}(\mathbb{A})$ associated with unipotent orbits of Sp_{2k}, and then consider certain Fourier coefficients, which will be used in this paper.

2.1. **Fourier coefficients associated to unipotent orbits.** Over the algebraic closure $\overline{F}$ of F, unipotent orbits in $\mathrm{Sp}_{2k}(\overline{F})$ correspond to symplectic partitions of $2k$, i.e. to partitions having the property that odd numbers occur with even multiplicity. To a given symplectic partition $\underline{p}$ of $2k$, denote the corresponding orbit by $\mathcal{O}_{\underline{p}}$. The set of F-rational points of $\mathcal{O}_{\underline{p}}$

is a union of unipotent orbits in $\mathrm{Sp}_{2k}(F)$. We will denote the set of these orbits by $\mathcal{O}_{\underline{p}}(F)$ and call its elements F-stable unipotent orbits associated to $\underline{p}$.

Let π be an automorphic representation of $\mathrm{Sp}_{2k}(\mathbb{A})$. As in [GRS03], but with a slightly different notation, for each unipotent orbit $\mathcal{O}$ in $\mathrm{Sp}_{2k}(F)$, the Fourier coefficient of $\varphi_\pi \in \pi$, attached to $\mathcal{O}$, is defined by

$$\mathcal{F}^{\psi_{\mathcal{O}}}(\varphi_\pi) := \int_{V_{\mathcal{O}}(F)\backslash V_{\mathcal{O}}(\mathbb{A})} \varphi_\pi(v)\psi_{\mathcal{O}}(v)dv. \tag{2.1}$$

Here, $V_{\mathcal{O}}$ is the F-unipotent subgroup attached to $\mathcal{O}$ and $\psi_{\mathcal{O}}$ is a character of $V_{\mathcal{O}}(\mathbb{A})$, attached to $\mathcal{O}$, which is trivial on $V_{\mathcal{O}}(F)$, and depends on a choice of a nontrivial character ψ of $F\backslash\mathbb{A}$, which we now fix. (See [GRS03] for the precise definitions; there, $V_{\mathcal{O}}$ is denoted by $V_2(\mathcal{O})$.) Note that $V_{\mathcal{O}}$ depends only on the symplectic partition, while $\psi_{\mathcal{O}}$ depends on the orbit $\mathcal{O}$. In the sequel, for specific Fourier coefficients, we will distinguish the various characters according to the corresponding orbits. If $\mathcal{F}^{\psi_{\mathcal{O}}}(\varphi_\pi)$ is nonzero for some $\varphi_\pi \in \pi$, we say that π has a nonzero $\psi_{\mathcal{O}}$-Fourier coefficient. Similar notions and notation hold for the metaplectic group $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$.

2.2. **Fourier coefficients corresponding to certain partitions.** In this paper, we consider two families of Fourier coefficients.

The first family is attached to the symplectic partitions $[(2r)1^{2(k-r)}]$ of $2k$ (that is the partition of $2k$ whose parts are $2r$ and 1 repeated $2(k-r)$ times). Let V_r^{2k} be the unipotent subgroup $V_{[(2r)1^{2(k-r)}]}$ attached to the partition $[(2r)1^{2(k-r)}]$. In matrices, the elements of V_r^{2k} can be written as follows,

$$v = v(u,x,z) = \begin{pmatrix} u & x & z \\ & \mathrm{I}_{2(k-r)} & x' \\ & & u^* \end{pmatrix} \in \mathrm{Sp}_{2k}, \tag{2.2}$$

where $u \in U_r$, which is the maximal upper unipotent subgroup of GL_r, and $x \in \mathrm{Mat}_{r\times 2(k-r)}$ is such that its r-th row is zero. The $\mathrm{Sp}_{2k}(F)$-orbits corresponding to the partition $[(2r)1^{2(k-r)}]$ are parametrized by square classes $(F^*)^2\backslash F^*$. For a representative a of such a square class, the corresponding character $\psi_{V_r^{2k},a}$, attached to the partition $[(2r)1^{2(k-r)}]$, is given by

$$\psi_{V_r^{2k},a}(v(u,x,z)) := \psi(u_{1,2} + \cdots + u_{r-1,r} + az_{r,1}). \tag{2.3}$$

For an automorphic representation π of $\mathrm{Sp}_{2k}(\mathbb{A})$, the set of Fourier coefficients, attached to this partition, is parametrized by

$$\mathcal{F}^{\psi_{V_r^{2k},a}}(\varphi_\pi) := \int_{V_r^{2k}(F)\backslash V_r^{2k}(\mathbb{A})} \varphi_\pi(v)\psi_{V_r^{2k},a}(v)dv, \tag{2.4}$$

where a varies over a set of representatives of square classes in F^*. Here, φ_π is a vector in the space of π. In the special case when $r = k$, V_k^{2k} is the standard maximal unipotent subgroup of Sp_{2k}, and $\mathcal{F}^{\psi_{V_r^{2k},a}}(\varphi_\pi)$ is the Whittaker-Fourier coefficient of φ_π with respect to the Whittaker character (see (2.3)) of $V_k^{2k}(\mathbb{A})$, which we denote by ψ^a,

$$\psi^a(v) = \psi(v_{1,2} + \cdots + v_{k-1,k} + av_{k,k+1}), \quad v \in V_k^{2k}(\mathbb{A}).$$

Thus, the ψ^a-Whittaker coefficient corresponds to the partition $[(2k)]$. If the ψ^a-Whittaker coefficient is nontrivial on π, we say that π is ψ^a-generic. As usual, these notions apply to metaplectic groups as well.

The second family of Fourier coefficients that we consider is attached to the symplectic partition $[(2n)^2 1^{2r}]$ of $4n+2r$. In Sp_{4n+2r}, $V_{[(2n)^2 1^{2r}]}$ is an F-subgroup of the unipotent radical of the standard parabolic subgroup of Sp_{4n+2r}, whose Levi part is $\mathrm{GL}_2^n \times \mathrm{Sp}_{2r}$. To define characters of $V_{[(2n)^2 1^{2r}]}$, we identify

$$V_{[(2n)^2 1^{2r}]}/[V_{[(2n)^2 1^{2r}]}, V_{[(2n)^2 1^{2r}]}] \cong \mathrm{Mat}_{2\times 2}^{n-1} \times \mathrm{Mat}_{2\times 2}^{0}, \tag{2.5}$$

with

$$\mathrm{Mat}_{2\times 2}^{0} = \{A \in \mathrm{Mat}_{2\times 2} \ : \ J_2 A = A^t J_2\}.$$

The Levi subgroup $\mathrm{GL}_2^n \times \mathrm{Sp}_{2r}$ acts on $V_{[(2n)^2 1^{2r}]}/[V_{[(2n)^2 1^{2r}]}, V_{[(2n)^2 1^{2r}]}]$ by

$$(h_1, \cdots, h_n; g) \circ (X_1, \cdots, X_{n-1}; X_0) = (h_1 X_1 h_2^{-1}, \cdots, h_{n-1} X_{n-1} h_n^{-1}; h_n X_0 {}^t h_n). \tag{2.6}$$

Representatives of generic $\mathrm{GL}_2^n(F) \times \mathrm{Sp}_{2r}(F)$-orbits (i.e. the F-stable orbits, which correspond to the Zariski open orbit over $\overline{F}$) on the quotient

$$V_{[(2n)^2 1^{2r}]}(F)/[V_{[(2n)^2 1^{2r}]}(F), V_{[(2n)^2 1^{2r}]}(F)]$$

are given by

$$(\mathrm{I}_2, \cdots, \mathrm{I}_2; \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}), \tag{2.7}$$

where $a, b \in (F^*)^2 \backslash F^*$ (We often identify an element of a square class with the square class itself.) Define a character of $V_{[(2n)^2 1^{2r}]}(\mathbb{A})$ as follows. Given $v \in V_{[(2n)^2 1^{2r}]}(\mathbb{A})$, let v_1 be its projection on $V_{[(2n)^2 1^{2r}]}(\mathbb{A})/[V_{[(2n)^2 1^{2r}]}(\mathbb{A}), V_{[(2n)^2 1^{2r}]}(\mathbb{A})]$. Identifying v_1 with an element $(X_1, \cdots, X_{n-1}; X_0)$, as above, we define

$$\psi_{[(2n)^2 1^{2r}]; b,a}(v) := \psi(\mathrm{tr}(X_1 + \cdots + X_{n-1} + \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} X_0)). \tag{2.8}$$

For an automorphic representation π of $\mathrm{Sp}_{4n+2r}(\mathbb{A})$, the set of Fourier coefficients attached to the partition $[(2n)^2 1^{2r}]$ is given by (notation as before)

$$\mathcal{F}^{\psi_{[(2n)^2 1^{2r}]; b,a}}(\varphi_\pi) := \int_{V_{[(2n)^2 1^{2r}]}(F) \backslash V_{[(2n)^2 1^{2r}]}(\mathbb{A})} \varphi_\pi(v) \psi_{[(2n)^2 1^{2r}]; b,a}(v) dv. \tag{2.9}$$

2.3. **Certain residues of Eisenstein series.** Let τ be an irreducible, unitary, cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$, and let $\widetilde{\pi}$ be an irreducible, genuine, cuspidal automorphic representation of the metaplectic group $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. Assume that $\widetilde{\pi}$ has a weak functorial lift, with respect to ψ, to the representation τ. Recall that there is no canonical way to determine Satake parameters of $\widetilde{\pi}$ at the places v, which are "unramified" for $\widetilde{\pi}$. For such a place v, we know that $\widetilde{\pi}_v$ is a constituent of a representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, induced from a character of the inverse image of the standard Borel subgroup, $\widetilde{B}_v$. In order to write such a character, we will use the Weil factor γ_{ψ_v}, associated to ψ_v. It is a function on F_v^*, taking values in the group of fourth roots of unity, and satisfies

$$\gamma_{\psi_v}(xy) = \gamma_{\psi_v}(x)\gamma_{\psi_v}(y)(x,y), \quad x, y \in F_v^*,$$

where $(,)$ is the Hilbert symbol of F_v. Now the character above of $\widetilde{B}_v$ is determined on the inverse image of the diagonal subgroup, on which it is of the form

$$(\mathrm{diag}(t_1, ..., t_n, t_n^{-1}, ..., t_1^{-1}), \epsilon) \mapsto \epsilon \gamma_{\psi_v}(t_1 \cdot ... \cdot t_n) \mu_1(t_1) \cdot ... \cdot \mu_n(t_n),$$

where $\mu_1, ..., \mu_n$ are unramified characters of F_v^*. Here we realized $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, using the Rao cocycle. See, for example [JS07]. When we say that $\widetilde{\pi}$ has a weak functorial lift, with respect to ψ, to the representation τ, we mean that at almost all unramified places v, as above, τ_v is the unramified constituent of the representation of $\mathrm{GL}_{2n}(F_v)$ induced from the standard Borel subgroup and the following character determined by $\widetilde{\pi}_v$, given (with the same notation as above) on the diagonal subgroup by

$$\mathrm{diag}(t_1, ..., t_{2n}) \mapsto \mu_1(t_1) \cdot ... \cdot \mu_n(t_n)\mu_n^{-1}(t_{n+1}) \cdot ... \cdot \mu_1^{-1}(t_{2n}).$$

It is clear that τ_v is self dual, with trivial central character, at almost all places v, and, hence, by the strong multiplicity one property of automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A})$, we get that $\tau = \hat{\tau}$ is self-dual. Similarly, τ has a trivial central character. Let S be any finite set of places of F, containing those at infinity, outside which $\widetilde{\pi}_v$ is unramified, v is odd and ψ_v is normalized. Then we have the equality of the partial tensor product L-functions

$$L_\psi^S(s, \widetilde{\pi} \times \tau) = L^S(s, \tau \times \tau),$$

and we conclude that $L_\psi^S(s, \widetilde{\pi} \times \tau)$ has a pole at $s = 1$. Indeed, since τ is self-dual, $L^S(s, \tau \times \tau)$ has a pole at $s = 1$. As before, once we fixed ψ, then we can associate Satake parameters to $\widetilde{\pi}$ at all places outside S. By definition, for v outside S, the Satake parameter of $\widetilde{\pi}_v$, is the conjugacy class of the following element in $\mathrm{Sp}_{2n}(\mathbb{C})$ (using the notation above)

$$\mathrm{diag}(\mu_1(p), ..., \mu_n(p), \mu_n^{-1}(p), ..., \mu_1^{-1}(p)),$$

where p is a generator of the prime ideal of the ring of integers in F_v. This enables us to define L-functions for metaplectic groups. See [GRS98] and [GJRS09], where certain global integrals of Shimura type are introduced and shown to represent the standard partial L-functions with respect to ψ, $L_\psi^S(s, \widetilde{\pi} \times \sigma)$, for any irreducible, automorphic, cuspidal representation σ of $\mathrm{GL}_k(\mathbb{A})$ (any k).

Consider an Eisenstein series on $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$, $\widetilde{E}(g, \phi_{\tau,\widetilde{\pi};s})$, corresponding to a holomorphic section $\phi_{\tau,\widetilde{\pi};s}$ in

$$\mathrm{Ind}_{\widetilde{P}_{2n}^{6n}(\mathbb{A})}^{\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})}(\gamma_\psi \cdot \tau |\det|^s \otimes \widetilde{\pi}), \tag{2.10}$$

where P_{2n}^{6n} is the standard parabolic subgroup of Sp_{6n}, whose Levi part is isomorphic to $\mathrm{GL}_{2n} \times \mathrm{Sp}_{2n}$. By considering the constant term along P_{2n}^{6n}, it is easy to see that $\widetilde{E}(g, \phi_{\tau,\widetilde{\pi};s})$ has a pole at $s = 1$. Denote by $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ the corresponding residual representation of $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$.

The following theorem on Fourier coefficients of the residual representation $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ is crucial to this paper.

Theorem 2.1. *With notation as in the last paragraph, for all integers l, such that $n < l \leq 3n$, the residual representation $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ has no nonzero Fourier coefficient attached to the symplectic partition*

$$[(2l)1^{2(3n-l)}].$$

Also, $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ has a nonzero Fourier coefficient associated with any choice of representative of the unipotent orbit $[(2n)1^{4n}]$, i.e. for all $a \in F^*$, the Fourier coefficient $\mathcal{F}^{\psi_{V_n^{6n},a}}$, defined in (2.4), is nontrivial on $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$.

Proof. The fact that $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ has no nonzero Fourier coefficient corresponding to any unipotent orbit attached to $[(2l)1^{2(3n-l)}]$, for $n < l \leq 3n$, follows from Lemma 3.1(2) and Lemma 3.3

in [GRS05] (for $k=1$). Thus, it remains to prove the statement about the orbit $[(2n)1^{4n}]$. We need to prove that for all $a\in F^*$, the integral

$$\mathcal{F}^{\psi_{V_n^{6n},a}}(\widetilde{\xi}_{\tau,\widetilde{\pi}})=\int_{V_n^{6n}(F)\backslash V_n^{6n}(\mathbb{A})}\widetilde{\xi}_{\tau,\widetilde{\pi}}(v)\psi_{V_n^{6n},a}(v)dv \tag{2.11}$$

is not identically zero, as $\widetilde{\xi}_{\tau,\widetilde{\pi}}$ varies in $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$. This is equivalent, by Lemma 1.1 in [GRS03], to the nonvanishing of the following Fourier coefficient on $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$

$$\mathcal{F}^{\psi_{(V')_n^{6n},a}}(\widetilde{\xi}_{\tau,\widetilde{\pi}})=\int_{(V')_n^{6n}(F)\backslash (V')_n^{6n}(\mathbb{A})}\widetilde{\xi}_{\tau,\widetilde{\pi}}(v)\psi_{(V')_n^{6n},a}(v)dv, \tag{2.12}$$

where $(V')_n^{6n}$ is the group of the following elements in Sp_{6n},

$$v(u,x,z)=\begin{pmatrix}u & x & z\\ & \mathrm{I}_{4n} & x'\\ & & u^*\end{pmatrix},$$

where $u\in U_n$, $x\in \mathrm{Mat}_{n\times 4n}$ is such that $x_{n,1}=\cdots=x_{n,2n}=0$, and

$$\psi_{(V')_n^{6n},a}(v(u,x,z))=\psi(u_{1,2}+\cdots+u_{n-1,n}+az_{n,1}).$$

Thus, the nonvanishing, for some choice of data, of the Fourier coefficient (2.11) is equivalent to that of the following coefficient

$$\mathcal{F}^{\psi_{\widetilde{V}_n^{6n},a}}(\widetilde{\xi}_{\tau,\widetilde{\pi}})=\int_{\widetilde{V}_n^{6n}(F)\backslash \widetilde{V}_n^{6n}(\mathbb{A})}\widetilde{\xi}_{\tau,\widetilde{\pi}}(v)\psi_{\widetilde{V}_n^{6n},a}(v)dv, \tag{2.13}$$

where $\widetilde{V}_n^{6n}$ is the subgroup of $v(u,x,z)\in (V')_n^{6n}$, such that $x_{n,2n+1}=\cdots=x_{n,3n}=0$, and $\psi_{\widetilde{V}_n^{6n},a}$ is defined by restriction of $\psi_{(V')_n^{6n},a}$ to $\widetilde{V}_n^{6n}(\mathbb{A})$. Indeed, if (2.13) is identically zero, then, being an inner integration of (2.12), we see that (2.12) is identically zero. Conversely, if (2.12) is identically zero, then (2.11) is identically zero, and this being an inner integration of (2.13), we see that (2.13) is identically zero. The nontriviality of the coefficient (2.13) will follow from the nontriviality of the following Fourier coefficient on our residual representation.

$$\int_{V_n^{4n}(F)\backslash V_n^{4n}(\mathbb{A})}\int_{\widetilde{V}_n^{6n}(F)\backslash \widetilde{V}_n^{6n}(\mathbb{A})}\widetilde{\xi}_{\tau,\widetilde{\pi}}(vv_1)\psi_{\widetilde{V}_n^{6n},a}(v)\psi_{V_n^{4n},-a}(v_1)dvdv_1. \tag{2.14}$$

The proof that (2.14) is nontrivial is very similar to that in [GRS99b], Sec. 5. Let $\widetilde{\omega}$ be the Weyl element of GL_{2n} defined in (4.31) in [GRS99b],

$$\begin{aligned}\widetilde{\omega}_{2i,i}&=1, && i=1,...,n,\\ \widetilde{\omega}_{2i-1,i+n}&=1, && i=1,...,n,\\ \widetilde{\omega}_{i,j}&=0, && \text{otherwise.}\end{aligned}$$

Put

$$\omega=\begin{pmatrix}\widetilde{\omega} & & \\ & \mathrm{I}_{2n} & \\ & & \widetilde{\omega}^*\end{pmatrix}\in \mathrm{Sp}_{6n}(F). \tag{2.15}$$

We identify $\mathrm{Sp}_{6n}(F)$ with the subgroup $\mathrm{Sp}_{6n}(F)\times 1$ of $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$. Let $R=\widetilde{V}_n^{6n}V_n^{4n}$, and consider

$$B=\omega R\omega^{-1}.$$

The metaplectic cover splits over $R(\mathbb{A})$ and $R(\mathbb{A}) \times 1$ is a subgroup of $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$. We identify these two subgroups. The same is true for $B(\mathbb{A})$. Using the left invariance to rational elements of $\widetilde{\xi}_{\tau,\widetilde{\pi}}$, the integral (2.14) is equal to

$$\int_{B(F)\backslash B(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(v\omega)\chi_{\psi,a}(v)dv. \tag{2.16}$$

Here we simply used conjugation by ω inside (2.14). The group B consists of the following elements in Sp_{6n}.

$$v(T,C,Z) = \begin{pmatrix} T & C & Z \\ & \mathrm{I}_{2n} & C' \\ & & T^* \end{pmatrix}, \tag{2.17}$$

where the last two rows of C are zero and $T \in \mathrm{GL}_{2n}$ has the form (4.34) in [GRS99b], namely, if we write T as an $n \times n$ matrix of 2×2 block matrices $T = ([T]_{i,j})$, $1 \leq i, j \leq n$, then the blocks $[T]_{i,j}$ have the following form.

$$[T]_{n,1} = \cdots = [T]_{n,n-1} = 0, \quad [T]_{n,n} = \mathrm{I}_2;$$

$[T]_{i,i}$ is lower unipotent, for $i < n$. Finally, for $i < j$, $[T]_{i,j}$ is lower triangular, and for $j < i < n$, $[T]_{i,j}$ is lower nilpotent. Denote this group of matrices T by $T(n)$. With these notations, $\chi_{\psi,a}$ is the character of $B(\mathbb{A})$ given by

$$\psi(\mathrm{tr}([T]_{1,2} + [T]_{2,3} + \cdots + [T]_{n-1,n}) + a(Z_{2n,1} - Z_{2n-1,2})). \tag{2.18}$$

Now, exactly the same steps as in [GRS99b], p. 894-895, show that the integral (2.16) is equal to

$$\int_{Y(\mathbb{A})} \int_{E(F)\backslash E(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(vy\omega)\psi'_{E,a}(v)dvdy, \tag{2.19}$$

where Y is the subgroup of lower unipotent matrices in B, E is the unipotent group of Sp_{6n}, which corresponds to the symplectic partition $[(2n)^2 1^{2n}]$, and $\psi'_{E,a} = \psi_{[(2n)^2 1^{2n}];a,-a}$, the associated character (2.8). Thus, the dv integration in (2.19) is the application of the Fourier coefficient $\mathcal{F}^{\psi_{[(2n)^2 1^{2n}];a,-a}}$.

