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On the Fundamental Automorphic L-Functions of $SO(2n+1)$

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1 Introduction

The fundamental automorphic L-functions of SO_{2n+1} are by definition the Langlands automorphic L-functions attached to irreducible cuspidal automorphic representations σ of $SO_{2n+1}(\mathbb{A})$ and the fundamental complex representations $\rho_1, \rho_2, \dots, \rho_n$ of the complex dual group $Sp_{2n}(\mathbb{C})$ of SO_{2n+1}, where $\mathbb{A}$ is the ring of adeles of the number field k. These L-functions are denoted by $L(s, \sigma, \rho_j)$ which is given by a Euler product of all local L-factors. The precise definition will be given in Section 2.

It is known by a theorem of Langlands that the L-functions $L(s, \sigma, \rho_j)$ converge absolutely for the real part of s large [6]. The Langlands conjecture asserts that $L(s, \sigma, \rho_j)$ should have meromorphic continuation to the whole complex plane $\mathbb{C}$, satisfy a functional equation relating the value at s to the value at $1-s$, and have finitely many poles on the real line. When $j = 1$, $L(s, \sigma, \rho_1)$ is the standard L-function. In this case, the Langlands conjecture has been verified through the doubling method of Gelbart, Piatetski-Shapiro, and Rallis [11] and through the Langlands-Shahidi method [12]. For $j \geq 2$, it is clear that the Langlands conjecture for $L(s, \sigma, \rho_j)$ is beyond reach via the Langlands-Shahidi method or via any currently known integral representation of the Rankin-Selberg-type (except for certain cases of $n \leq 4$ and $j = 2$ [7]).

One of the results in this paper is to verify the Langlands conjecture for the case when $j = 2$ and σ is generic. It uses the recent results on local and global Langlands

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functoriality [8, 9, 15, 24, 25] and the local Langlands correspondence for general linear groups [18, 19]. Such an application of the progress in the Langlands functoriality seems to be expected.

Theorem 1.1 (Theorem 2.1). Let σ be an irreducible generic unitary cuspidal automorphic representation of $SO_{2n+1}(\mathbb{A})$ and let ρ_2 be the second fundamental complex representation of the complex dual group $Sp_{2n}(\mathbb{C})$ of SO_{2n+1}. Then the second fundamental automorphic L-function $L(s, \sigma, \rho_2)$ converges absolutely and is nonzero for the real part of s greater than one, has meromorphic continuation to the whole complex plane $\mathbb{C}$, and satisfies the functional equation relating s to $1-s$. Moreover, $L(s, \sigma, \rho_2)$ has possible poles at $s = 0, 1$, besides other possible poles in the open interval $(0, 1)$. □

From the proof we give in Section 2, the finiteness of the possible poles of $L(s, \sigma, \rho_2)$ is essentially related to the zeros of other L-functions, including the Dedekind zeta function, in the open interval $(0, 1)$. This indicates that the finiteness of poles of automorphic L-functions is a deep and difficult problem. In some special cases when the L-functions are in the Shahidi list or can be represented by integrals of Rankin-Selberg-type, the finiteness of poles of the automorphic L-functions is expected to follow essentially from the finiteness of poles of normalized Eisenstein series.

In a recent paper [30], it is proved that almost all automorphic L-functions occurring in the Shahidi list have at most simple pole at $s = 1$. Of course, many examples show that automorphic L-functions may have higher-order poles at $s = 1$ even for irreducible generic unitary cuspidal automorphic representations. For example, the degree 16 automorphic L-functions for irreducible generic unitary cuspidal automorphic representations of $GSp(4) \times GSp(4)$ may have a pole at $s = 1$ of order at most two. The occurrence of the different orders of the pole at $s = 1$ indicates different endoscopy structures of the cuspidal automorphic representations under consideration, some more details of which can be found in the introduction in [23] and can now be proved by means of [4]. Another well-known example is that an irreducible unitary cuspidal automorphic representation π of $GL_{2n}(\mathbb{A})$ is an endoscopy lifting from $SO_{2n+1}(\mathbb{A})$ if and only if the exterior square L-function $L(s, \pi, \Lambda^2)$ has a pole at $s = 1$, which is known to be a simple pole. We give precise account of this fact in Section 2. Some other special cases of the same nature are studied by many authors. We refer to [7] for preliminarily discussions in this aspect.