Let us sketch the steps, as in loc. cit., which lead to (2.19). For this, consider the following sequences of root subgroups of Sp_{6n}. Let, for $1 \leq r, s \leq n$,

$$\begin{aligned} Y^{(r,s)} &= \{y^{(r,s)}(t) = v(\mathrm{I}_{2n} + te_{2r,2s-1}, 0, 0)\} \\ X^{(r,s)} &= \{x^{(r,s)}(t) = v(\mathrm{I}_{2n} + te_{2r-1,2s}, 0, 0)\}, \end{aligned} \tag{2.20}$$

where $e_{i,j}$ denotes the $2n \times 2n$ matrix, which has 1 in the (i,j) coordinate, and zero elsewhere. We used the notation of (2.17). Let, for $1 \leq j \leq i \leq n-1$, $(B')^{(i,j)}$ be the group of all $v(T,C,Z)$, such that the last two rows of C are zero, and T has the following form

$$T = \begin{pmatrix} \mathrm{I}_2 & * & * & * \\ & \ddots & * & * \\ & & \mathrm{I}_2 & * \\ & & & T' \end{pmatrix},$$

where $T' \in T(n-j+1)$, and $T_{\ell,2j-1}=0$, for all $\ell > 2i$. Now, define

$$\begin{array}{rcl} B^{(i,j)} & = & (B')^{(i,j)} \prod_{s=i+2}^{n} X^{(j,s)}, \\ C^{(i,j)} & = & \{b \in B^{(i,j)} | b_{2i,2j-1}=0\}, \\ D^{(j,i+1)} & = & C^{(i,j)} X^{(j,i+1)}, \\ A^{(j,i+1)} & = & D^{(j,i+1)} Y^{(i,j)}. \end{array} \tag{2.21}$$

We let $X^{(j,n+1)}$ and $T(1)$ be the trivial groups. Let $\chi_{\psi,a}^{(i,j)}$ be the following character of $C^{(i,j)}(\mathbb{A})$

$$\chi_{\psi,a}^{(i,j)}(c) = \psi((c_{1,3}+c_{2,4})+(c_{3,5}+c_{4,6})+\cdots+(c_{2n-3,2n-1}+c_{2n-2,2n})+a(c_{2n,4n+1}-c_{2n-1,4n+2})). \tag{2.22}$$

The groups $A^{(j,i+1)}, B^{(i,j)}, C^{(i,j)}, D^{(j,i+1)}, Y^{(i,j)}, X^{(j,i+1)}$ and the character $\chi_{\psi,a}^{(i,j)}$ satisfy the five conditions of the lemma in Sec. 2.2 of [GRS99b]. In particular, since $X^{(j,i+1)}$ and $Y^{(i,j)}$ normalize $C^{(i,j)}$, and their Adele points preserve $\chi_{\psi,a}^{(i,j)}$, upon conjugation, we may extend $\chi_{\psi,a}^{(i,j)}$ to a character of $B^{(i,j)}(\mathbb{A})$ by making it trivial on $Y^{(i,j)}(\mathbb{A})$ and also to a character of $D^{(j,i+1)}(\mathbb{A})$ by making it trivial on $X^{(j,i+1)}(\mathbb{A})$. We will keep denoting these extensions (on each group) by $\chi_{\psi,a}^{(i,j)}$, for simplicity. Finally, define, for $h \in \widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$,

$$\begin{aligned} S_{i,j}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(h) &= \int_{B^{(i,j)}(F)\backslash B^{(i,j)}(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(vh\omega)\chi_{\psi,a}^{(i,j)}(v)dv, \\ S'_{j,i+1}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(h) &= \int_{D^{(j,i+1)}(F)\backslash D^{(j,i+1)}(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(vh\omega)\chi_{\psi,a}^{(i,j)}(v)dv. \end{aligned} \tag{2.23}$$

Then

$$S_{i,j}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(1) = \int_{Y^{(i,j)}(\mathbb{A})} S'_{j,i+1}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(y)dy. \tag{2.24}$$

The proof of this fact appears on p. 374-375 in [S05] (although for specific groups $A, B, ..., D$, but the proof uses only the five properties mentioned before). A general formulation with a proof appears in Lemma 7.1 in [GRS11]. In loc. cit. it is also proved that $S_{i,j}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(1)$ is not identically zero, if and only if $S'_{j,i+1}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(1)$ is not identically zero (as $\widetilde{\xi}_{\tau,\widetilde{\pi}}$ varies). Denote $S_{i,j}(\widetilde{\xi}_{\tau,\widetilde{\pi}}) = S_{i,j}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(1)$, $S'_{j,i+1}(\widetilde{\xi}_{\tau,\widetilde{\pi}}) = S'_{j,i+1}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(1)$. Then on $\widetilde{\mathcal{E}}_{\tau,\tilde{\pi}}$,

$$S_{i,j} \neq 0 \Leftrightarrow S'_{j,i+1} \neq 0. \tag{2.25}$$

The proofs in loc. cit. show even more. Namely, given $\widetilde{\xi}_{\tau,\widetilde{\pi}}$, there is $\widetilde{\xi}'_{\tau,\widetilde{\pi}}$ (in $\widetilde{\mathcal{E}}_{\tau,\tilde{\pi}}$), such that, for all $h \in \widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, (with Sp_{2n} embedded in the middle block inside Sp_{6n}),

$$\int_{Y^{(i,j)}(\mathbb{A})} \int_{D^{(j,i+1)}(F)\backslash D^{(j,i+1)}(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(vyh\omega)\chi_{\psi,a}^{(i,j)}(v)dvdy = \int_{D^{(j,i+1)}(F)\backslash D^{(j,i+1)}(\mathbb{A})} \widetilde{\xi}'_{\tau,\widetilde{\pi}}(vh)\chi_{\psi,a}^{(i,j)}(v)dv. \tag{2.26}$$

This follows from Corollary 7.2 in [GRS11]. The argument appears also in the proof of Lemma 1, p. 895, in [GRS99b]. The idea is that by the lemma of Dixmier–Malliavin, we may assume that $\widetilde{\xi}_{\tau,\widetilde{\pi}}$ has the form $\varphi * \widetilde{\epsilon}_{\tau,\widetilde{\pi}}$, where $\varphi \in \mathcal{S}(X^{(j,i+1)}(\mathbb{A}))$ and the convolution is along $X^{(j,i+1)}(\mathbb{A})$ given by

$$\varphi * \widetilde{\epsilon}_{\tau,\widetilde{\pi}} = \int_{X^{(j,i+1)}(\mathbb{A})} \varphi(x)\rho(x)(\widetilde{\epsilon}_{\tau,\widetilde{\pi}})dx,$$

where $\rho(x)$ denotes right translation by x. Let $\phi \in \mathcal{S}(Y^{(i,j)}(\mathbb{A}))$ be the Fourier transform of φ, with respect to $\chi^{(i,j)}_{\psi,a}$; the root groups $Y^{(i,j)}$ and $X^{(j,i+1)}$ are in duality under the commutator. Now, by an easy calculation, we get

$$\int_{Y^{(i,j)}(\mathbb{A})} \int_{D^{(j,i+1)}(F)\backslash D^{(j,i+1)}(\mathbb{A})} \varphi * \widetilde{\epsilon}_{\tau,\tilde{\pi}}(vyh)\chi^{(i,j)}_{\psi,a}(v)dvdy \tag{2.27}$$

$$= \int_{D^{(j,i+1)}(F)\backslash D^{(j,i+1)}(\mathbb{A})} \phi \star \widetilde{\epsilon}_{\tau,\tilde{\pi}}(vh)\chi^{(i,j)}_{\psi,a}(v)dv,$$

where the convolution $\phi \star \widetilde{\epsilon}_{\tau,\tilde{\pi}}$ is, of course, along $Y^{(i,j)}(\mathbb{A})$. We ignored the fixed right translation by ω. Note that (2.24), (2.26) imply that the space spanned by the functions $S_{i,j}(\widetilde{\xi}_{\tau,\tilde{\pi}})(h)$ (functions of $h \in \widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$) is equal to the space spanned by the functions $S'_{j,i+1}(\widetilde{\xi}_{\tau,\tilde{\pi}})(h)$ (and so we may "ignore" the dy-integration).

We say that in the passage in (2.24), and in (2.25), from the integral $S_{i,j}$ to the integral $S'_{j,i+1}$, we "exchanged the roots" $Y^{i,j}$ and $X^{j,i+1}$. Note that $B^{(n-1,1)} = B$, and for $2 \leq i \leq n-1$, $D^{(1,i+1)} = B^{(i-1,1)}$. Using this and (2.25), we get that $S_{i,1}(\widetilde{\xi}_{\tau,\tilde{\pi}})$ is not identically zero if and only if $S_{i-1,1}(\widetilde{\xi}_{\tau,\tilde{\pi}})$ is not identically zero ($i = n-1, ..., 2$). Since the integral (2.16) is $S_{n-1,1}(\widetilde{\xi}_{\tau,\tilde{\pi}})$, and since $D^{(1,2)} = B^{(n-1,2)}$, we get that the integral (2.16) is not identically zero if and only if $S_{n-1,2}(\widetilde{\xi}_{\tau,\tilde{\pi}})$ is not identically zero. In general, we have that $D^{(j,i+1)} = B^{(i-1,j)}$, for $i \geq j+1$, and $D^{(j,j+1)} = B^{(n-1,j+1)}$. We use (2.25) repeatedly as before, and get that the functionals $S_{n-1,j}$, $j = 1, 2, ..., n-1$ are all together nontrivial, or all together trivial. Apply (2.25) one more time and we get that the integral (2.16) is not identically zero if and only if $S'_{n-1,n}(\widetilde{\xi}_{\tau,\tilde{\pi}})$ is not identically zero. $D^{(n-1,n)}$ is the unipotent group E in (2.19), and the character $\chi^{(n-1,n)}_{\psi,a}$ of $D^{(n-1,n)}(\mathbb{A})$ is the character $\psi'_{E,a}$. In order to get the equality of (2.16) and (2.19), we need to replace each of the repeated applications of (2.25) by the identity (2.24), namely

$$S_{n-1,1}(\widetilde{\xi}_{\tau,\tilde{\pi}})(1) = \int_{Y^{(n-1,1)}(\mathbb{A})} S_{n-2,1}(\widetilde{\xi}_{\tau,\tilde{\pi}})(y)dy$$

$$= \int_{Y^{(n-1,1)}(\mathbb{A})} \int_{Y^{(n-2,1)}(\mathbb{A})} S_{n-3,1}(\widetilde{\xi}_{\tau,\tilde{\pi}})(y_2y_1)dy_2dy_1 = \tag{2.28}$$

$$... = \int_{Y(\mathbb{A})} S'_{n-1,n}(\widetilde{\xi}_{\tau,\tilde{\pi}})(y)dy,$$

where Y is the product of the unipotent groups $Y^{(i,j)}$ used at each step

$$Y^{(n-1,n-1)}Y^{(n-2,n-2)}Y^{(n-1,n-2)} \cdots Y^{(2,2)}Y^{(3,2)} \cdots Y^{(n-1,2)}Y^{(1,1)}Y^{(2,1)} \cdots Y^{(n-1,1)}.$$

We conclude that the integrals (2.16) and (2.19) are equal, and the integral (2.16) is not identically zero if and only if the inner integral of (2.19), which is (up to a right translation by $y\omega$)

$$\mathcal{F}^{\psi_{[(2n)^2 1^{2n}];a,-a}}(\widetilde{\xi}_{\tau,\tilde{\pi}}) \tag{2.29}$$

is not identically zero on $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$. Moreover, application of (2.26), at each step, shows that for a given $\widetilde{\xi}_{\tau,\widetilde{\pi}}$, there is $\widetilde{\xi}'_{\tau,\widetilde{\pi}}$ (in $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$) such that for all $h \in \widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$,

$$\int_{Y(\mathbb{A})} \int_{E(F)\backslash E(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(vyh\omega)\psi'_{E,a}(v)dvdy = \int_{E(F)\backslash E(\mathbb{A})} \widetilde{\xi}'_{\tau,\widetilde{\pi}}(vh)\psi'_{E,a}(v)dv. \tag{2.30}$$

Let $b = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, and as in (4.8) in [GRS99b], consider

$$\hat{b} = \mathrm{diag}(b, ..., b, \mathrm{I}_{2n}, b^*, ..., b^*) \in \mathrm{Sp}_{6n}(F). \tag{2.31}$$

Then

$$\mathcal{F}^{\psi_{[(2n)^2 1^{2n}];a,-a}}(\widetilde{\xi}_{\tau,\widetilde{\pi}}) = \int_{E(F)\backslash E(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(v\hat{b})\psi_{E,a}(v)dv, \tag{2.32}$$

where $\psi_{E,a}$ is the character $\psi'_{E,a}$, conjugated by $\hat{b}$. Write an element of the unipotent subgroup E in the form

$$v(A, C, Z) = \begin{pmatrix} A & C & Z \\ & \mathrm{I}_{2n} & C' \\ & & A^* \end{pmatrix}, \tag{2.33}$$

where the last two rows of C are zero, and if we write A as an $n \times n$ matrix of 2×2 blocks, then

$$A = \begin{pmatrix} \mathrm{I}_2 & A_{1,2} & ... & A_{1,n} \\ & \mathrm{I}_2 & ... & A_{2,n} \\ & & \ddots & \\ & & & \mathrm{I}_2 \end{pmatrix}. \tag{2.34}$$

Then we have with the notation in (2.33) and (2.34) the following expression:

$$\psi_{E,a}(v(A, C, Z)) = \psi(\mathrm{tr}(A_{1,2} + A_{2,3} + \cdots A_{n-1,n}) - aZ_{2n-1,1}). \tag{2.35}$$

By Lemma 1.1 in [GRS03], it follows that the integral on the right hand side of (2.32) is not identically zero if and only if the following integral

$$\mathcal{F}^{\psi_{E',a}}(\widetilde{\xi}_{\tau,\widetilde{\pi}}) = \int_{E'(F)\backslash E'(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(v)\psi_{E',a}(v)dv, \tag{2.36}$$

is not identically zero on $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$. Here E' is the unipotent F-group consisting of the elements of the form (2.33), with A as in (2.34), and on C we require that only its last row is zero; the character $\psi_{E',a}$ of $E'(\mathbb{A})$ is defined by formula (2.35). (Note that E' is of the form $V_1([(2n)^2 1^{2n}])$, in the notation of [GRS03], p. 3, corresponding to the character (2.35).) From this point, the proof continues exactly as on p. 890 in [GRS99b], starting from (5.3), using the same steps. Let ν be the Weyl element in $\mathrm{Sp}_{4n}(F)$ defined in (4.9) in [GRS99b].

$$\begin{aligned} \nu_{i,2i-1} &= 1, \quad i = 1, ..., 2n, \\ \nu_{2n+i,2i} &= -1, \quad i = 1, ..., n, \\ \nu_{2n+i,2i} &= 1, \quad i = n+1, ..., 2n, \\ \nu_{i,j} &= 0, \quad \textit{otherwise}. \end{aligned}$$

Write

$$\nu = \begin{pmatrix} \nu_1 & \nu_2 \\ \nu_3 & \nu_4 \end{pmatrix},$$

where ν_i are $2n \times 2n$ matrices. Let

$$\nu' = \begin{pmatrix} \nu_1 & & \nu_2 \\ & \mathrm{I}_{2n} & \\ \nu_3 & & \nu_4 \end{pmatrix}. \tag{2.37}$$

Then

$$\mathcal{F}^{\psi_{E',a}}(\widetilde{\xi}_{\tau,\widetilde{\pi}}) = \int_{B'(F)\backslash B'(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(v\nu')\psi_{B',a}(v)dv, \tag{2.38}$$

where $B' = \nu' E' \nu^{-1}$. The elements in B' have the following form

$$v = \begin{pmatrix} u_1 & u_2 & c & z_1 & z_2 \\ 0 & u_3 & 0 & 0 & z'_1 \\ 0 & d' & \mathrm{I}_{2n} & 0 & c' \\ y_1 & y_2 & d & u_3^* & u'_2 \\ 0 & y'_1 & 0 & 0 & u_1^* \end{pmatrix} \in \mathrm{Sp}_{6n}(F), \tag{2.39}$$

where u_1, u_3 are $n \times n$ upper unipotent, z_1, y_1 are $n \times n$ upper nilpotent, and the last row of d is zero. In the notation of (2.39),

$$\psi_{B',a}(v) = \psi((u_1)_{1,2} + \cdots + (u_1)_{n-1,n} - a(u_2)_{n,1} - (u_3)_{1,2} - \cdots - (u_3)_{n-1,n}). \tag{2.40}$$

Now we carry out the process of exchanging roots, exactly as we did in [GRS99b], from (5.3) till (5.16). In this process, we need to use the property that, for all $n < l \leq 3n$, $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ has no nonzero Fourier coefficient attached to $[(2l)1^{2(3n-l)}]$. Note that we did not use this property up to this point in the proof. We get that the right hand side of (2.38) is equal to

$$\int_{L(\mathbb{A})} \int_{V_{2n}^{6n}(F)\backslash V_{2n}^{6n}(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(vy\nu')\psi'_a(v)dvdy, \tag{2.41}$$

where L is the subgroup consisting of lower unipotent matrices in B', and ψ'_a is the character of $V_{2n}^{6n}(\mathbb{A})$ given by the same formula as (2.40), namely

$$\psi'_a(v) = \psi(v_{1,2} + \cdots + v_{n-1,n} - av_{n,n+1} - v_{n+1,n+2} - \cdots - v_{2n-1,2n}). \tag{2.42}$$

We sketch the proof of the equality of (2.38) and (2.41). Denote by Z_{2n} the subgroup $v(U_{2n}, 0, 0)$ of B'. Recall that U_{2n} denotes the standard maximal unipotent subgroup of GL_{2n}. For $1 \leq i \leq j \leq 2n$, define

$$\begin{aligned} X_{i,j} &= \{v(\mathrm{I}_{2n}, 0, t(e_{i,j} + e_{2n+1-j,2n+1-i})\} \\ Y_{i,j} &= \{\bar{v}(\mathrm{I}_{2n}, 0, t(e_{i,j} + e_{2n+1-j,2n+1-i})\}, \end{aligned}$$

where

$$\bar{v}(A, C, Z) = \begin{pmatrix} A & & \\ C & \mathrm{I}_{2n} & \\ Z & C' & A^* \end{pmatrix} \in \mathrm{Sp}_{6n}, \quad A \in \mathrm{GL}_{2n}.$$

For $1 \leq i, j \leq 2n$, define

$$\begin{aligned} X'_{i,j} &= \{v(\mathrm{I}_{2n}, te_{i,j}, 0)\} \\ Y'_{i,j} &= \{\bar{v}(\mathrm{I}_{2n}, te_{i,j}, 0)\}. \end{aligned}$$

The group B' is generated by Z_{2n}, $X_{i,j}$, and $Y_{i,j}$, for $1 \leq i < j \leq 2n$, by $X'_{i,j}$, for $1 \leq i \leq n$ and $1 \leq j \leq 2n$, and by $Y'_{i,j}$, for $1 \leq i \leq 2n$ and $n+1 \leq j \leq 2n$. Let $1 \leq i < j \leq n+1$. Assume that $i+1 \leq j-1$. We define $C_{i,j}$ to be the group generated by Z_{2n}, the groups $X'_{r,s}, Y'_{k,\ell} \subset B'$, the groups $Y_{r,\ell} \subset B'$, except the indices (r, ℓ), where $r, \ell \leq j-1$ and the indices (r,j), where $r \geq i$, and the groups $X_{r,\ell}$, with $\ell \leq r < j-2$, or $r = j-1$ and $i+1 \leq \ell \leq j-1$. When $i = 1 = j$ (and $1 \leq i < j \leq n+1$), the definition of $C_{i,j}$ is the same, except that in the list of subgroups $X_{r,\ell}$, we take the indices (r,ℓ) as follows: $\ell \leq r \leq j-2$. Denote $B_{i,j} = C_{i,j}Y_{i,j}$, $D_{i,j} = C_{i,j}X_{j-1,i}$, $A_{i,j} = D_{i,j}Y_{i,j}$. We denote by $\eta_{\psi,a}$ the restriction of the right hand side of (2.40) to $Z_{2n}(\mathbb{A})$. Then we can exchange the roots $Y_{i,j}$, $X_{j-1,i}$, as we did in (2.24), (2.25), applying Lemma 7.1 in [GRS11]. Indeed the groups $A_{i,j}, B_{i,j}, C_{i,j}, D_{i,j}, Y_{i,j}, X_{j-1,i}$ satisfy the five conditions of the lemma in Sec. 2.2 of [GRS99b]. In particular, we may extend $\eta_{\psi,a}$ from $Z_{2n}(\mathbb{A})$ to $C_{i,j}(\mathbb{A})$ and also to $B_{i,j}(\mathbb{A}), D_{i,j}(\mathbb{A})$, so that it is trivial on the corresponding subgroups $X_{r,\ell}(\mathbb{A})$, $Y_{r,\ell}(\mathbb{A})$. We denote each such extension by $\eta^{(i,j)}_{\psi,a}$. Define, as in (2.23), for $h \in \widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$,

$$R_{i,j}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(h) = \int_{B_{i,j}(F)\backslash B_{i,j}(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(vh\nu')\eta^{(i,j)}_{\psi,a}(v)dv,$$

$$R'_{i,j}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(h) = \int_{D_{i,j}(F)\backslash D_{i,j}(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(vh\nu')\eta^{(i,j)}_{\psi,a}(v)dv. \tag{2.43}$$

Then

$$R_{i,j}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(1) = \int_{Y_{i,j}(\mathbb{A})} R'_{i,j}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(y)dy. \tag{2.44}$$

Denote $R_{i,j}(\widetilde{\xi}_{\tau,\widetilde{\pi}}) = R_{i,j}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(1)$, $R'_{i,j}(\widetilde{\xi}_{\tau,\widetilde{\pi}}) = R'_{i,j}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(1)$. Then, as in (2.25), we have on $\widetilde{\mathcal{E}}_{\tau,\tilde{\pi}}$,

$$R_{i,j} \neq 0 \Leftrightarrow R'_{i,j} \neq 0. \tag{2.45}$$

Note that $B_{1,2} = B'$, and the right hand side of (2.38) is $R_{1,2}(\widetilde{\xi}_{\tau,\widetilde{\pi}})$. Note also that, for $2 \leq i < j \leq n+1$, $D_{i,j} = B_{i-1,j}$, and for $1 \leq j \leq n$, $D_{1,j} = B_{j,j+1}$. Using this and (2.45) repeatedly as before, we get that $R_{1,2}(\widetilde{\xi}_{\tau,\widetilde{\pi}})$ is not identically zero if and only if $R'_{1,j}(\widetilde{\xi}_{\tau,\widetilde{\pi}})$ is not identically zero ($j = 2, ..., n+1$). Thus, $R_{1,2}(\widetilde{\xi}_{\tau,\widetilde{\pi}})$ is not identically zero if and only if $R'_{1,n+1}(\widetilde{\xi}_{\tau,\widetilde{\pi}})$ is not identically zero. When we replace each of the repeated applications of (2.45) by the identity (2.44), we get the identity

$$R_{1,2}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(1) = \int_{Y(\mathbb{A})} R'_{1,n+1}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(y)dy, \tag{2.46}$$

where Y is the product of the unipotent groups $Y_{i,j}$ used at each step

$$Y_{1,n+1}Y_{2,n+1}\cdots Y_{n,n+1}\cdots Y_{1,3}Y_{2,3}Y_{1,2}.$$

Let $x_{n+1,n}(t) = v(\mathrm{I}_{2n}, 0, te_{n+1,n}) \in X_{n+1,n}(\mathbb{A})$, and consider the smooth function on $F\backslash\mathbb{A}$, $t \mapsto R'_{1,n+1}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(x_{n+1,n}(t))$, and its Fourier expansion. Each nontrivial Fourier coefficient of this function is a Fourier coefficient of $\widetilde{\xi}_{\tau,\widetilde{\pi}}$ associated to the partition $[2n+2, 1^{4n-2}]$, and hence is zero, by the first part of the theorem. Thus, only the trivial character contributes to the Fourier expansion, and we get that

$$R'_{1,n+1}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(h) = \int_{X_{n+1,n}(F)\backslash X_{n+1,n}(\mathbb{A})} R'_{1,n+1}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(xh)dx. \tag{2.47}$$

Extend $\eta^{1,n+1}_{\psi,a}$ from $D_{1,n+1}(\mathbb{A})$ to $D_{1,n+1}X_{n+1,n}(\mathbb{A})$ by the trivial character of $X_{n+1,n}(\mathbb{A})$. To lighten notation, we will denote this extension and the ones which will follow by $\eta'_{\psi,a}$.

Now, we exchange roots as before, exchanging $Y_{n-1,n+2}$, $Y_{n-2,n+2}$,...,$Y_{1,n+2}$ with $X_{n+1,n-1}$, $X_{n+1,n-2}$,...,$X_{n+1,1}$, respectively. Then we go on exchanging $Y'_{2n,n+2}$, $Y'_{2n-1,n+2}$,..., $Y'_{1,n+2}$ with $X'_{1,2n}$, $X'_{1,2n-1}$,...,$X'_{1,1}$, respectively. Denote by $D_{1,n+2}$ the unipotent group obtain by the following sequence of operations. Start with $D_{1,n+1}X_{n+1,n}$, "take off" $Y_{n-1,n+2}$, meaning: consider the subgroup $C_{n-1,n+2}$ of $D_{1,n+1}X_{n+1,n}$ generated by all its root subgroups except $Y_{n-1,n+2}$, and "add" $X_{n+1,n-1}$, meaning: consider the group generated by $C_{n-1,n+2}$ and $X_{n+1,n-1}$ (this is $C_{n-1,n+2}X_{n+1,n-1}$.) Denote this group by $D_{n-1,n+2}$. Now, from $D_{n-1,n+2}$, "take off" $Y_{n-2,n+2}$ and then "add" $X_{n+1,n-2}$. Denote the resulting group by $D_{n-2,n+2}$, and so on, until we exchange $Y_{1,n+2}$ with $X_{n+1,1}$, and get $D_{1,n+2}$. We continue like this with the roots $Y'_{2n-i,n+2}, X'_{1,2n-i}$, $i = 0, 1, ..., 2n-1$, and denote the resulting unipotent group by $D'_{1,n+2}$. All the root exchanges above are possible, in the sense that Lemma 7.1 in [GRS11] is applicable. Denote

$$R'_{1,n+2}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(h) = \int_{D'_{1,n+2}(F)\backslash D'_{1,n+2}(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(vh\nu')\eta'_{\psi,a}(v)dv.$$

Denote also $R'_{1,n+2}(\widetilde{\xi}_{\tau,\widetilde{\pi}}) = R'_{1,n+2}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(1)$. Then from (2.47) and successive applications of the root exchanges which we just made, we conclude, as in (2.45),

$$R_{1,2} \neq 0 \Leftrightarrow R'_{1,n+2} \neq 0. \tag{2.48}$$

Now we repeat the step (2.47) by considering $x_{n+2,n-1}(t) = v(\mathrm{I}_{2n}, 0, te_{n+2,n-1}) \in X_{n+2,n-1}(\mathbb{A})$ and the Fourier expansion of the smooth function, on $F\backslash\mathbb{A}$, $t \mapsto R'_{1,n+2}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(x_{n+2,n-1}(t))$. Each nontrivial Fourier coefficient of this function is a Fourier coefficient of $\widetilde{\xi}_{\tau,\widetilde{\pi}}$ associated to the partition $[2n+4, 1^{4n-4}]$, and hence is zero, by the first part of the theorem. Thus, only the trivial character contributes to the Fourier expansion, and we get that

$$R'_{1,n+2}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(h) = \int_{X_{n+2,n-1}(F)\backslash X_{n+2,n-1}(\mathbb{A})} R'_{1,n+2}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(xh)dx. \tag{2.49}$$

Now we exchange the roots $Y_{n-2,n+3}$, $Y_{n-3,n+3}$,...,$Y_{1,n+3}$, $Y'_{2n,n+3}$, $Y'_{2n-1,n+3}$,..., $Y'_{1,n+3}$ with $X_{n+2,n-2}$, $X_{n+2,n-3}$,...,$X_{n+2,1}$, $X'_{2,2n}$, $X'_{2,2n-1}$,...,$X'_{2,1}$, respectively. We continue like this until we "exhaust" all the roots $Y_{i,j}, Y'_{r,s}$ in B'; the column $Y_{n-i+1,n+i}$, $Y_{n-i,n+i}$,...,$Y_{1,n+i}$, $Y'_{2n,n+i}$, $Y'_{2n-1,n+i}$,..., $Y'_{1,n+i}$ is exchanged with the row $X_{n+i-1,n-i+1}$, $X_{n+i-1,n-i}$,...,$X_{n+i-1,1}$, $X'_{i-1,2n}$, $X'_{i-1,2n-1}$,...,$X'_{i-1,1}$, respectively. We get the group $D'_{1,n+i}$, as above. Then define the integral $R'_{1,n+i}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(h)$ as above, and using the property that, for all $n < l \leq 3n$, $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ has no nonzero Fourier coefficient attached to $[(2l)1^{2(3n-l)}]$, we conclude that $R'_{1,n+i}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(h)$ is left $X_{n+i,n-i+1}(\mathbb{A})$-invariant. We do this for $i = 2, 3, ..., n$. The unipotent group V_{2n}^{6n} in (2.41) is $D'_{1,2n}X_{2n,1}$. The inner integral in (2.41) is $R'_{1,2n}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(y)$; $R'_{1,2n}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(h)$ is left $X_{2n,1}(\mathbb{A})$-invariant. All in all, we conclude that $R_{1,2}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(1)$, the integral (2.38), is not identically zero, if and only if the integral

$$\int_{V_{2n}^{6n}(F)\backslash V_{2n}^{6n}(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(v)\psi'_a(v)dv \tag{2.50}$$

is not identically zero (as $\widetilde{\xi}_{\tau,\widetilde{\pi}}$ varies in $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$). Moreover, by (2.46), and the similar identities obtained with the further root exchanges, we get the equality of (2.38) and (2.41). Note, also, that we may repeat, at each step, the argument in (2.27) and obtain, as in (2.30), that, given $\widetilde{\xi}_{\tau,\widetilde{\pi}}$, there is $\widetilde{\xi}'_{\tau,\widetilde{\pi}}$, such that, for all $h \in \widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$,

$$\int_{L(\mathbb{A})}\int_{V_{2n}^{6n}(F)\backslash V_{2n}^{6n}(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(vyh\nu')\psi'_a(v)dvdy = \int_{V_{2n}^{6n}(F)\backslash V_{2n}^{6n}(\mathbb{A})} \widetilde{\xi}'_{\tau,\widetilde{\pi}}(vh)\psi'_a(v)dv. \tag{2.51}$$

The same argument shows that (2.50) is equal to

$$\int_{U_{2n}^{6n}(F)\backslash U_{2n}^{6n}(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(v)\psi''_a(v)dv, \tag{2.52}$$

where U_{2n}^{6n} is the unipotent radical of the standard parabolic subgroup, whose Levi part is isomorphic to $GL_1^{2n} \times \mathrm{Sp}_{2n}$, and ψ''_a is the character of $U_{2n}^{6n}(\mathbb{A})$ given by the same formula as (2.42). To show this, we realize the compact abelian group $F^{2n}\backslash\mathbb{A}^{2n}$ in terms of Heisenberg group $\mathcal{H}_{2n+1}$ in $2n+1$ variable as follows. Let $\mathcal{Z}$ be the center of $\mathcal{H}_{2n+1}$. Then $\mathcal{X} := \mathcal{H}_{2n+1}/\mathcal{Z}$ is an affine space of dimension $2n$. Thus, we may identify the quotient $F^{2n}\backslash\mathbb{A}^{2n}$ with $\mathcal{X}(F)\backslash\mathcal{X}(\mathbb{A})$, which we embed inside Sp_{6n} as

$$i : (x;t) \mapsto \begin{pmatrix} \mathrm{I}_{2n-1} & & & & \\ & 1 & x & t & \\ & & \mathrm{I}_{2n} & x' & \\ & & & 1 & \\ & & & & \mathrm{I}_{2n-1} \end{pmatrix}. \tag{2.53}$$

Now, we consider the function

$$x \mapsto (x;t) \mapsto \int_{V_{2n}^{6n}(F)\backslash V_{2n}^{6n}(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(vi(x;t))\psi'_a(v)dv, \tag{2.54}$$

for $x \in \mathbb{A}^{2n}$. Note again that the last integral is $i(\mathcal{Z}(\mathbb{A}))$-invariant. The integral (2.54) is a smooth function on $F^{2n}\backslash\mathbb{A}^{2n}$. Consider, as before, its Fourier expansion. Each Fourier coefficient with respect to a nontrivial character of $F^{2n}\backslash\mathbb{A}^{2n}$, is zero on $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$, since it can be expressed as a sum of coefficients corresponding to $[(2l)1^{2(3n-l)}]$, for $l > 2n$. See Lemma 3.4 in [GRS05]. The integral (2.52) is not identically zero, since it is equal to

$$\int_{U_{2n}(F)\backslash U_{2n}(\mathbb{A})} \mathcal{C}_{N_{2n}^{6n}}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(u)\psi'_{U_{2n},a}(u)du. \tag{2.55}$$

Recall that U_{2n} denotes the standard maximal unipotent subgroup of GL_{2n}. The character $\psi'_{U_{2n},a}$ is the Whittaker character of $U_{2n}(\mathbb{A})$ given by the formula (2.42); N_{2n}^{6n} is the unipotent radical of P_{2n}^{6n}, the standard parabolic subgroup, whose Levi part is isomorphic to $\mathrm{GL}_{2n} \times \mathrm{Sp}_{2n}$, and $\mathcal{C}_{N_{2n}^{6n}}(\widetilde{\xi}_{\tau,\widetilde{\pi}})$ is the constant term along N_{2n}^{6n}, applied to $\widetilde{\xi}_{\tau,\widetilde{\pi}}$. Let us write $\widetilde{\xi}_{\tau,\widetilde{\pi}}$ as the residue at $s=1$ of an Eisenstein series corresponding to a holomorphic section $\phi_{\tau,\tilde{\pi};s}$ in (2.10). Then

$$\mathcal{C}_{N_{2n}^{6n}}(\widetilde{\xi}_{\tau,\widetilde{\pi}}) = Res_{s=1}\mathcal{C}_{N_{2n}^{6n}}(\widetilde{E}(\cdot,\phi_{\tau,\tilde{\pi};s})).$$

Since the Eisenstein series $\widetilde{E}(\cdot,\phi_{\tau,\tilde{\pi};s})$ is induced from a cuspidal representation on the maximal parabolic subgroup $\widetilde{P}_{2n}^{6n}(\mathbb{A})$, the constant term $\mathcal{C}_{N_{2n}^{6n}}(\widetilde{E}(\cdot,\phi_{\tau,\tilde{\pi};s}))$ is a sum of two terms: the first term is the section $\phi_{\tau,\tilde{\pi};s}$, which is holomorphic and does not contribute to the residue at $s=1$, and the second term is the intertwining operator M_s, corresponding to the long Weyl element, modulo the Levi part of P_{2n}^{6n}, applied to $\phi_{\tau,\tilde{\pi};s}$. See [MW95], II.1.7. Thus,

$$\mathcal{C}_{N_{2n}^{6n}}(\widetilde{\xi}_{\tau,\widetilde{\pi}}) = Res_{s=1}M_s(\phi_{\tau,\tilde{\pi};s}).$$

The right hand side of the last equality takes values in the space of

$$\gamma_\psi(\det)|\det|^{2n-\frac{1}{2}}\tau\otimes\widetilde{\pi}.$$

The integral (2.55) is then an application of the Whittaker coefficient with respect to the character $\psi'_{U_{2n},a}$ to elements of τ, and hence is not identically zero. This completes the proof of the theorem. □

Note that in the last proof we proved that the Fourier coefficient (2.11) is nontrivial (on our residual representation) if and only if the Fourier coefficient (2.55) is nontrivial (for any given $a \in F^*$). In Theorem 5.1 we will write a precise identity relating these two coefficients, by keeping track of the various identities, such as (2.46) etc.

2.4. **Fourier-Jacobi coefficients and descent.** The Fourier coefficient $\mathcal{F}^{\psi_{V_r^{2k},a}}$, defined in (2.4), is closely related to a Fourier-Jacobi coefficient. Such coefficients, when applied to certain residual representations yield automorphic descents. See [GRS99a] and [GRS05], and also see [I94], [GRS98], [GRS03], and [GJR04].

Let, for $1 \le r \le k$, U_r^{2k} denote the unipotent radical of the standard parabolic subgroup Q_r^{2k} of Sp_{2k}, whose Levi part is isomorphic to $\mathrm{GL}_1^r \times \mathrm{Sp}_{2(k-r)}$. Recall that V_r^{2k} is the unipotent subgroup defined in (2.2). Note that U_{r-1}^{2k} is the subgroup of elements $v \in V_r^{2k}$, such that $v_{r,2k-r+1} = 0$. Of course, V_r^{2k} is a subgroup of U_r^{2k}. The Heisenberg group of dimension $2(k-r)+1$, $\mathcal{H}_{2(k-r)+1}$, is isomorphic to the quotient group U_r^{2k}/U_{r-1}^{2k} and embeds in Sp_{2k} like in (2.53). It is isomorphic to $U_1^{2(k-r+1)}$. Denote the projection composed with the embedding by

$$\ell_{k-r}\ : U_r^{2k} \to U_r^{2k}/U_{r-1}^{2k} \cong \mathcal{H}_{2(k-r)+1} \cong U_1^{2(k-r+1)}. \tag{2.56}$$

Denote by $\psi_{U_r^{2k}}$ the following character of $U_r^{2k}(\mathbb{A})$

$$\psi_{U_r^{2k}}(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{r-1,r}). \tag{2.57}$$

This character of $U_r^{2k}(\mathbb{A})$ is obtained first by restricting the standard Whittaker character of the standard maximal unipotent subgroup of Sp_{2k} to $U_{r-1}^{2k}(\mathbb{A})$, and then extending this restriction to $U_r^{2k}(\mathbb{A})$ by the trivial character.