Another result in this paper is to show that the order of the pole at $s = 1$ of $L(s, \sigma, \rho_2)$ carries information of the endoscopy structure of σ. More precisely, we prove the following theorem.

Theorem 1.2 (Theorem 3.2)**.** Let σ be an irreducible generic unitary cuspidal automorphic representation of $SO_{2n+1}(\mathbb{A})$ and let ρ_2 be the second fundamental complex representation of $Sp_{2n}(\mathbb{C})$.

(1) The second fundamental automorphic L-function $L(s,\sigma,\rho_2)$ may have a possible pole at $s=1$ of order from 0 to $n-1$.

(2) $L(s,\sigma,\rho_2)$ has a pole at $s=1$ of order $r-1$ if and only if there exists a partition $n=\sum_{j=1}^r n_j$ with $n_j>0$ such that σ is an endoscopy lifting (see Section 3 for definition) from an irreducible, generic, unitary, stable, cuspidal automorphic representation $\sigma_1\otimes\cdots\otimes\sigma_r$ of an endoscopy group $SO_{2n_1+1}(\mathbb{A})\times\cdots\times SO_{2n_r+1}(\mathbb{A})$. Moreover, the endoscopy data are uniquely determined by the given σ (see Section 3 for detail).

(3) The order of the pole at $s=1$ of $L(s,\sigma,\rho_2)$ is equal to the multiplicity of the trivial representation of $H^\vee_{[n_1,\ldots,n_r]}=Sp_{2n_1}(\mathbb{C})\times\cdots\times Sp_{2n_r}(\mathbb{C})$ occurring in the restriction of ρ_2 to $H^\vee_{[n_1,\ldots,n_r]}$. □

By definition, an irreducible generic unitary cuspidal automorphic representation σ of $SO_{2n+1}(\mathbb{A})$ is *stable* if it is not an image of any nontrivial endoscopy transfer. The definition of endoscopy transfers in the context of this paper will be given in Section 3. It follows from the theorem that an irreducible generic unitary cuspidal automorphic representation σ of $SO_{2n+1}(\mathbb{A})$ is stable if and only if the second fundamental automorphic L-function $L(s,\sigma,\rho_2)$ is holomorphic at $s=1$.

If the complete L-function $L(s,\sigma,\rho_2)$ is replaced by a partial L-function $L^S(s,\sigma,\rho_2)$ for a finite set S of local places including all Archimedean local places, then the theorem can be proved essentially by a reformulation of the explicit Langlands functorial transfer from $SO_{2n+1}(\mathbb{A})$ to $GL_{2n}(\mathbb{A})$ for irreducible generic unitary cuspidal automorphic representations, which has been established in [9, 15, 24, 25]. We give details of the proof of this theorem in Section 3.

It is expected that analogues of the above theorems for other classical groups can be easily formulated based on the extensions of the work in [24, 25] from SO_{2n+1} to all classical groups, and the work in [3, 9, 29, 39]. Guided by the Langlands philosophy, Theorem 1.2 gives a new and interesting example of relation between the poles of certain L-functions and the endoscopy structure of irreducible cuspidal automorphic representations under consideration. A special case when $r=n$ of Theorem 1.2 is a part of [13, Conjecture 1]. It is not clear if there exists a nonvanishing criterion in terms of a certain period to characterize endoscopy structures of σ as given in the theorem, although it is expected for the special case when $r=n$ in [13]. The results for nongeneric cuspidal automorphic representations are currently still beyond reach in general. However, from the forthcoming work [26], a certain family of cuspidal automorphic representations of Ramanujan-type can be treated in the same way.