For a given $a \in F^*$, denote by ω_{ψ^a} the Weil representation of the semi-direct product

$$\mathcal{H}_{2(k-r)+1}(\mathbb{A}) \cdot \widetilde{\mathrm{Sp}}_{2(k-r)}(\mathbb{A}),$$

attached to the character ψ^a. This representation acts on $\mathcal{S}(\mathbb{A}^{k-r})$, the space of all Schwartz-Bruhat functions in $(k-r)$-variables. For $\phi \in \mathcal{S}(\mathbb{A}^{k-r})$, $l \in \mathcal{H}_{2(k-r)+1}(\mathbb{A})$ and $h \in \widetilde{\mathrm{Sp}}_{2(k-r)}(\mathbb{A})$, denote by $\widetilde{\theta}_{\phi,k-r}^{\psi^a}(lh)$ the corresponding theta function. This is an automorphic function on $\mathcal{H}_{2(k-r)+1}(\mathbb{A}) \cdot \widetilde{\mathrm{Sp}}_{2(k-r)}(\mathbb{A})$.

For an automorphic form φ on $\mathrm{Sp}_{2k}(\mathbb{A})$, the Fourier-Jacobi coefficient of φ (with respect to ψ^a and $k-r$) is defined by the following integral

$$\mathrm{FJ}_{\phi,k-r}^{\psi^a}(\varphi)(h) := \int_{U_r^{2k}(F)\backslash U_r^{2k}(\mathbb{A})} \varphi(uh)\widetilde{\theta}_{\phi,k-r}^{\psi^a}(\ell_{k-r}(u)h)\psi_{U_r^{2k}}(u)du. \tag{2.58}$$

It defines an automorphic form on $\widetilde{\mathrm{Sp}}_{2(k-r)}(\mathbb{A})$. Let π be an automorphic representation of $\mathrm{Sp}_{2k}(\mathbb{A})$. We denote by $\widetilde{\mathcal{D}}^{2k}_{2(k-r),\psi^a}(\pi)$ the automorphic representation of $\widetilde{\mathrm{Sp}}_{2(k-r)}(\mathbb{A})$ generated by all Fourier-Jacobi coefficients $\mathrm{FJ}^{\psi^a}_{\phi,k-r}(\varphi_\pi)$, for all vectors φ_π in the space of π and all ϕ in $\mathcal{S}(\mathbb{A}^{k-r})$. We refer to $\widetilde{\mathcal{D}}^{2k}_{2(k-r),\psi^a}(\pi)$ as the ψ^a-descent of π from $\mathrm{Sp}_{2k}(\mathbb{A})$ to $\widetilde{\mathrm{Sp}}_{2(k-r)}(\mathbb{A})$.

Similarly, for an automorphic form $\widetilde{\varphi}$ on $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$, the Fourier-Jacobi coefficient of $\widetilde{\varphi}$ (with respect to ψ^a and $k-r$) is defined by the following integral

$$\mathrm{FJ}^{\psi^a}_{\phi,k-r}(\widetilde{\varphi})(h) := \int_{U_r^{2k}(F)\backslash U_r^{2k}(\mathbb{A})} \widetilde{\varphi}(uh)\widetilde{\theta}^{\psi^a}_{\phi,k-r}(\ell_{k-r}(u)h)\psi_{U_r^{2k}}(u)du. \tag{2.59}$$

It defines an automorphic form on $\mathrm{Sp}_{2(k-r)}(\mathbb{A})$. Let $\widetilde{\pi}$ be an automorphic representation of $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$. We denote by $\mathcal{D}^{2k}_{2(k-r),\psi^a}(\widetilde{\pi})$ the automorphic representation of $\mathrm{Sp}_{2(k-r)}(\mathbb{A})$ generated by all Fourier-Jacobi coefficients $\mathrm{FJ}^{\psi^a}_{\phi,k-r}(\widetilde{\varphi}_{\widetilde{\pi}})$, for all vectors $\widetilde{\varphi}_{\widetilde{\pi}}$ in the space of $\widetilde{\pi}$ and all ϕ in $\mathcal{S}(\mathbb{A}^{k-r})$. We refer to $\mathcal{D}^{2k}_{2(k-r),\psi^a}(\widetilde{\pi})$ as the ψ^a-descent of $\widetilde{\pi}$ from $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$ to $\mathrm{Sp}_{2(k-r)}(\mathbb{A})$.

Note that when we factor the integrations in (2.58) and (2.59) through $V_r^{2k}(F)\backslash V_r^{2k}(\mathbb{A})$, then the inner integration becomes an application of the Fourier coefficient $\mathcal{F}^{\psi_{V_r^{2k},a}}$ in (2.4) to φ. The following Lemma is proved in [I94]. See also Lemma 1.1 in [GRS03].

Lemma 2.2. *Let $a \in F^*$. For any automorphic representation π of $\mathrm{Sp}_{2k}(\mathbb{A})$, the Fourier coefficient $\mathcal{F}^{\psi_{V_r^{2k},a}}$ is not identically zero on π if and only if the Fourier-Jacobi coefficient $\mathrm{FJ}^{\psi^a}_{\phi,k-r}(\varphi)$ is not identically zero, as φ varies in the space of π and as ϕ varies in $\mathcal{S}(\mathbb{A}^{k-r})$. A similar statement holds for automorphic representations of $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$.*

Let τ be an irreducible, unitary, cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$, such that $L^S(s,\tau,\wedge^2)$ has a pole at $s=1$, and $L^S(\frac{1}{2},\tau) \neq 0$. Note that these assumptions for partial L-functions are equivalent to the same assumptions for the corresponding full L-functions. (At the places of S, we take Shahidi's definition of the local L-function.) This follows from the structure of the exponents of the representations in the local unitary dual of general linear groups and analytic properties (e.g. holomorphic for $\mathrm{Re}(s) > 0$ when the representation is tempered) of the corresponding local L-functions at the ramified local places. The result is that the local L-function $L(s,\tau_v,\wedge^2)$ is holomorphic at $\mathrm{Re}(s) \geq 1$ and $L(s,\tau_v)$ is holomorhic at $\mathrm{Re}(s) \geq \frac{1}{2}$. Consider an Eisenstein series $E(g,\phi_{\tau,s})$ on $\mathrm{Sp}_{4n}(\mathbb{A})$ associated to a holomorphic section $\phi_{\tau,s}$ in

$$\mathrm{Ind}^{\mathrm{Sp}_{4n}(\mathbb{A})}_{P^{4n}_{2n}(\mathbb{A})}(\tau|\det|^s)$$

where P^{4n}_{2n} is the Siegel parabolic subgroup of Sp_{4n}. By our assumptions on τ, $E(g,\phi_{\tau,s})$ has a simple pole at $s=\frac{1}{2}$ (as the section varies). See Prop. 1 in [GRS99a]. We denote by $\mathcal{E}_\tau$ the residual representation generated by the residues

$$\xi_\tau(g) := \mathrm{Res}_{s=\frac{1}{2}} E(g,\phi_{\tau,s}). \tag{2.60}$$

The following theorem is a summary of results proved in [GRS99a], [GRS99b], [GRS99c], [GRS02], [JS03], and [JS04] (with ψ replaced by ψ^{-1}).

Theorem 2.3. *The ψ-descent from $\mathrm{Sp}_{4n}(\mathbb{A})$ to $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ of $\mathcal{E}_\tau$, $\widetilde{\mathcal{D}}^{4n}_{2n,\psi}(\mathcal{E}_\tau)$, is an irreducible, genuine, ψ^{-1}-generic, cuspidal automorphic representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which lifts weakly to τ with respect to ψ.*

For the notion of weak lift with respect to ψ, see the beginning of Sec. 2.3. We also know the converse (see [S05]).

Theorem 2.4. *Assume that $\widetilde{\pi}$ is an irreducible, genuine, cuspidal automorphic representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which is ψ^{-1}-generic. Assume that $\widetilde{\pi}$ lifts weakly, with respect to ψ, to an irreducible cuspidal automorphic representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$. Then $L^S(s,\tau,\wedge^2)$ has a pole at $s=1$, and $L^S(\frac{1}{2},\tau)\neq 0$, and hence the residual representation $\mathcal{E}_\tau$ exists. Similarly, the residual representation of $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$, $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ exists.*

The third part of the next theorem relates the two residual representations $\mathcal{E}_\tau$ and $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$.

Theorem 2.5. *Let $\widetilde{\pi}$ be an irreducible, genuine, cuspidal automorphic representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which has a ψ-weak lift to an irreducible cuspidal automorphic representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$. Then the following hold.*

1. *As an automorphic representation of $\mathrm{Sp}_{4n}(\mathbb{A})$, the descent $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ is nontrivial and square-integrable. Moreover, it is a subrepresentation in the space of the automorphic discrete spectrum of $\mathrm{Sp}_{4n}(\mathbb{A})$.*
2. *This descent $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ is cuspidal if and only if $\widetilde{\pi}$ is not ψ^{-1}-generic.*
3. *If $\widetilde{\pi}$ is ψ^{-1}-generic, then the descent $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ is a direct sum of the residual representation $\mathcal{E}_\tau$ and a cuspidal automorphic representation of $\mathrm{Sp}_{4n}(\mathbb{A})$.*

Proof. The nonvanishing of the descent $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ in Part 1 follows from Theorem 2.1 and Lemma 2.2. Next, we will prove the square integrability of the descent $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$. We will do this by computing the constant terms and automorphic exponents of the descent along all maximal parabolic subgroups, and then we will use the criterion for square-integrability of automorphic forms in [MW95], I.4.11. The calculations of these constant terms will also prove all the remaining assertions of the theorem.

Let P^{4n}_r (with $1\leq r\leq 2n$) be the standard maximal parabolic subgroup of Sp_{4n} whose Levi part isomorphic to $\mathrm{GL}_r\times\mathrm{Sp}_{2(2n-r)}$. We denote by N^{4n}_r its unipotent radical. Let $\widetilde{\xi}_{\tau,\widetilde{\pi}}\in\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$. Denote the constant term of the Fourier-Jacobi coefficient $\mathrm{FJ}^{\psi^{-1}}_{\phi,2n}(\widetilde{\xi}_{\tau,\widetilde{\pi}})$ along P^{4n}_r by

$$\mathcal{C}_{N^{4n}_r}(\mathrm{FJ}^{\psi^{-1}}_{\phi,2n}(\widetilde{\xi}_{\tau,\widetilde{\pi}})). \tag{2.61}$$

We can write a general formula for the constant term (2.61) evaluated at the identity. It has the following form :

$$\sum_{j=0}^{r}\sum_{\gamma\in P^1_{r-j,1^j}(F)\backslash\mathrm{GL}_r(F)}\int_{L(\mathbb{A})}\phi_1(i(\lambda))\mathrm{FJ}^{\psi^{-1}}_{\phi_2,2n-r}(\mathcal{C}_{N^{6n}_{r-j}}(\widetilde{\xi}_{\tau,\widetilde{\pi}}))(\hat{\gamma}\lambda\beta)d\lambda. \tag{2.62}$$

Here, we assume that $\phi=\phi_1\otimes\phi_2$ with $\phi_1\in\mathcal{S}(\mathbb{A}^r)$ and $\phi_2\in\mathcal{S}(\mathbb{A}^{2n-r})$; the subgroup $P^1_{r-j,1^j}$ of GL_r consists of all matrices of the form

$$\begin{pmatrix} g & x\\ 0 & z\end{pmatrix}\in\mathrm{GL}_r,$$

where $z \in U_j$ (the standard maximal unipotent subgroup of GL_j); for $a \in \mathrm{GL}_k$, $k \leq 3n$, we denote $\hat{a} = \mathrm{diag}(a, I_{2(3n-k)}, a^*)$; the group L is unipotent and consists of all matrices

$$\lambda = \begin{pmatrix} \mathrm{I}_r & 0 \\ x & \mathrm{I}_n \end{pmatrix}^{\wedge} \in \mathrm{Sp}_{6n},$$

and in this notation $i(\lambda)$ is the last row of x; and finally $\beta = \beta_r = \begin{pmatrix} & \mathrm{I}_r \\ \mathrm{I}_n & \end{pmatrix}^{\wedge}$. In the last formula, we restrict $\mathcal{C}_{N^{6n}_{r-j}}(\widetilde{\xi}_{\tau,\widetilde{\pi}})$ to $\widetilde{\mathrm{Sp}}_{6n-2r+2j}(\mathbb{A})$, and then we apply the Fourier-Jacobi coefficient $\mathrm{FJ}^{\psi^{-1}}_{\phi_2,2n-r}$, which takes automorphic forms on $\widetilde{\mathrm{Sp}}_{6n-2r+2j}(\mathbb{A})$ to those on $\mathrm{Sp}_{4n-2r}(\mathbb{A})$.

The proof of the formula follows exactly the same steps in [GRS99a], p. 844-847. The difference is that at the last reference, at each step in the development of this computation, when we reached a constant term (of $\mathcal{E}_\tau$), we knew it was zero, and then we did not need to write its contribution. See also the proof of Prop. 5.2 in [GRS05]. The proof of this formula appears in general and in detail in Sec. 7.6 in the book [GRS11].

Now, we have a similar case here, since $\mathcal{C}_{N^{6n}_{r-j}}(\widetilde{\xi}_{\tau,\widetilde{\pi}})$ is identically zero, unless $j = r$ or $r - j = 2n$. This is due to the cuspidality of τ and $\widetilde{\pi}$. Since $r \leq 2n$, the second case is possible only for $j = 0$. In the first case, $j = r$, the corresponding term in (2.62), which is the integral of $\mathrm{FJ}^{\psi^{-1}}_{\phi_2,2n-r}((\widetilde{\xi}_{\tau,\widetilde{\pi}}))$ along $L(\mathbb{A})$, is identically zero by Theorem 2.1, since $\mathrm{FJ}^{\psi^{-1}}_{\phi_2,2n-r}((\widetilde{\xi}_{\tau,\widetilde{\pi}}))$ involves a Fourier coefficient corresponding to $[2(n+r), 1^{2(2n-r)}]$ applied to $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ (as an inner integral). Thus, all constant terms above, corresponding to $r < 2n$, are zero, and for $r = 2n$, only the term corresponding to $j = 0$ remains in (2.62).

It follows that the constant term along P^{4n}_{2n} (evaluated at the identity) is equal to

$$\int_{L(\mathbb{A})} \phi_1(i(\lambda)) \mathrm{FJ}^{\psi^{-1}}_{\phi_2,0}(\mathcal{C}_{N^{6n}_{2n}}(\widetilde{\xi}_{\tau,\widetilde{\pi}}))(\lambda\beta_{2n}) d\lambda. \tag{2.63}$$

The Fourier-Jacobi coefficient in the integrand is just the ψ^{-1}-Whittaker coefficient, applied to $\mathcal{C}_{N^{6n}_{2n}}(\widetilde{\xi}_{\tau,\widetilde{\pi}})$, when restricted to $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ (there is no ϕ_2 now; $\phi = \phi_1$). Clearly, the integral (2.63) is not identically zero if and only if the ψ^{-1}-Whittaker coefficient is nontrivial on $\mathcal{C}_{N^{6n}_{2n}}(\widetilde{\xi}_{\tau,\widetilde{\pi}})$, when restricted to $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. We know that this constant term is equal to the residue at $s = 1$ of the intertwining operator corresponding to the long Weyl element, modulo the Weyl group of the Levi part of P^{6n}_{2n}, and it takes values in the space of

$$\delta^{\frac{1}{2}}_{P^{6n}_{2n}} \gamma_\psi(\det) |\det|^{-1} \tau \otimes \widetilde{\pi}.$$

See the explanation right after (2.55). We conclude that $\widetilde{\pi}$ is not ψ^{-1}-generic if and only if (2.63) is identically zero. This means that $\widetilde{\pi}$ is not ψ^{-1}-generic if and only if all the constant terms of the descent $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ are zero, i.e. the descent $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ is cuspidal. This completes the proof of Part 2.

Assume now that $\widetilde{\pi}$ is ψ^{-1}-generic. In this case, the above discussion shows that the descent $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ has only one constant term, which is (2.63), the one along the parabolic subgroup P^{4n}_{2n}.

In order to calculate the automorphic exponents attached to this nontrivial constant term (see [MW95] I.3.3) we compute the action of $g \in \mathrm{GL}_{2n}(\mathbb{A})$, viewed as the Levi part of $P^{4n}_{2n}(\mathbb{A})$.

Note that $\beta_{2n} = \begin{pmatrix} & \mathrm{I}_{2n} \\ \mathrm{I}_n & \end{pmatrix}^{\wedge}$. The conjugation of g by β_{2n} is denoted by $\hat{g}$, which can be viewed as an element in the Levi part of $P_{2n}^{4n}(\mathbb{A})$. Using the fact that g acts on $\phi = \phi_1$ by right translation twisted by $|\det(g)|^{\frac{1}{2}}$ and the Weil factor of $\det(g)$, with respect to ψ^{-1}, and then changing variables in (2.63), $\lambda \mapsto \lambda g^{-1}$, we conclude that g acts on (2.63) by $\tau(g)$ twisted by the character

$$|\det(g)|^n = \delta_{P_{2n}^{4n}}(m(g))^{\frac{1}{2}}|\det(g)|^{-\frac{1}{2}}, \tag{2.64}$$

where $m(g) = \begin{pmatrix} g & \\ & g^* \end{pmatrix} \in \mathrm{Sp}_{4n}(\mathbb{A})$. Therefore, we showed that the descent $\mathcal{D}_{4n,\psi^{-1}}^{6n}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ has only one nontrivial constant term, and it provides us with a unique exponent, namely (2.64), and this unique exponent is negative. By the square-integrability criterion of Langlands ([MW95], Sec. I.4.11) the descent $\mathcal{D}_{4n,\psi^{-1}}^{6n}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ is square-integrable as an automorphic representation of $\mathrm{Sp}_{4n}(\mathbb{A})$. Moreover, since this unique constant term of $\mathcal{D}_{4n,\psi^{-1}}^{6n}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ affords the representation $\delta_{P_{2n}^{4n}}^{\frac{1}{2}}|\det\cdot|^{-\frac{1}{2}} \otimes \tau$ of $\mathrm{GL}_{2n}(\mathbb{A})$, we conclude that

$$\mathcal{D}_{4n,\psi^{-1}}^{6n}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}) \subset L_d^2(\mathrm{Sp}_{4n}(F)\backslash\mathrm{Sp}_{4n}(\mathbb{A})),$$

that is, the descent $\mathcal{D}_{4n,\psi^{-1}}^{6n}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ (is non-cuspidal and) appears in the discrete automorphic spectrum. By Theorem 2.4, the residual representation $\mathcal{E}_\tau$ of $\mathrm{Sp}_{4n}(\mathbb{A})$ is nonzero if $\widetilde{\pi}$ is ψ^{-1}-generic. Note that the residual representation $\mathcal{E}_\tau$ of $\mathrm{Sp}_{4n}(\mathbb{A})$ has only one nonzero constant term, which is the one along P_{2n}^{4n}, and this nonzero constant term has exactly the same exponent as $\mathcal{D}_{4n,\psi^{-1}}^{6n}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$, namely the one that appears in (2.64). Furthermore, this residual representation $\mathcal{E}_\tau$ is irreducible. Hence the descent $\mathcal{D}_{4n,\psi^{-1}}^{6n}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ can be written as a direct sum of the residual representation $\mathcal{E}_\tau$ and a cuspidal representation of $\mathrm{Sp}_{4n}(\mathbb{A})$ (which may be zero, irreducible, or reducible). This completes the proof of Part 3.

The irreducibility of the residual representation $\mathcal{E}_\tau$ of $\mathrm{Sp}_{4n}(\mathbb{A})$ can be deduced from the facts that the residual representation $\mathcal{E}_\tau$ is square-integrable, and at each place v,

$$\mathrm{Ind}_{P_{2n}^{4n}(F_v)}^{\mathrm{Sp}_{4n}(F_v)}(\tau_v|\det|^{\frac{1}{2}})$$

has a unique irreducible quotient. The latter follows from the fact that τ_v, being generic, can be written as the full parabolic induction from its Langlands data, and since it is also unitary, we also know (see [Tm86]) that its exponents are in the open interval $(-\frac{1}{2}, \frac{1}{2})$. (This argument was explained to us by Erez Lapid, and we thank him for that.) This completes the proof of the theorem. □

3. Certain Near-Equivalence Sets

Let $\mathcal{A}_d(G)$ be the set of all irreducible, automorphic representations of $G(\mathbb{A})$, occurring as subrepresentations in the space of square-integrable automorphic functions of $G(\mathbb{A})$. In this paper, $G(\mathbb{A})$ is either $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$ or $\mathrm{Sp}_{2k}(\mathbb{A})$. Recall that two irreducible automorphic representations $\pi_i = \otimes_v \pi_{i,v}$, for $i = 1, 2$, of $G(\mathbb{A})$ are said to be near-equivalent if the local components $\pi_{1,v}$ and $\pi_{2,v}$ are equivalent representations of $G(F_v)$, at almost all places v of F. We are going to define certain near-equivalence subsets in $\mathcal{A}_d(G)$, which will be the main objects studied in the following sections.

3.1. **Certain near-equivalence subsets in $\mathcal{A}_d(\widetilde{\mathrm{Sp}}_{2n})$.** From this point, till the end of this paper, we fix an irreducible unitary cuspidal automorphic representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$, such that

$$\text{(3.1)} \qquad \begin{array}{l} L^S(s,\tau,\wedge^2) \textit{ has a simple pole at } s=1, \textit{and} \\ \textit{there exists } a\in F^*, \textit{ such that } L^S(\tfrac{1}{2},\tau\otimes\chi_a)\neq 0, \end{array}$$

where χ_a is the quadratic character given by the global Hilbert symbol $(\cdot,a)$.

Definition 3.1. *For τ as above, denote by $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$ the set of all irreducible (genuine) cuspidal automorphic representations of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which lift weakly to τ with respect to ψ.*

From the definition, it is clear that the set $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$ consists of certain near-equivalent members of $\mathcal{A}_d(\widetilde{\mathrm{Sp}}_{2n})$. In fact, for any two members $\widetilde{\pi}_1$, $\widetilde{\pi}_2$ in $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$, at any place v of F, where both $\widetilde{\pi}_{1,v}$ and $\widetilde{\pi}_{2,v}$ are unramified, they share the same local Satake parameter (with respect to ψ_v) with τ_v. Hence $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ are near-equivalent.

Note that for τ and $a\in F^*$ as in (3.1),

$$\text{(3.2)} \qquad \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)=\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau\otimes\chi_a,\psi^a).$$

This follows from the following property of the Weil factors

$$\gamma_{\psi^a}(x)=\gamma_\psi(x)\chi_a(x).$$

Proposition 3.2. *The set $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$ is not empty. Moreover, there is an element $\widetilde{\pi}$ in $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$ which is (globally) generic.*

Proof. For the given representation τ, fix $a\in F^*$, such that $L^S(\frac{1}{2},\tau\otimes\chi_a)\neq 0$. Thus, the residual representation $\mathcal{E}_{\tau\otimes\chi_a}$ of $\mathrm{Sp}_{4n}(\mathbb{A})$ exists (is nonzero). We can consider the following ψ^a-descent from $\mathrm{Sp}_{4n}(\mathbb{A})$ to $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$

$$\widetilde{\pi}=\widetilde{\mathcal{D}}^{4n}_{2n,\psi^a}(\mathcal{E}_{\tau\otimes\chi_a}).$$

By Theorem 2.3, $\widetilde{\pi}$ is an irreducible, genuine, ψ^{-a}-generic, cuspidal automorphic representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, and lies in the set $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau\otimes\chi_a,\psi^a)$. By (3.2), it follows that $\widetilde{\pi}$ is ψ^{-a}-generic member in $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$. □

3.2. **Certain near-equivalence sets in $\mathcal{A}_d(\mathrm{Sp}_{4n})$.** Recall that an irreducible cuspidal automorphic representation π of $\mathrm{Sp}_{4n}(\mathbb{A})$ is a CAP representation, with respect to the CAP-datum

$$\text{(3.3)} \qquad (\mathrm{GL}_{2n},\tau,\frac{1}{2})$$

if at almost all finite places v of F, where the local components π_v and τ_v are unramified, π_v is isomorphic to the irreducible unramified constituent of

$$\mathrm{Ind}^{\mathrm{Sp}_{4n}(F_v)}_{P^{4n}_{2n}(F_v)}(\tau_v|\det|^{\frac{1}{2}}).$$

If $\pi\in\mathcal{A}_d(\mathrm{Sp}_{4n})$ (cuspidal or not) satisfies the property above, we will say that π is of type $(\mathrm{GL}_{2n},\tau,\frac{1}{2})$.

Definition 3.3. *Let τ be as in* (3.1). *We denote by* $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$ *the set of all representations* $\pi \in \mathcal{A}_d(\mathrm{Sp}_{4n})$, *such that* π *is of type* $(\mathrm{GL}_{2n}, \tau, \frac{1}{2})$, *and such that it has a nonzero Fourier coefficient with respect to the character* $\psi_{\mathrm{V}_n^{4n},1}$, *i.e. the Fourier coefficient* $\mathcal{F}^{\psi_{\mathrm{V}_n^{4n},1}}$ *defined in* (2.4) *is nontrivial on the space of* π.

The first result is

Proposition 3.4. *The set* $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$ *is not empty.*

Proof. By Proposition 3.2, $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$ is not empty. Let $\widetilde{\pi}$ be an element in this set. Then $\widetilde{\pi}$ lifts weakly to τ with respect to ψ. Hence the residual representation $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ of $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$ exists (nonzero). By Theorem 2.5, the descent $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$, as an automorphic representation of $\mathrm{Sp}_{4n}(\mathbb{A})$, is nonzero and can be expressed as a direct sum of irreducible square-integrable automorphic representations of $\mathrm{Sp}_{4n}(\mathbb{A})$, that is,

$$\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}) = \pi_1 \oplus \pi_2 \oplus \cdots \tag{3.4}$$

where all irreducible summands π_i belong to $\mathcal{A}_d(\mathrm{Sp}_{4n})$, the discrete spectrum of $\mathrm{Sp}_{4n}(\mathbb{A})$. Moreover, all the summands are cuspidal if $\widetilde{\pi}$ is not ψ^{-1}-generic. Otherwise, there is one non-cuspidal summand, which is the residual representation $\mathcal{E}_\tau$. Furthermore, by the calculation of Proposition 5.4 in [GRS05] (for $k = 1$), every irreducible summand in (3.4) is of type $(\mathrm{GL}_{2n}, \tau, \frac{1}{2})$. See also Lemma 3.1 in [GRS05].

To complete the proof, it is enough to show that one of the π_i's belongs to the set $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$. This means that one of the π_i's has a nonzero Fourier coefficient $\mathcal{F}^{\psi_{\mathrm{V}_n^{4n},1}}(\varphi_{\pi_i})$ for some φ_{π_i} in the space of the π_i. In fact, we show that if π_i is any summand in (3.4), then the Fourier coefficient $\mathcal{F}^{\psi_{\mathrm{V}_n^{4n},1}}(\varphi_{\pi_i})$ is not identically zero on the space of π_i.

Assume, first, that π_i is cuspidal. From the definition of π_i, we deduce that the integral

$$< \varphi_{\pi_i}, \mathrm{FJ}^{\psi^{-1}}_{\phi,2n}(\widetilde{\xi}_{\tau,\widetilde{\pi}}) >= \int_{\mathrm{Sp}_{4n}(F)\backslash \mathrm{Sp}_{4n}(\mathbb{A})} \varphi_{\pi_i}(h)\overline{\mathrm{FJ}^{\psi^{-1}}_{\phi,2n}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(h)}dh \tag{3.5}$$

is not zero for some elements $\varphi_{\pi_i} \in \pi_i$, $\widetilde{\xi}_{\tau,\widetilde{\pi}} \in \widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$. Replacing in (3.5) the residue $\widetilde{\xi}_{\tau,\widetilde{\pi}}$ by a corresponding Eisenstein series $\widetilde{E}(g, \phi_{\tau\otimes\widetilde{\pi}}; s)$, we deduce that the following integral

$$< \varphi_{\pi_i}, \mathrm{FJ}^{\psi^{-1}}_{\phi,2n}(\widetilde{E}(\cdot, \phi_{\tau\otimes\widetilde{\pi}}; s)) >= \int_{\mathrm{Sp}_{4n}(F)\backslash \mathrm{Sp}_{4n}(\mathbb{A})} \varphi_{\pi_i}(h)\overline{\mathrm{FJ}^{\psi^{-1}}_{\phi,2n}(\widetilde{E}(\cdot, \phi_{\tau\otimes\widetilde{\pi}}; s)(h)}dh, \tag{3.6}$$

is a nonzero meromorphic function, for some choice of data. As in the proof of Proposition 6.6 in [GJR04], we unfold the integral and obtain the following integral

$$< \mathrm{FJ}^{\psi}_{\phi,n}(\varphi_{\pi_i}), \widetilde{\varphi}_{\widetilde{\pi}} >= \int_{\mathrm{Sp}_{2n}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \mathrm{FJ}^{\psi}_{\phi,n}(\varphi_{\pi_i})(g)\overline{\widetilde{\varphi}_{\widetilde{\pi}}(g)}dg \tag{3.7}$$

as inner integration, for appropriate cusp forms $\widetilde{\varphi}_{\widetilde{\pi}}$ in $\widetilde{\pi}$. See [GJRS09] for the theory of global integrals (3.6) in general. We conclude that $\mathrm{FJ}^{\psi}_{\phi,n}(\varphi_{\pi_i})$ is not identically zero on π_i. From Lemma 2.2, it follows that $\mathcal{F}^{\psi_{\mathrm{V}_n^{4n},1}}(\varphi_{\pi_i})$ is nonzero for some choice of data.

Assume, next, that $\pi_i = \mathcal{E}_\tau$ is a summand of (3.4). It follows from Theorem 2.3 (applied with ψ^{-1} instead of ψ) that the Fourier coefficient $\mathcal{F}^{\psi_{\mathrm{V}_n^{4n},1}}(\mathcal{E}_\tau)$ is not identically zero on $\mathcal{E}_\tau$, and now apply Lemma 2.2 as before.

This proves that all the irreducible summands in (3.4) belong to the set $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$, and in particular, it is not empty. □

By combining Theorem 2.5 with the proof of Proposition 3.4, we obtain the following refinement of Proposition 3.4.

Corollary 3.5. *Let τ satisfy* (3.1). *Suppose that $L^S(\frac{1}{2},\tau)=0$. Then the set $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$ contains at least one irreducible cuspidal automorphic representation of $\mathrm{Sp}_{4n}(\mathbb{A})$. If $L^S(\frac{1}{2},\tau)$ is nonzero, then the set $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$ contains at least the residual representation $\mathcal{E}_\tau$ of $\mathrm{Sp}_{4n}(\mathbb{A})$.*

Remark 3.6. *In our definitions the representations in $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$ and $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$ appear with multiplicities. One expects that the multiplicity one property in the automorphic discrete spectrums of $\mathrm{Sp}_{2m}(\mathbb{A})$ and SO_{2m+1} holds in general. See* [A05]. *This would also imply the multiplicity one property for the cuspidal spectrum of $\widetilde{\mathrm{Sp}}_{2m}(\mathbb{A})$, by using the theta correspondence. See Theorems 1.1, 1.3 in* [JS07]. *In this paper, we do not use these multiplicity one properties. We analyze each automorphic representation in our sets in its concrete realization as an irreducible submodule in the space of square-integrable automorphic functions on $G(\mathbb{A})$, where $G(\mathbb{A})$ is either $\mathrm{Sp}_{4n}(\mathbb{A})$ or $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$.*

4. Formulation of the Main Theorems

The proof of Proposition 3.4 shows more, namely, with $\widetilde{\pi}\in\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$, each irreducible subrepresentation of the descent $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ lies in $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$. Let $\mathcal{N}^0_{\mathrm{Sp}_{4n}}(\tau,\psi)$ be the subset of cuspidal representations in $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$. Define $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$ to be the following subset of $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$.

If $L^S(\frac{1}{2},\tau)=0$, then we define

$$\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi):=\mathcal{N}^0_{\mathrm{Sp}_{4n}}(\tau,\psi),$$

and if $L^S(\frac{1}{2},\tau)\neq 0$, then we define

$$\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi):=\mathcal{N}^0_{\mathrm{Sp}_{4n}}(\tau,\psi)\cup\{\mathcal{E}_\tau\}.$$

Note that when $L^S(\frac{1}{2},\tau)\neq 0$, the residual representation $\mathcal{E}_\tau$ exists, since we always assume that τ satisfies (3.1). By Theorem 2.5, for each $\widetilde{\pi}\in\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$, all irreducible subrepresentations of $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ lie in $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$. We expect that

$$\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)=\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi).$$

In fact, this will be a consequence of Arthur's theorem on the structure of the discrete spectrum of $\mathrm{Sp}_{4n}(\mathbb{A})$ ([A05]) and the generalized Ramanujan conjecture. However, it seems difficult to prove this equality directly.

Define, for $\widetilde{\pi}\in\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$,

$$\Phi(\widetilde{\pi}):=\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}). \tag{4.1}$$

Thus, $\Phi(\widetilde{\pi})$ is a nontrivial square integrable automorphic representation of $\mathrm{Sp}_{4n}(\mathbb{A})$, and each irreducible subrepresentation of $\Phi(\widetilde{\pi})$ lies in $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$.

We have a mapping in the reverse sense, namely the descent map $\widetilde{\mathcal{D}}^{4n}_{2n,\psi}$ applied to the elements π of $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$. Here, the results of [GRS99b] and [GRS02] apply directly, since the proof that $\widetilde{\mathcal{D}}^{4n}_{2n,\psi}(\pi)$ is cuspidal depends only on the structure of the local component of

π at one unramified place. Recall that at each unramified finite place v, the local component π_v is isomorphic to the unramified constituent of

$$\mathrm{Ind}_{P_{2n}^{4n}(F_v)}^{\mathrm{Sp}_{4n}(F_v)}(\tau_v|\det|^{\frac{1}{2}}),$$

and the calculation of the Satake parameter at each unramified local place v of an irreducible summand of $\widetilde{\mathcal{D}}_{2n,\psi}^{4n}(\pi)$ depends only on π_v. Since $\widetilde{\mathcal{D}}_{2n,\psi}^{4n}(\pi)$ is nontrivial by definition, we get that every irreducible subrepresentation of $\widetilde{\mathcal{D}}_{2n,\psi}^{4n}(\pi)$ lies in $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$.

Define

$$\Psi(\pi) := \widetilde{\mathcal{D}}_{2n,\psi}^{4n}(\pi). \tag{4.2}$$

Thus, $\Psi(\pi)$ is a nontrivial, genuine, cuspidal automorphic representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, and each irreducible subrepresentation of $\Psi(\pi)$ lies in $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$.

Define

$$\Psi' := \Psi|_{\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)}, \tag{4.3}$$

the restriction of Ψ to the subset $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$.

One of our main results is

Theorem 4.1. *Let π be in the set $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$. Then the descent*

$$\Psi'(\pi) = \Psi(\pi) = \widetilde{\mathcal{D}}_{2n,\psi}^{4n}(\pi)$$

is an irreducible automorphic representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$.

We will prove this theorem in Section 6.

This theorem provides a mapping,

$$\Psi' : \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi) \to \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi). \tag{4.4}$$

Theorem 4.1 generalizes the irreducibility of the descent in Theorem 2.3. In fact, if $L^S(\frac{1}{2},\tau) \neq 0$ (and, as always, we assume that τ satisfies (3.1)), then the representation

$$\widetilde{\pi}_\psi(\tau) = \Psi(\mathcal{E}_\tau) \tag{4.5}$$

is irreducible (and nonzero). In this sense, the proof of Theorem 4.1 provides a new proof of this deep result, which was proved in a different way in [JS03]. (By reviewing the proofs in Sec. 5, 6, it is not hard to check that the proof of Theorem 4.1 does not use Theorem 2.3.) The representation $\widetilde{\pi}_\psi(\tau)$ is also called the descent (with respect to ψ) of τ (as well as the descent of $\mathcal{E}_\tau$) to $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$.

The mapping Φ provides a sort of inverse to Ψ'. We will prove

Theorem 4.2. *For any $\widetilde{\pi}$ in the set $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$, we have the equality*

$$\Psi(\Phi(\widetilde{\pi})) = \widetilde{\pi}$$

as subspaces in the space of square-integrable automorphic functions on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. In particular, for each $\widetilde{\pi} \in \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$, there is $\pi \in \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$, such that $\Psi'(\pi) = \widetilde{\pi}$ (as subspaces).

Here, we conveniently extended the definition of Ψ, so that if we write $\Phi(\widetilde{\pi})$ as a sum of irreducible subrepresentations

$$\Phi(\widetilde{\pi}) = \pi_1 \oplus \pi_2 \oplus \cdots ,$$

then

$$\Psi(\Phi(\widetilde{\pi})) = \Psi(\pi_1) + \Psi(\pi_2) + \cdots$$

This extension of the definition of Ψ makes sense. Recall that each irreducible subrepresentation π of $\Phi(\widetilde{\pi})$ lies in $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$, and hence, $\Psi(\pi) = \Psi'(\pi)$, so that in Theorem 4.2, $\Psi(\Phi(\widetilde{\pi})) = \Psi'(\Phi(\widetilde{\pi}))$.

In the reverse direction, we can prove, at this point,

Theorem 4.3. *For each representation $\pi \in \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$, π is a subrepresentation of $\Phi(\Psi'(\pi))$, which is realized as an inclusion of the two subspaces in the space of square-integrable automorphic functions on* $\mathrm{Sp}_{4n}(\mathbb{A})$.

In order to have the full analogue of Theorem 4.2, we need the following result, which we formulate as an assumption.

Assumption (A): *For a representation $\widetilde{\pi}$ in $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$, the residual representation $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ of* $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$ *is irreducible.*

The need of Assumption (A) will become clear in the course of our proof. We remark that this assumption is an expected theorem when admitting Arthur's conjectures. See Theorem 7.1 in [M09] and also discussions in [M08]. We also remark that Assumption (A) holds if we know that the weak ψ-lift from $\widetilde{\pi}$ to τ is compatible with the local Langlands functorial lift at all local places, which is the case when $\widetilde{\pi}$ is generic. The proof follows by the same argument used in the end of the proof of Theorem 2.5, showing there that $\mathcal{E}_\tau$ is irreducible.

Theorem 4.4. *Suppose that* Assumption (A) *holds. Then, for $\pi \in \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$,*

$$\Phi(\Psi'(\pi)) = \pi.$$

Thus, with Assumption (A), Φ defines a bijection

$$\Phi\ :\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi) \to \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi). \tag{4.6}$$

These theorems can be summarized together as our main theorem.

Theorem 4.5. *Suppose that* Assumption (A) *holds. For each $\pi \in \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$, $\Psi'(\pi)$ is irreducible, and for each $\widetilde{\pi} \in \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$, $\Phi(\widetilde{\pi})$ is irreducible. Moreover, the mappings*

$$\Psi' :\ \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi) \to \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$$

and

$$\Phi\ :\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi) \to \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$$

are bijective, and satisfy

$$\Psi' \circ \Phi = \mathrm{Id}_{\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)}, \quad \Phi \circ \Psi' = \mathrm{Id}_{\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)}.$$

We conjecture that Theorem 4.5 holds with $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$ instead of $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$, and without adding Assumption (A).

We obtain the following interesting application.

Theorem 4.6. *Let $a \in F^*$ be such that $L^S(\frac{1}{2},\tau\otimes\chi_a) \neq 0$. Then the set $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$ contains a unique representation which is ψ^{-a}- (globally) generic, namely the representation (4.5) $\widetilde{\pi}_{\psi^a}(\tau\otimes\chi_a)$. This representation is the unique, irreducible, genuine, ψ^{-a}-generic,* *cuspidal automorphic representation, which lifts weakly to τ with respect to ψ. In particular, $\widetilde{\pi}_{\psi^a}(\tau\otimes\chi_a)$ occurs with multiplicity one in the subspace of ψ^{-a}-generic cusp forms on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$.*

Proof. Assume first that $a = 1$. Since $L^S(\frac{1}{2}, \tau) \neq 0$, we know from Theorem 2.3 that $\widetilde{\pi}_\psi(\tau)$ is ψ^{-1}-generic. Recall that Assumption (A) holds when $\widetilde{\pi}$ is generic and so Theorem 4.4 holds for such representations, and, in particular, $\Phi(\widetilde{\pi})$ is irreducible, for (globally) generic $\widetilde{\pi}$. Assume now that $\widetilde{\pi} \in \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$ is ψ^{-1}-generic. By the proof of Theorem 2.5, we know that $\mathcal{E}_\tau$ is a subrepresentation of $\Phi(\widetilde{\pi})$, realized in space of square-integrable automorphic functions on $\mathrm{Sp}_{4n}(\mathbb{A})$. By Theorem 4.5, $\Phi(\widetilde{\pi})$ is irreducible, and hence we must have $\Phi(\widetilde{\pi}) = \mathcal{E}_\tau$. This implies that

$$\widetilde{\pi} = \Psi(\Phi(\widetilde{\pi})) = \Psi(\mathcal{E}_\tau) = \widetilde{\pi}_\psi(\tau).$$

Consider now the general case. Assume that $\widetilde{\pi} \in \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$ is ψ^{-a}-generic. Then by (3.2), $\widetilde{\pi} \in \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau \otimes \chi_a, \psi^a)$. By the previous case (replacing ψ by ψ^a) we conclude that $\widetilde{\pi} = \widetilde{\pi}_{\psi^a}(\tau \otimes \chi_a)$ and the multiplicity one property follows. □

We remark that $a \in F^*$ is such that $L^S(\frac{1}{2}, \tau \otimes \chi_a) \neq 0$, if and only if $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$ contains a representation which is ψ^{-a}- generic. This follows from Theorems 2.3 and 2.4.

We regard Theorem 4.6 as an automorphic version of Shahidi's conjecture [Sh88]. The uniqueness of $\widetilde{\pi}_\psi(\tau)$ up to isomorphism follows from [JS03] and [JS04]. Theorem 4.6 says that $\widetilde{\pi}_\psi(\tau)$ is unique (multiplicity one) within the set of all ψ^{-1}-generic, irreducible, cuspidal automorphic representations of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. See also Theorem 1.6 in the introduction and the discussion preceding this theorem.

5. Two Basic Identities and the Proof of Theorem 4.2

We keep the notation in previous sections. The goal of this section is to establish two identities which interpret the composition of descent constructions as indicated in Diagram (1.6) in terms of certain Fourier coefficients. These identities are new and important to the proofs of Theorems 4.1 and 4.2.

The first identity gives a precise formula for the composition of the two descent maps

$$\mathrm{FJ}^{\psi}_{\phi_1,n} \circ \mathrm{FJ}^{\psi^{-1}}_{\phi_2,2n},$$

applied to $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$. The identity is a refined version of the proof of Theorem 2.1, in case $a = -1$, in the sense that at each step in the proof of Theorem 2.1, where we claimed the equivalence of the nonvanishing property of two integrals, we will now write the precise identity relating these two integrals (e.g. the equality of the right hand side of (2.32) and (2.36), the equality of (2.38) and (2.41) etc.). Of course, we also incorporate Lemma 2.2. As in the proof of Theorem 2.1, the proof of the following identity is very similar to the proof of identity (5.27) in [GRS99b], which gives a formula for the ψ^{-1}-Whittaker coefficient of the descent $\Psi(\mathcal{E}_\tau)$ to $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$.

Theorem 5.1. *Let $\widetilde{\pi}$ belong to the set $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$. Let $\phi_1 \in \mathcal{S}(\mathbb{A}^n)$, $\phi_2 \in \mathcal{S}(\mathbb{A}^{2n})$, and $\widetilde{\xi}_{\tau,\widetilde{\pi}} \in \widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$. Assume that $\phi_2 = \phi_{21} \otimes \phi_{22}$ with $\phi_{21}, \phi_{22} \in \mathcal{S}(\mathbb{A}^n)$. Then the following identity* *holds as functions in* $h \in \widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$*:*

$$\begin{aligned}&(\mathrm{FJ}^{\psi}_{\phi_1,n} \circ \mathrm{FJ}^{\psi^{-1}}_{\phi_2,2n})(\widetilde{\xi}_{\tau,\widetilde{\pi}})(h)\\ =\ &\int_{\mathbb{A}^n}\int_{Y(\mathbb{A})}\int_{\mathbb{A}^{2n}}\int_{L(\mathbb{A})} \mathcal{C}^{\psi'_{U_{2n},-1}}_{N^{6n}_{2n}}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(hy'\nu'\widehat{l_2}\hat{b}y\omega\widehat{l_1})\phi(l_2,l_1)dy'dl_2dydl_1.\end{aligned}$$

Here, $\phi \in \mathcal{S}(\mathbb{A}^{3n})$ *can be written explicitly in terms of* $\phi_1, \phi_{21}, \phi_{22}$*; the definitions of* $\widehat{l_1}, \widehat{l_2}$ *are given in the proof; and the remaining notation is as in the proof of Theorem 2.1 and will be specifically indicated in the course of the proof.*

Proof. By definition of Fourier-Jacobi coefficients ((2.58) and (2.59)), the composition of the Fourier-Jacobi coefficients $\mathrm{FJ}^{\psi}_{\phi_1,n}$ and $\mathrm{FJ}^{\psi^{-1}}_{\phi_2,2n}$ is given by

$$\begin{aligned}&(\mathrm{FJ}^{\psi}_{\phi_1,n} \circ \mathrm{FJ}^{\psi^{-1}}_{\phi_2,2n})(\widetilde{\xi}_{\tau,\widetilde{\pi}})(h)\\ =\ &\int_{U^{4n}_n(F)\backslash U^{4n}_n(\mathbb{A})}\int_{U^{6n}_n(F)\backslash U^{6n}_n(\mathbb{A})} \widetilde{\xi}_{\tau,\widetilde{\pi}}(uvh)\widetilde{\theta}^{\psi^{-1}}_{\phi_2,2n}(\ell_{2n}(u)vh)\psi_{U^{6n}_n}(u)du\widetilde{\theta}^{\psi}_{\phi_1,n}(\ell_n(v)h)\psi_{U^{4n}_n}(v)dv\end{aligned} \tag{5.1}$$

where $h \in \widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$.

We first unfold the theta series $\widetilde{\theta}^{\psi^{-1}}_{\phi_2,2n}(\ell_{2n}(u)vh)$. Write

$$\ell_{2n}(u) = (l_1, l_2, l_3; z)$$

where $l_1, l_3 \in \mathbb{A}^n$, $l_2 \in \mathbb{A}^{2n}$ and $z \in \mathbb{A}$. Then

$$\widetilde{\theta}^{\psi^{-1}}_{\phi_2,2n}(\ell_{2n}(u)vh) = \sum_{\xi_1,\xi_2 \in F^n} \omega^{2n}_{\psi^{-1}}((\xi_1,0,0;0)(l_1,l_2,l_3,z)vh)\phi_2(0,\xi_2),$$

where $\omega^{2n}_{\psi^{-1}}$ is the Weil representation of $\widetilde{\mathrm{Sp}}_{4n}(\mathbb{A})$ attached to the character ψ^{-1}. Plug this into integral (5.1). Next, we collapse summation with integration in the variables l_1 and ξ_1, and conjugate it to the right. Thus, integral (5.1) is equal to

$$\begin{aligned}&\int_{U^{4n}_n(F)\backslash U^{4n}_n(\mathbb{A})}\int_{U^{6n}_{n,1}(F)\backslash U^{6n}_{n,1}(\mathbb{A})}\int_{\mathbb{A}^n}\sum_{\xi_2\in F^n} \omega^{2n}_{\psi^{-1}}(\ell_{2n}(u)vh(l_1,0,0,0))\phi_2(0,\xi_2)\\ &\qquad \widetilde{\theta}^{\psi}_{\phi_1,n}(\ell_n(v)h)\widetilde{\xi}_{\tau,\widetilde{\pi}}(uvh\widehat{l_1})\psi_{U^{6n}_n}(u)du\psi_{U^{4n}_n}(v)dl_1dv.\end{aligned} \tag{5.2}$$

Here, $U^{6n}_{n,1}$ is the subgroup of U^{6n}_n consisting of all $u = (u_{i,j}) \in U^{6n}_n$, such that $u_{n,j} = 0$, for all $n+1 \le j \le 2n$. Also, for $l_1 = (r_1, ..., r_n) \in \mathbb{A}^n$,

$$\widehat{l_1} = \mathrm{I}_{6n} + r_1e'_{n,n+1} + \cdots + r_ne'_{n,2n},$$

where $e'_{i,j}$ is the matrix of size $6n$ defined by $e'_{i,j} = e_{i,j} - e_{6n-j+1,6n-i+1}$. Henceforth, we shall write l_1 for $(l_1, 0, 0, 0)$.

Note that, following the above notation, we have, for $u \in U^{6n}_{n,1}$,

$$\ell_{2n}(u) = (0, l_2, l_3; z).$$

It follows from the well known action of the Weil representation that

$$\sum_{\xi_2\in F^n} \omega^{2n}_{\psi^{-1}}((0,l_2,l_3;z)vhl_1)\phi_2(0,\xi_2) = \sum_{\xi_2\in F^n} \omega^{2n}_{\psi^{-1}}((0,l_2,0;z)hl_1)\phi_2(0,\xi_2) = \sum_{\xi_2\in F^n} \omega^{2n}_{\psi^{-1}}((0,l_2,0;z)h)\phi_2(l_1,\xi_2).$$

Since $\phi_2 = \phi_{21}\otimes\phi_{22}$, we have

$$\sum_{\xi_2\in F^n} \omega^{2n}_{\psi^{-1}}((0,l_2,0;z)h)\phi_2(l_1,\xi_2) = \widetilde{\theta}^{\psi^{-1}}_{\phi_{22},n}((l_2;z)h)\phi_{21}(l_1).$$

Denote $\ell_n(u) = (l_2;z)$. Then integral (5.2) is equal to

$$\int_{U^{4n}_n(F)\backslash U^{4n}_n(\mathbb{A})}\int_{U^{6n}_{n,1}(F)\backslash U^{6n}_{n,1}(\mathbb{A})}\int_{\mathbb{A}^n} \widetilde{\theta}^{\psi^{-1}}_{\phi_{22},n}(\ell_n(u)h)\widetilde{\theta}^{\psi}_{\phi_1,n}(\ell_n(v)h)$$

$$\widetilde{\xi}_{\tau,\widetilde{\pi}}(uvh\widehat{l_1})\phi_{21}(l_1)dl_1\psi_{U^{6n}_n}(u)du\psi_{U^{4n}_n}(v)dv. \tag{5.3}$$

Note that this integral is equal to

$$\int_{U^{4n}_n(F)\backslash U^{4n}_n(\mathbb{A})}\int_{U^{6n}_{n,1}(F)\backslash U^{6n}_{n,1}(\mathbb{A})} \widetilde{\theta}^{\psi^{-1}}_{\phi_{22},n}(\ell_n(u)h)\widetilde{\theta}^{\psi}_{\phi_1,n}(\ell_n(v)h)$$

$$\widetilde{\xi}'_{\tau,\widetilde{\pi}}(uvh)\psi_{U^{6n}_n}(u)du\psi_{U^{4n}_n}(v)dv, \tag{5.4}$$

where $\widetilde{\xi}'_{\tau,\widetilde{\pi}}$ is the convolution of $\widetilde{\xi}'_{\tau,\widetilde{\pi}}$ against ϕ_{21}, along $\mathbb{A}^n$ (in the variable l_1).

To proceed, we now switch the order of integration (easily justified) in (5.3), and rewrite it as

$$\int_{\mathbb{A}^n}\int_{U^{4n}_n(F)\backslash U^{4n}_n(\mathbb{A})}\int_{U^{6n}_{n,1}(F)\backslash U^{6n}_{n,1}(\mathbb{A})} \widetilde{\theta}^{\psi^{-1}}_{\phi_{22},n}(\ell_n(u)h)\widetilde{\theta}^{\psi}_{\phi_1,n}(\ell_n(v)h)$$

$$\widetilde{\xi}_{\tau,\widetilde{\pi}}(uvh\widehat{l_1})\psi_{U^{6n}_n}(u)du\psi_{U^{4n}_n}(v)dv\phi_{21}(l_1)dl_1. \tag{5.5}$$

Let ω be the Weyl element (2.15) of Sp_{6n}. As in (2.16), the integral (5.5) is equal to

$$\int_{\mathbb{A}^n}\int_{B(F)\backslash B(\mathbb{A})} \widetilde{\theta}^{\psi^{-1}}_{\phi_{22},n}(\ell''(v)h)\widetilde{\theta}^{\psi}_{\phi_1,n}(\ell'(v)h)\widetilde{\xi}_{\tau,\widetilde{\pi}}(vh\omega\widehat{l_1})\psi_1(v)\phi_{21}(l_1)dvdl_1. \tag{5.6}$$

Here, $B = \omega(U^{6n}_{n,1}U^{4n}_n)\omega^{-1}$. This is the group of elements $v = v(T,C,Z)$ as in (2.17), where the only difference is that now we do not require that the last two rows of C are zero, and then

$$\ell'(v) = (C_{2n-1};Z_{2n-1,2}), \qquad \ell''(v) = (C_{2n};Z_{2n,1}),$$

(C_i is the i-th row of C; $Z_{i,j}$ is the (i,j)-th entry of Z); $\psi_1(v)$ is given by

$$\psi(\mathrm{tr}([T]_{1,2}+\cdots+[T]_{n-1,n}))$$

with notation of (2.17) and (2.18).

Next, we apply the exact same steps which led to (2.19), that is, the steps (2.20)–(2.28). These steps are to preform a series of Fourier expansions along the variables in T, below its diagonal, and "exchanging" them with root coordinates above the diagonal. The analogue of (2.19) is that the integral (5.6) is equal to

$$\int_{\mathbb{A}^n}\int_{Y(\mathbb{A})}\int_{\mathcal{E}(F)\backslash\mathcal{E}(\mathbb{A})} \widetilde{\theta}^{\psi^{-1}}_{\phi_{22},n}(\ell''_n(v)h)\widetilde{\theta}^{\psi}_{\phi_1,n}(\ell'_n(v)h)\widetilde{\xi}_{\tau,\widetilde{\pi}}(vhy\omega\widehat{l_1})\psi_1(v)\phi_{21}(l_1)dvdydl_1. \tag{5.7}$$

Here, $\mathcal{E}$ is the unipotent radical of the standard parabolic subgroup of Sp_{6n}, whose Levi part is isomorphic to $\mathrm{GL}_2^n \times \mathrm{Sp}_{2n}$. We use this notation in order to stress the resemblance to (2.19). Note that when we apply the same steps, which led to (2.30), to the integral (5.4), we get that for a given $\widetilde{\xi}'_{\tau,\widetilde{\pi}}$, there is $\widetilde{\xi}''_{\tau,\widetilde{\pi}}$, such that (5.4) is equal to

$$\int_{\mathcal{E}(F)\backslash\mathcal{E}(\mathbb{A})} \widetilde{\theta}^{\psi^{-1}}_{\phi_{22},n}(\ell''_n(v)h)\widetilde{\theta}^{\psi}_{\phi_1,n}(\ell'_n(v)h)\widetilde{\xi}''_{\tau,\widetilde{\pi}}(vh)\psi_1(v)dv. \tag{5.8}$$

Write an element in $\mathcal{E}$ in the form $v = v(A, C, Z)$, as in (2.33) and (2.34). Then $\ell'_n(v)$ and $\ell''_n(v)$ are given by the same formulae for $\ell'(v)$ and $\ell''(v)$ above, and $\psi_1(v)$ is given by

$$\psi(\mathrm{tr}(A_{1,2} + \cdots + A_{n-1,n}))$$

with notation of (2.34). Finally, the unipotent group Y is the one in (2.19).

By using the properties of theta functions (it takes a "direct sum" embedding of symplectic/ Heisenberg groups to a product of theta series), we have the identity

$$\widetilde{\theta}^{\psi^{-1}}_{\phi_{22},n}((x_1, y_1, ; z_1)h)\widetilde{\theta}^{\psi}_{\phi_1,n}((x_2, y_2, ; z_2)h) = \widetilde{\theta}^{\psi}_{\phi_3,2n}((x_1, x_2, -y_2, y_1; z_2 - z_1)\widetilde{h}). \tag{5.9}$$

Here, $\phi_3 = \phi_{22} \otimes \phi_1$, $x_1, x_2, y_1, y_2 \in \mathbb{A}^n$,

$$\widetilde{h} = \left(\begin{pmatrix} A & & & -B \\ & A & B & \\ & C & D & \\ -C & & & D \end{pmatrix}, 1\right) \text{ and } h = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \epsilon\right).$$

Let

$$\gamma = \begin{pmatrix} I_n & 0 & -\frac{1}{2}I_n & 0 \\ I_n & 0 & \frac{1}{2}I_n & 0 \\ 0 & -I_n & 0 & \frac{1}{2}I_n \\ 0 & I_n & 0 & \frac{1}{2}I_n \end{pmatrix}.$$

Then $\gamma \in \mathrm{Sp}_{4n}(F)$ and has the property that

$$\begin{aligned} (x_1, x_2, -y_2, y_1)\gamma &= (x_1 + x_2, y_1 + y_2, -\frac{1}{2}(x_1 - x_2), \frac{1}{2}(y_1 - y_2)), \\ \hat{h} &:= \gamma^{-1}\widetilde{h} = \left(\begin{pmatrix} h & \\ & h^* \end{pmatrix}, 1\right). \end{aligned}$$

We conclude that

$$\widetilde{\theta}^{\psi}_{\phi_3,2n}((x_1, x_2, -y_2, y_1; z_1 - z_2)\widetilde{h}) = \widetilde{\theta}^{\psi}_{\phi'_3,2n}((x_1 + x_2, y_1 + y_2, -\frac{1}{2}(x_1 - x_2), \frac{1}{2}(y_1 - y_2))\hat{h}), \tag{5.10}$$

where $\phi'_3 = \omega_{\psi^{-1},2n}(\gamma^{-1})\phi_3$. Let $\hat{b} \in \mathrm{Sp}_{6n}(F)$ be the matrix (2.31). Then as in (2.32), after a change of variables (due to conjugation by $\hat{b}$) and using (5.10), the integral (5.7) becomes

$$\int_{\mathbb{A}^n} \int_{Y(\mathbb{A})} \int_{\mathcal{E}(F)\backslash\mathcal{E}(\mathbb{A})} \widetilde{\theta}^{\psi}_{\phi'_3,2n}(\ell(v)\hat{h})\widetilde{\xi}_{\tau,\widetilde{\pi}}(vh\hat{b}y\omega\widehat{l_1})\psi_1(v)\phi_{21}(l_1)dvdydl_1. \tag{5.11}$$

Here, for $v = v(A, C, Z)$, written as before, let us write, for $i = 2n - 1, 2n$, $C_i = (x_i, y_i)$, where $x_i, y_i \in \mathbb{A}^n$. Then

$$\ell(v) = (x_{2n}, y_{2n}, \frac{1}{2}x_{2n-1}, -\frac{1}{2}y_{2n-1}; z_{2n-1,1}).$$

We unfold the theta series $\widetilde{\theta}^{\psi}_{\phi_3',2n}(\ell(v)\hat{h})$, as we did in the beginning of the proof. Note that now $\hat{h}$ acts linearly in the Weil representation. We get that integral (5.11) is equal to

$$\int_{\mathbb{A}^n}\int_{Y(\mathbb{A})}\int_{\mathbb{A}^{2n}}\int_{E'(F)\backslash E'(\mathbb{A})}\widetilde{\xi}_{\tau,\widetilde{\pi}}(vh\widehat{l_2}by\omega\widehat{l_1})\psi_{E',-1}(v)\phi(l_2,l_1)dvdl_2dydl_1. \tag{5.12}$$

Here E' and $\psi_{E',-1}$ are as in (2.36) and $\phi=\phi_3'\otimes\phi_{21}$. The embedding of $l_2=(m_1,...,m_{2n})\in\mathbb{A}^{2n}$ inside $\mathrm{Sp}_{6n}(\mathbb{A})$ is denoted by $\widehat{l_2}$ and is given by

$$\widehat{l_2}=I_{6n}+\sum_{i=1}^{2n}m_ie'_{2n,2n+i}.$$

Applying the steps (5.9)–(5.12) to (5.8), and using the same argument which yielded (5.4), we get that for a given $\widetilde{\xi}''_{\tau,\widetilde{\pi}}$, there is $\widetilde{\xi}'''_{\tau,\widetilde{\pi}}$, such that the integral (5.8) is equal to

$$\int_{E'(F)\backslash E'(\mathbb{A})}\widetilde{\xi}'''_{\tau,\widetilde{\pi}}(vh)\psi_{E',-1}(v)dv. \tag{5.13}$$

In the notation of (2.36), the integral (5.12) can we rewritten as

$$\int_{\mathbb{A}^n}\int_{Y(\mathbb{A})}\int_{\mathbb{A}^{2n}}\mathcal{F}^{\psi_{E',-1}}(h\widehat{l_2}by\omega\widehat{l_1}\star\widetilde{\xi}_{\tau,\widetilde{\pi}})\phi(l_2,l_1)dl_2dydl_1, \tag{5.14}$$

where the star denotes the action by right translation. By (2.41), the integral (5.14) is equal to

$$\int_{\mathbb{A}^n}\int_{Y(\mathbb{A})}\int_{\mathbb{A}^{2n}}\int_{L(\mathbb{A})}\mathcal{F}^{\psi'_{V^{6n}_{2n},-1}}(hy'\nu'\widehat{l_2}by\omega\widehat{l_1}\star\widetilde{\xi}_{\tau,\widetilde{\pi}})\phi(l_2,l_1)dy'dl_2dydl_1. \tag{5.15}$$

Here, $\mathcal{F}^{\psi'_{V^{6n}_{2n},-1}}$ denotes the operation of taking a Fourier coefficient along V^{6n}_{2n}, with respect to the character $\psi'_{V^{6n}_{2n},-1}$ given by (2.42), with $a=-1$. Similarly, when we apply (2.51) to (5.13), we get that for a given $\widetilde{\xi}'''_{\tau,\widetilde{\pi}}$, there is $\widetilde{\epsilon}_{\tau,\widetilde{\pi}}$ (in $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$) such that, for all $h\in\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$,

$$\int_{E'(F)\backslash E'(\mathbb{A})}\widetilde{\xi}'''_{\tau,\widetilde{\pi}}(vh)\psi_{E',-1}(v)dv=\mathcal{F}^{\psi'_{V^{6n}_{2n},-1}}(\widetilde{\epsilon}_{\tau,\widetilde{\pi}})(h) \tag{5.16}$$

Now, (2.55) shows that the integral (5.15) is equal to

$$\begin{aligned}\int_{\mathbb{A}^n}\int_{Y(\mathbb{A})}\int_{\mathbb{A}^{2n}}\int_{L(\mathbb{A})}\int_{U_{2n}(F)\backslash U_{2n}(\mathbb{A})}&\mathcal{C}_{N^{6n}_{2n}}(hy'\nu'\widehat{l_2}by\omega\widehat{l_1}\star\widetilde{\xi}_{\tau,\widetilde{\pi}})(u)\\&\psi'_{U_{2n},-1}(u)du\phi(l_2,l_1)dudy'dl_2dy'dl_1dy.\end{aligned} \tag{5.17}$$

We used the notation of (2.55). This proves our identity, when we denote

$$\mathcal{C}^{\psi'_{U_{2n},-1}}_{N^{6n}_{2n}}(\widetilde{\xi}_{\tau,\widetilde{\pi}})=\int_{U_{2n}(F)\backslash U_{2n}(\mathbb{A})}\mathcal{C}_{N^{6n}_{2n}}(\widetilde{\xi}_{\tau,\widetilde{\pi}})(u)\psi'_{U_{2n},-1}(u)du.$$

Similarly, we get that in (5.16),

$$\int_{E'(F)\backslash E'(\mathbb{A})}\widetilde{\xi}'''_{\tau,\widetilde{\pi}}(vh)\psi_{E',-1}(v)dv=\mathcal{C}^{\psi'_{U_{2n},-1}}_{N^{6n}_{2n}}(\widetilde{\epsilon}_{\tau,\widetilde{\pi}})(h). \tag{5.18}$$

□

As a first consequence, we have the following precise results for the composition of the two consecutive descents of the residual representation $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ of $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$.

Proposition 5.2. *The space of automorphic forms on* $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ *generated by all elements*

$$(\mathrm{FJ}^{\psi}_{\phi_1,n}\circ \mathrm{FJ}^{\psi^{-1}}_{\phi_2,2n})(\widetilde{\xi}_{\tau,\widetilde{\pi}})(h),$$

as in (5.1)*, is equal to the space of the automorphic representation* $\widetilde{\pi}$*. In other words, the space of the double descent*

$$\widetilde{\mathcal{D}}^{4n}_{2n,\psi}\circ\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$$

of the residual representation $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ *and the space of* $\widetilde{\pi}$ *are equal.*

Proof. Denote the right hand side of the identity in Theorem 5.1 by $I(\widetilde{\xi}_{\tau,\widetilde{\pi}},\phi)$. By (5.4), (5.8), (5.13), (5.16), (5.18), it follows that the space generated by the functions $I(\widetilde{\xi}_{\tau,\widetilde{\pi}},\phi)$ is contained in the space generated by the functions $\mathcal{C}^{\psi'_{U_{2n},-1}}_{N^{6n}_{2n}}(\widetilde{\epsilon}_{\tau,\widetilde{\pi}})$. As we explained in the end of the proof of Theorem 2.1, the functions of $h\in\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, $\mathcal{C}^{\psi'_{U_{2n},-1}}_{N^{6n}_{2n}}(\widetilde{\epsilon}_{\tau,\widetilde{\pi}})(h)$, lie in the space of $\widetilde{\pi}$. Since $\widetilde{\pi}$ is irreducible, and since $\widetilde{\mathcal{D}}^{4n}_{2n,\psi}\circ\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})$ is nontrivial, we get that $\widetilde{\mathcal{D}}^{4n}_{2n,\psi}\circ\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}})=\widetilde{\pi}$. □

Now Theorem 4.2 follows from Theorem 5.1 and Proposition 5.2. Namely, for any representation $\widetilde{\pi}\in\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$, we have

$$\Psi(\Phi(\widetilde{\pi})=\widetilde{\pi}.$$

The second identity relates two expressions of elements in representations π in $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$, which are obtained in two different ways. One is to use the composition of two consecutive descents and the other is to use certain Fourier coefficients. In more details, given such a π in $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$, we form the Eisenstein series $E(g,\phi_{\tau,\pi;s})$ associated to a holomorphic section $\phi_{\tau,\pi;s}$ in

$$\mathrm{Ind}^{\mathrm{Sp}_{8n}(\mathbb{A})}_{P^{8n}_{2n}(\mathbb{A})}(\tau|\det|^s\otimes\pi). \tag{5.19}$$

Recall that P^{8n}_{2n} is the standard parabolic subgroup of Sp_{8n}, whose Levi part is isomorphic to $\mathrm{GL}_{2n}\times\mathrm{Sp}_{4n}$. Because π is of type $(\mathrm{GL}_{2n},\tau,\frac{1}{2})$, the partial tensor product L-function $L^S(s,\tau\times\pi)$ has a pole at $s=\frac{3}{2}$. This implies that the Eisenstein series $E(g,\phi_{\tau,\pi;s})$ has a simple pole at $s=\frac{3}{2}$. To explain this, consider the constant term $\mathcal{C}_{N^{8n}_{2n}}(E(\cdot,\phi_{\tau,\pi;s}))$ along P^{8n}_{2n}. Then, due to the cuspidality of τ, we have, for $\mathrm{Re}(s)$ large, and $g\in\mathrm{GL}_{2n}(\mathbb{A})\times\mathrm{Sp}_{4n}(\mathbb{A})$ (identified with the Levi part of $P^{8n}_{2n}(\mathbb{A})$)

$$\begin{aligned}\mathcal{C}_{N^{8n}_{2n}}(E(\cdot,\phi_{\tau,\pi;s}))(g) &= \phi_{\tau,\pi;s}(g)\\ &+\sum_{\gamma\in P^{4n}_{2n}(F)\backslash\mathrm{Sp}_{4n}(F)}\int_{Z_{2n}(\mathbb{A})}\int_{N^{4n}_{2n}(F)\backslash N^{4n}_{2n}(\mathbb{A})}\phi_{\tau,\pi;s}(v'w_1z\gamma'g)dvdz\\ &+M(w_0)(\phi_{\tau,\pi;s})(g),\end{aligned} \tag{5.20}$$

where the notation is as follows. $M(w_0)$ is the intertwining operator with respect to the corresponding long Weyl element

$$w_0=\begin{pmatrix} & & I_{2n}\\ & I_{4n} & \\ -I_{2n} & & \end{pmatrix};$$

w_1 is the Weyl element $\mathrm{diag}(\omega_1, \omega_1^*)$, where

$$\omega_1 = \begin{pmatrix} & I_{2n} \\ I_{2n} & \end{pmatrix};$$

for $\gamma \in \mathrm{Sp}_{4n}$, $\gamma' = \mathrm{diag}(I_{2n}, \gamma, I_{2n})$; Z_{2n} is the subgroup of elements $\mathrm{diag}(u, u^*)$, where u is of the form

$$u = \begin{pmatrix} I_{2n} & \star \\ & I_{2n} \end{pmatrix}.$$

Note that the inner dv-integration in the second term of (5.20) provides the constant term on π along P_{2n}^{4n}. This is of course zero, unless $\pi = \mathcal{E}_\tau$, in which case, we must have $L^S(\frac{1}{2}, \tau) \neq 0$. Here, we use the fact that π is of type $(\mathrm{GL}_{2n}, \tau, \frac{1}{2})$. The normalizing factor of $M(w_0)(\phi_{\tau,\pi;s})$ (outside S) is

$$\frac{L^S(\pi \times \tau, s) L^S(\tau, \wedge^2, 2s)}{L^S(\pi \times \tau, s+1) L^S(\tau, \wedge^2, 2s+1)}.$$

Since π is of type $(\mathrm{GL}_{2n}, \tau, \frac{1}{2})$, this is equal to

$$\frac{L^S(\tau \times \tau, s+\frac{1}{2}) L^S(\tau \times \tau, s-\frac{1}{2}) L^S(\tau, s) L^S(\tau, \wedge^2, 2s)}{L^S(\tau \times \tau, s+\frac{3}{2}) L^S(\tau \times \tau, s+\frac{1}{2}) L^S(\tau, s+1) L^S(\tau, \wedge^2, 2s+1)},$$

and now we see that $s = \frac{3}{2}$ is a pole of $L^S(\tau \times \tau, s - \frac{1}{2})$, and clearly, at this point all the remaining factors are holomorphic and nonzero. This shows that $M(w_0)(\phi_{\tau,\pi;s})$ has a pole at $s = \frac{3}{2}$. In the case $\pi = \mathcal{E}_\tau$, it is easy to see that the second term in (5.20) does not have a pole at $s = \frac{3}{2}$. Thus, the Eisenstein series above has a (simple) pole at $s = \frac{3}{2}$. Denote the corresponding residual representation by $\mathcal{E}_{\tau,\pi}$. Note that what we just explained also proves

Proposition 5.3. *Let $\pi \in \mathcal{N}_{\mathrm{Sp}_{4n}}(\tau, \psi)$. Then we have the following equality of the spaces of automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A}) \times \mathrm{Sp}_{4n}(\mathbb{A})$,*

$$\mathcal{C}_{N_{2n}^{8n}}(\mathcal{E}_{\tau,\pi}) = \delta_{P_{2n}^{8n}}^{\frac{1}{2}} |\det|^{-\frac{3}{2}} \tau \otimes \pi.$$

The analogue of Theorem 2.1 works with almost the same proof, except for obvious modifications, for the representation $\mathcal{E}_{\tau,\pi}$. We formulate the analogous theorem and list the main steps of the proof, which follow the main steps in the proof of Theorem 2.1, each of thich carries the same proof, as in Theorem 2.1.

Theorem 5.4. *For all integers l, such that $n < l \leq 4n$, the residual representation $\mathcal{E}_{\tau,\pi}$ has no nonzero Fourier coefficient attached to the symplectic partition*

$$[(2l)1^{2(4n-l)}].$$

Also, $\mathcal{E}_{\tau,\pi}$ has a nonzero Fourier coefficient associated with any choice of representative of the unipotent orbit $[(2n)1^{6n}]$, i.e. for all $a \in F^*$, the Fourier coefficient $\mathcal{F}^{\psi_{V_n^{8n},a}}$, defined in (2.4), is nontrivial on $\mathcal{E}_{\tau,\pi}$.

Proof. The fact that $\mathcal{E}_{\tau,\pi}$ has no nonzero Fourier coefficient corresponding to any unipotent orbit attached to $[(2l)1^{2(4n-l)}]$, for $n < i \leq 4n$, follows from Lemma 3.1(1) and Lemma 3.3 in [GRS05] (for $k = 2$). Thus, it remains to prove the statement about the orbit $[(2n)1^{6n}]$. We need to prove that for all $a \in F^*$, the integral

$$\mathcal{F}^{\psi_{V_n^{8n},a}}(\xi_{\tau,\pi}) = \int_{V_n^{8n}(F)\backslash V_n^{8n}(\mathbb{A})} \xi_{\tau,\pi}(v) \psi_{V_n^{8n},a}(v) dv \tag{5.21}$$

is not identically zero, as $\xi_{\tau,\pi}$ varies in $\mathcal{E}_{\tau,\pi}$. This is equivalent, by Lemma 1.1 in [GRS03], to the nonvanishing of the following Fourier coefficient on $\mathcal{E}_{\tau,\pi}$

$$\mathcal{F}^{\psi_{(V')^{8n}_n,a}}(\xi_{\tau,\pi}) = \int_{(V')^{8n}_n(F)\backslash (V')^{8n}_n(\mathbb{A})} \xi_{\tau,\pi}(v)\psi_{(V')^{8n}_n,a}(v)dv, \tag{5.22}$$

where $(V')^{8n}_n$ is the group of the following elements in Sp_{8n},

$$v(u,x,z) = \begin{pmatrix} u & x & z \\ & I_{6n} & x' \\ & & u^* \end{pmatrix},$$

where $u \in U_n$, $x \in \mathrm{Mat}_{n\times 6n}$ is such that $x_{n,1} = \cdots = x_{n,3n} = 0$, and

$$\psi_{(V')^{8n}_n,a}(v(u,x,z)) = \psi(u_{1,2} + \cdots + u_{n-1,n} + az_{n,1}).$$

Thus, the nonvanishing, for some choice of data, of the Fourier coefficient (5.21) is equivalent to that of the following coefficient

$$\mathcal{F}^{\psi_{\widetilde{V}^{8n}_n,a}}(\xi_{\tau,\pi}) = \int_{\widetilde{V}^{8n}_n(F)\backslash \widetilde{V}^{8n}_n(\mathbb{A})} \xi_{\tau,\pi}(v)\psi_{\widetilde{V}^{8n}_n,a}(v)dv, \tag{5.23}$$

where $\widetilde{V}^{8n}_n$ is the subgroup of $v(u,x,z) \in (V')^{8n}_n$, such that $x_{n,3n+1} = \cdots = x_{n,5n} = 0$, and $\psi_{\widetilde{V}^{8n}_n,a}$ is defined by restriction of $\psi_{(V')^{8n}_n,a}$ to $\widetilde{V}^{8n}_n(\mathbb{A})$. The nontriviality of the coefficient (5.23) will follow from the nontriviality of the following Fourier coefficient on our residual representation

$$\int_{V^{6n}_n(F)\backslash V^{6n}_n(\mathbb{A})} \int_{\widetilde{V}^{8n}_n(F)\backslash \widetilde{V}^{8n}_n(\mathbb{A})} \xi_{\tau,\pi}(vv_1)\psi_{\widetilde{V}^{8n}_n,a}(v)\psi_{V^{6n}_n,-a}(v_1)dvdv_1. \tag{5.24}$$

As in the proof of (2.14), the proof of (5.24) is very similar to [GRS99b], Sec. 5. Let

$$\omega' = \begin{pmatrix} \widetilde{\omega} & & \\ & I_{4n} & \\ & & \widetilde{\omega}^* \end{pmatrix} \in \mathrm{Sp}_{6n}(F), \tag{5.25}$$

where $\widetilde{\omega}$ is defined in (2.15). Let $R_1 = \widetilde{V}^{8n}_n V^{6n}_n$, and consider

$$B_1 = \omega' R_1 (\omega')^{-1}.$$

The integral (5.24) is equal to

$$\int_{B_1(F)\backslash B_1(\mathbb{A})} \xi_{\tau,\pi}(v\omega')\chi_{\psi,a}(v)dv. \tag{5.26}$$

The group B_1 consists of the following elements in Sp_{8n}.

$$v(T,C,Z) = \begin{pmatrix} T & C & Z \\ & I_{4n} & C' \\ & & T^* \end{pmatrix}, \tag{5.27}$$

where the last two rows of C are zero and $T \in \mathrm{GL}_{2n}$ has the form described right after (2.17), and the character $\chi_{\psi,a}$ is given by (2.18).

Next, the integral (5.26) is equal to

$$\int_{Y_1(\mathbb{A})} \int_{E_1(F)\backslash E_1(\mathbb{A})} \xi_{\tau,\pi}(vy\omega')\psi'_{E_1,a}(v)dvdy, \tag{5.28}$$

where Y_1 is the subgroup of lower unipotent matrices in B_1, E_1 is the unipotent group of Sp_{8n}, which corresponds to the symplectic partition $[(2n)^2 1^{4n}]$ and $\psi'_{E_1,a} = \psi_{[(2n)^2 1^{4n}];a,-a}$, the associated character (2.8). Thus, the dv integration in (5.28) is the application of the Fourier coefficient $\mathcal{F}^{\psi_{[(2n)^2 1^{4n}];a,-a}}$. The proof is exactly the same as the one for (2.19) (with the same root subgroups $X^{(i,j)}, Y^{(i,j)}$, but only difference that their matrices have I_{4n} as a middle block, instead of I_{2n}). Moreover, the integral (5.26) is not identically zero if and only if the inner integral of (5.28), which is (up to a right translation by $y\omega'$) $\mathcal{F}^{\psi_{[(2n)^2 1^{2n}];a,-a}}(\xi_{\tau,\pi})$, is not identically zero on $\mathcal{E}_{\tau,\pi}$. Exactly as in (2.30), we also get that for a given $\xi_{\tau,\pi}$, there is $\xi'_{\tau,\pi}$ (in $\mathcal{E}_{\tau,\pi}$) such that, for all $h \in \mathrm{Sp}_{4n}(\mathbb{A})$,

$$\int_{Y_1(\mathbb{A})} \int_{E_1(F)\backslash E_1(\mathbb{A})} \xi_{\tau,\pi}(vyh\omega')\psi'_{E_1,a}(v)dvdy = \int_{E_1(F)\backslash E_1(\mathbb{A})} \xi'_{\tau,\pi}(vh)\psi'_{E_1,a}(v)dv. \tag{5.29}$$

Let $\hat{b}'$ be the matrix defined as in (2.31), except that the middle block I_{2n} is replaced by I_{4n}. Then

$$\mathcal{F}^{\psi_{[(2n)^2 1^{2n}];a,-a}}(\xi_{\tau,\pi}) = \int_{E_1(F)\backslash E_1(\mathbb{A})} \xi_{\tau,\pi}(v\hat{b}')\psi_{E_1,a}(v)dv, \tag{5.30}$$

where $\psi_{E_1,a}$ is the character $\psi'_{E_1,a}$, conjugated by $\hat{b}'$. Note that the elements of the unipotent subgroup E have the form (2.33), (2.34), with the middle block I_{2n} replaced by I_{4n}, and then the character $\psi_{E_1,a}$ is given by (2.35). By Lemma 1.1 in [GRS03], it follows that the integral on the right hand side of (5.30) is not identically zero, if and only if the following integral is not identically zero on $\mathcal{E}_{\tau,\pi}$:

$$\mathcal{F}^{\psi_{E'_1,a}}(\xi_{\tau,\pi}) = \int_{E'_1(F)\backslash E'_1(\mathbb{A})} \xi_{\tau,\pi}(v)\psi_{E'_1,a}(v)dv, \tag{5.31}$$

where E'_1 is the unipotent F-group consisting of the elements of the form (2.33), with the only difference that the middle block I_{2n} is replaced by I_{4n}, and with A as in (2.34), but on C we require that only its last row is zero and the character $\psi_{E'_1,a}$ of $E'_1(\mathbb{A})$ is defined by formula (2.35). Let ν'' be the Weyl element in $\mathrm{Sp}_{8n}(F)$ obtained by replacing I_{2n} by I_{4n} in ν', defined in (2.37). Then

$$\mathcal{F}^{\psi_{E'_1,a}}(\xi_{\tau,\pi}) = \int_{B'_1(F)\backslash B'_1(\mathbb{A})} \xi_{\tau,\pi}(v\nu')\psi_{B'_1,a}(v)dv, \tag{5.32}$$

where $B'_1 = \nu'' E'_1(\nu'')^{-1}$. The elements in B'_1 have the form (2.39), with I_{2n} replaced by I_{4n}, and $\psi_{B'_1,a}$ is given by (2.40). Now we get the analogue of (2.41). The proof is almost a repetition of that which yields (2.41). Here, we use the property that, for all $n < l \leq 4n$, $\mathcal{E}_{\tau,\pi}$ has no nonzero Fourier coefficient attached to $[(2l)1^{2(4n-l)}]$. Thus, we get that the right hand side of (5.32) is equal to

$$\int_{L_1(\mathbb{A})} \int_{V_{2n}^{8n}(F)\backslash V_{2n}^{8n}(\mathbb{A})} \xi_{\tau,\pi}(vy\nu'')\psi'_a(v)dvdy, \tag{5.33}$$

where L_1 is the subgroup consisting of lower unipotent matrices in B'_1, and ψ'_a is the character of $V_{2n}^{8n}(\mathbb{A})$ given by

$$\psi'_a(v) = \psi(v_{1,2} + \cdots + v_{n-1,n} - av_{n,n+1} - v_{n+1,n+2} - \cdots - v_{2n-1,2n}). \tag{5.34}$$

Together with (5.33), we conclude that the integral (5.32) is not identically zero, if and only if the integral

$$\int_{V_{2n}^{8n}(F)\backslash V_{2n}^{8n}(\mathbb{A})} \xi_{\tau,\pi}(v)\psi'_a(v)dv \tag{5.35}$$

is not identically zero (as $\xi_{\tau,\pi}$ varies in $\mathcal{E}_{\tau,\pi}$). Moreover, as in (2.51), we get that for a given $\xi_{\tau,\pi}$, there is $\xi'_{\tau,\pi}$ such that, for all $h \in \mathrm{Sp}_{4n}(\mathbb{A})$,

$$\int_{L_1(\mathbb{A})}\int_{V_{2n}^{8n}(F)\backslash V_{2n}^{8n}(\mathbb{A})} \xi_{\tau,\pi}(vyh\nu'')\psi'_a(v)dvdy = \int_{V_{2n}^{8n}(F)\backslash V_{2n}^{8n}(\mathbb{A})} (\xi'_{\tau,\pi}(vh)\psi'_a(v)dv. \tag{5.36}$$

The argument similar to the one outlined after (2.52) shows that (5.35) is equal to

$$\int_{U_{2n}^{8n}(F)\backslash U_{2n}^{8n}(\mathbb{A})} \xi_{\tau,\pi}(v)\psi''_a(v)dv, \tag{5.37}$$

where U_{2n}^{8n} is the unipotent radical of the standard parabolic subgroup, whose Levi part is isomorphic to $\mathrm{GL}_1^{2n} \times \mathrm{Sp}_{4n}$, and ψ''_a is the character of $U_{2n}^{8n}(\mathbb{A})$ given by the same formula as (5.34). The integral (5.37) is equal to

$$\int_{U_{2n}(F)\backslash U_{2n}(\mathbb{A})} \mathcal{C}_{N_{2n}^{8n}}(\xi_{\tau,\pi})(u)\psi'_{U_{2n},a}(u)du, \tag{5.38}$$

where the character $\psi'_{U_{2n},a}$ is the Whittaker character of $U_{2n}(\mathbb{A})$ given by the formula (5.34); N_{2n}^{8n} is the unipotent radical of P_{2n}^{8n}; and $\mathcal{C}_{N_{2n}^{8n}}(\xi_{\tau,\pi})$ is the constant term along P_{2n}^{8n}, applied to $\xi_{\tau,\pi}$. By Proposition 5.3, we have the following equality of the spaces of automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A}) \times \mathrm{Sp}_{4n}(\mathbb{A})$,

$$\mathcal{C}_{N_{2n}^{8n}}(\mathcal{E}_{\tau,\pi}) = \delta_{P_{2n}^{8n}}^{\frac{1}{2}}|\det|^{-\frac{3}{2}}\tau \otimes \pi,$$

and now it is clear that (5.38) is not identically zero. □

Exactly in the same way that Theorem 5.1 follows by a precise book keeping of the steps of the proof of Theorem 2.1, we can do the same with the proof of the last theorem and obtain an identity, which is completely analogous to that of Theorem 5.1. It has the exact same structure. We will omit the proof. It is very similar to that of Theorem 5.1. One can see that these identities do generalize, but we prefer not to go into the general case in this paper, as this will require many more technical details and notations, and we want to keep this paper in a reasonable size. Here is the identity, analogous to Theorem 5.1.

Theorem 5.5. *Let π belong to the set $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$. Let $\phi_1 \in \mathcal{S}(\mathbb{A}^{2n})$, $\phi_2 \in \mathcal{S}(\mathbb{A}^{3n})$ and $\xi_{\tau,\pi} \in \mathcal{E}_{\tau,\pi}$. Assume that $\phi_2 = \phi_{21} \otimes \phi_{22}$, where $\phi_{21} \in \mathcal{S}(\mathbb{A}^n)$, $\phi_{22} \in \mathcal{S}(\mathbb{A}^{2n})$. Then the following identity holds, as functions in $h \in \mathrm{Sp}_{4n}(\mathbb{A})$,*

$$\begin{aligned}(5.39)\qquad &(\mathrm{FJ}^{\psi^{-1}}_{\phi_1,2n} \circ \mathrm{FJ}^{\psi}_{\phi_2,3n})(\xi_{\tau,\pi})(h) \\ &= \int_{\mathbb{A}^n}\int_{Y_1(\mathbb{A})}\int_{\mathbb{A}^{4n}}\int_{L_1(\mathbb{A})} \mathcal{C}_{N_{2n}^{8n}}(\xi_{\tau,\widetilde{\pi}})^{\psi'_{U_{2n},-1}}(hy'\nu''\widehat{l_2}\hat{b}'y\omega'\widehat{l_1})\phi(l_2,l_1)dy'dl_2dydl_1,\end{aligned}$$

where the notation can be explained as follows. By the definitions in (2.58) *and* (2.59),

$$(\mathrm{FJ}^{\psi^{-1}}_{\phi_1,2n} \circ \mathrm{FJ}^{\psi}_{\phi_2,3n})(\xi_{\tau,\pi})(h) := \int_{U_n^{6n}(F)\backslash U_n^{6n}(\mathbb{A})} \mathrm{FJ}^{\psi}_{\phi_2,3n}(\xi_{\tau,\pi})(vh)\widetilde{\theta}^{\psi^{-1}}_{\phi_1,2n}(\ell_{2n}(v)h)\psi_{U_n^{6n}}(v)dv,$$

with the function ϕ is in $\mathcal{S}(\mathbb{A}^{5n})$, which can be written explicitly in terms of $\phi_1, \phi_{21}, \phi_{22}$. The elements ω', $\hat{b}'$, ν'', the unipotent groups Y_1, L_1, and the Whittaker character $\psi'_{U_{2n}}$ for GL_{2n}, are all defined in the proof of Theorem 5.4. Finally, for $l_1 = (x_1, ..., x_n) \in \mathbb{A}^n$,

$$\widehat{l_1} = I_{8n} + \sum_{i=1}^{n} x_i e'_{n,n+i},$$

and for $l_2 = (y_1, ..., y_{4n}) \in \mathbb{A}^{4n}$,

$$\widehat{l_2} = I_{8n} + \sum_{i=1}^{4n} y_i e'_{2n,2n+i},$$

where $e'_{i,j} = e_{i,j} - e_{8n-j+1,8n-i+1}$, and $e_{i,j}$ is the $4n \times 4n$ matrix, which has 1 at the coordinate (i, j), and zero elsewhere.

Exactly as in the proof of Proposition 5.2, we conclude, using Proposition 5.3,

Proposition 5.6. *The space of automorphic forms on $\mathrm{Sp}_{4n}(\mathbb{A})$ generated by the elements*

$$(\mathrm{FJ}^{\psi^{-1}}_{\phi_1,2n} \circ \mathrm{FJ}^{\psi}_{\phi_2,3n})(\xi_{\tau,\pi})(h)$$

in (5.39) is equal to the space of π.

Theorem 5.5 and Proposition 5.6 prove the following analogue of Theorem 4.2,

Theorem 5.7. *For any representation $\pi \in \mathcal{N}_{\mathrm{Sp}_{4n}}(\tau, \psi)$, the space of the double descent is identically equal to the space of π:*

$$\mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{D}}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi})) = \pi.$$

For elements of $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau, \psi)$, we have the following analogue of Theorem 2.5.

Theorem 5.8. *Let $\pi \in \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau, \psi)$. Then the automorphic representation $\widetilde{\mathcal{D}}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi})$ is square-integrable. Moreover, there is an irreducible subrepresentation $\widetilde{\pi}$ of $\Psi(\pi)$, such that the space of the ψ-descent $\widetilde{\mathcal{D}}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi})$ has a nontrivial intersection with the space of the residual representation $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ of $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$.*

Proof. We follow the proof of Theorem 2.5. We have a formula, as in (2.62), for the constant term along P^{6n}_r, $1 \leq r \leq 3n$, of $\mathrm{FJ}^{\psi}_{\phi,3n}(\xi_{\tau,\pi})$ (evaluated at the identity). It reads

$$\sum_{j=0}^{r} \sum_{\gamma \in P^1_{r-j,1^j}(F)\backslash \mathrm{GL}_r(F)} \int_{L(\mathbb{A})} \phi_1(i(\lambda)) \mathrm{FJ}^{\psi}_{\phi_2,3n-r}(\mathcal{C}_{N^{8n}_{r-j}}(\xi_{\tau,\pi}))(\hat{\gamma}\lambda\beta) d\lambda. \tag{5.40}$$

Here, we assume that $\phi = \phi_1 \otimes \phi_2$ with $\phi_1 \in \mathcal{S}(\mathbb{A}^r)$ and $\phi_2 \in \mathcal{S}(\mathbb{A}^{3n-r})$; the subgroup $P^1_{r-j,1^j}$ is the one in (2.62); the rest of the notation is similar; for $a \in \mathrm{GL}_k$, $k \leq 4n$, we denote $\hat{a} = \mathrm{diag}(a, I_{2(4n-k)}, a^*)$; the group L is unipotent and consists of all matrices

$$\lambda = \begin{pmatrix} I_r & 0 \\ x & I_n \end{pmatrix}^{\wedge} \in \mathrm{Sp}_{8n},$$

and in this notation $i(\lambda)$ is the last row of x; and finally, $\beta = \beta_r = \begin{pmatrix} & I_r \\ I_n & \end{pmatrix}^{\wedge}$. The proof of the formula appears in detail in Sec. 7.6 in the book [GRS11].

The residual representation $\mathcal{E}_{\tau,\pi}$ has only one nontrivial constant term, namely the one along P^{8n}_{2n}, in the case that π is cuspidal. In the case that $\pi = \mathcal{E}_\tau$, it has an additional constant term, which is along P^{8n}_{4n}, the standard maximal parabolic subgroup whose Levi part GL_{4n}. Thus, in (5.40), either $r-j=0$, $r-j=2n$, or $r-j=4n$.

Since $r \leq 3n$, the last case is impossible. If $r-j=0$, then the corresponding term in (5.40) is zero. Indeed, in this case, $\mathrm{FJ}^{\psi}_{\phi_2,3n-r}(\mathcal{C}_{N^{8n}_{r-j}}(\xi_{\tau,\pi})) = \mathrm{FJ}^{\psi}_{\phi_2,3n-r}(\xi_{\tau,\pi})$, and since the Fourier-Jacobi coefficient $\mathrm{FJ}^{\psi}_{\phi_2,3n-r}$ involves the Fourier coefficient corresponding to $[2(n+r), 1^{2(3n-r)}]$ (Lemma 2.2), we see, by the first part of Theorem 5.4, that $\mathrm{FJ}^{\psi}_{\phi_2,3n-r}$ is zero on $\mathcal{E}_{\tau,\pi}$. Thus, $r-j=2n$. If $j \geq 1$, then the corresponding term in (5.40) is zero. For this, we will show that $\mathrm{FJ}^{\psi}_{\phi_2,3n-r}(\mathcal{C}_{N^{8n}_{2n}}(\xi_{\tau,\pi}))$ is identically zero on $\mathcal{E}_{\tau,\pi}$. By Proposition 5.3, the restriction of $\mathcal{C}_{N^{8n}_{2n}}(\xi_{\tau,\pi})$ to the Levi subgroup of $P^{8n}_{2n}(\mathbb{A})$, identified with $\mathrm{GL}_{2n}(\mathbb{A}) \times \mathrm{Sp}_{4n}(\mathbb{A})$, lies in $\delta^{\frac{1}{2}}_{P^{8n}_{2n}} |\det|^{-\frac{3}{2}} \tau \otimes \pi$. When we further apply $\mathrm{FJ}^{\psi}_{\phi_2,3n-r}$, we apply it to the second factor in the tensor product, namely to π. Note that this Fourier-Jacobi coefficient involves the Fourier coefficient corresponding to $[2(2n-t), 1^{2t}]$, where $t = 3n-r$. Since $r = 2n+j > 2n$, $t < n$. Since π is of type $(\mathrm{GL}_{2n}, \tau, \frac{1}{2})$, any such Fourier coefficient is trivial on π, and hence (Lemma 2.2) $\mathrm{FJ}^{\psi}_{\phi_2,3n-r}$ is trivial on π.

We conclude that (5.40) is zero, unless $r = 2n$, and then it reduces to just one term, namely the one corresponding to $j = 0$. In this case the constant term $\mathcal{C}_{N^{6n}_{2n}}(\mathrm{FJ}^{\psi}_{\phi,3n}(\xi_{\tau,\pi}))$ (evaluated at the identity) is, with notation as above, equal to

$$\int_{L(\mathbb{A})} \phi_1(i(\lambda)) \mathrm{FJ}^{\psi}_{\phi_2,n}(\mathcal{C}_{N^{8n}_{2n}}(\xi_{\tau,\pi}))(\lambda\beta_{2n}) d\lambda. \tag{5.41}$$

From (5.41), it is easy to see that, as a representation of $\mathrm{GL}_{2n}(\mathbb{A}) \times \widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$,

$$\mathcal{C}_{N^{6n}_{2n}}(\widetilde{D}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi})) = \gamma_\psi \delta^{\frac{1}{2}}_{P^{6n}_{2n}} |\det|^{-1} \tau \otimes \Psi(\pi). \tag{5.42}$$

Thus, $\widetilde{\mathcal{D}}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi})$ has a unique exponent and it is negative. In particular, it is square integrable, and if ρ is a non-cuspidal irreducible subrepresentation, then (5.42) implies that ρ must be an irreducible subrepresentation of $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$, for some irreducible subrepresentation $\widetilde{\pi}$ of $\Psi(\pi)$. This proves that the space of the descent $\widetilde{\mathcal{D}}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi})$ has a nontrivial intersection with the space of the residual representation $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$, when π is member in $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$. $\square$

Remark 5.9. *Note that in Theorem 5.8, we do not use* Assumption (A). *If we use* Assumption (A)*, then the residual representation $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ of $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$ is irreducible. In this case, Theorem 5.8 asserts that the ψ-descent $\widetilde{\mathcal{D}}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi})$ contains the residual representation $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ as an irreducible subrepresentation. This is the technical point where* Assumption (A) *is needed in the proof of Theorem 4.4 in §6.2.*

6. Proof of the Main Theorems

In this section, we are going to use the results established in the previous section and prove Theorems 4.1, 4.3, and 4.4. From these theorems, Theorem 4.5 follows.

6.1. **Proof of Theorem 4.1: Irreducibility of $\Psi'(\pi)$.** Let $\pi \in \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau, \psi)$. By Theorem 5.7,

$$\pi = \mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{D}}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi})). \tag{6.1}$$

By Theorem 5.8, there is an irreducible subrepresentation $\widetilde{\pi}$ of $\Psi(\pi)$, such that the space of $\widetilde{\mathcal{D}}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi})$ contains an irreducible subrepresentation σ of $\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$. From (6.1), since π is irreducible, we get that

$$\pi = \mathcal{D}^{6n}_{4n,\psi^{-1}}(\sigma). \tag{6.2}$$

By definition, $\mathcal{D}^{6n}_{4n,\psi^{-1}}(\sigma) \subset \Phi(\widetilde{\pi})$, and hence

$$\pi \subset \Phi(\widetilde{\pi}). \tag{6.3}$$

Applying Theorem 4.2, we get

$$\Psi(\pi) = \Psi'(\pi) \subset \Psi(\Phi(\widetilde{\pi})) = \widetilde{\pi}.$$

Since $\widetilde{\pi}$ is irreducible, we conclude that $\Psi(\pi) = \widetilde{\pi}$ is irreducible. This proves Theorem 4.1.

6.2. **Proof of Theorems 4.3, 4.4.** Let $\pi \in \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau, \psi)$. By Theorem 4.1, $\Psi(\pi) = \widetilde{\pi}$ is irreducible. By (6.3), we get

$$\pi \subset \Phi(\widetilde{\pi}) = \Phi(\Psi(\pi)).$$

This proves Theorem 4.3.

Suppose that Assumption (A) holds. Then in the proof of Theorem 4.1, we get that $\sigma = \widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}$ as in Remark 5.8, and then, by (6.2)

$$\pi = \mathcal{D}^{6n}_{4n,\psi^{-1}}(\widetilde{\mathcal{E}}_{\tau,\widetilde{\pi}}) = \Phi(\widetilde{\pi}) = \Phi(\Psi(\pi)).$$

This proves Theorem 4.4. Theorem 4.5 now follows.

We note that Assumption (A) is used only in the last step of the proof of Theorem 4.4.

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