In the last section we discuss certain relation between Theorem 1.2 and a Langlands problem (Problem 4.1) (or [33, Problem (II)]). After calculating explicitly the invariants in Section 4, we prove the following theorem.

Theorem 1.3 (Theorem 4.10)**.** Assume the validity of Assumption 4.2 for the L-functions attached to the fundamental representations $\rho_3, \rho_4, \ldots, \rho_n$ and Assumption 4.6. For a given irreducible generic unitary cuspidal automorphic representation σ of $SO_{2n+1}(\mathbb{A})$, the structure of the algebraic subgroup

$$H^{\vee}_{[n_1,\ldots,n_r]} = Sp_{2n_1}(\mathbb{C}) \times \cdots \times Sp_{2n_r}(\mathbb{C}), \tag{1.1}$$

that is, the partition $n = \sum_{j=1}^{r} n_j$, is completely determined by the order of the pole at $s = 1$ of the L-function $L(s, \sigma, \rho)$ for ρ being the fundamental representations $\rho_2, \rho_4, \ldots, \rho_{2[n/2]}$ of the complex dual group $Sp_{2n}(\mathbb{C})$. □

It is expected that the algebraic subgroup $H^{\vee}_{[n_1,\ldots,n_r]}$ for the given σ is closely related to the conjectural algebraic subgroup $\mathcal{H}_\sigma$ as given in the Langlands problem (Problem 4.1). Some details can be found in Section 4 although we are not be able to prove this here.

2 The second fundamental L-functions

Let k be a number field and $\mathbb{A} = \mathbb{A}_k$ be the ring of adeles of k. Let $G_n := SO(2n+1)$ be the k-split odd special orthogonal group, which is associated to a nondegenerate quadratic vector space of dimension $2n+1$ with Witt index n. The complex dual group ${}^LG_n^\circ = G_n^\vee$ of G_n is $Sp_{2n}(\mathbb{C})$. Since G_n is k-split, we may take $Sp_{2n}(\mathbb{C})$ as the Langlands dual group of G_n without loss of generality.

The fundamental representations of $Sp_{2n}(\mathbb{C})$ are the finite-dimensional complex representations associated to the fundamental weights. They can be constructed by the following (split) exact sequence:

$$0 \longrightarrow V^{(2n)}_{\rho_a} \longrightarrow \Lambda^a\left(\mathbb{C}^{2n}\right) \longrightarrow \Lambda^{a-2}\left(\mathbb{C}^{2n}\right) \longrightarrow 0, \tag{2.1}$$

where $\Lambda^a(\mathbb{C}^{2n})$ denotes the ath exterior power of $\mathbb{C}^{2n}$, the contraction map from $\Lambda^a(\mathbb{C}^{2n})$ onto $\Lambda^{a-2}(\mathbb{C}^{2n})$ is as defined in [16, page 236], and its kernel is denoted by $V^{(2n)}_{\rho_a}$. By [16, Theorem 5.1.8], $V^{(2n)}_{\rho_a}$ is the space of the irreducible representation ρ_a of $Sp_{2n}(\mathbb{C})$ with the ath fundamental weight. We prove in this section the Langlands conjecture on the analytic property of automorphic L-functions for the second fundamental L-function $L(s, \sigma, \rho_2)$.

Let ι be the natural embedding of $\mathrm{Sp}_{2n}(\mathbb{C})$ into $\mathrm{GL}_{2n}(\mathbb{C})$. Let Λ^2 be the exterior square representation of $\mathrm{GL}_{2n}(\mathbb{C})$ on the vector space $\Lambda^2(\mathbb{C}^{2n})$, which has dimension $2n^2 - n$. The composition $\Lambda^2 \circ \iota$ of Λ^2 with ι is a complex representation of $\mathrm{Sp}_{2n}(\mathbb{C})$. By (2.1) and by complete reducibility of representations of $\mathrm{Sp}_{2n}(\mathbb{C})$, we obtain

$$\Lambda^2 \circ \iota = \rho_2 \oplus \mathbf{1}_{\mathrm{Sp}_{2n}}, \tag{2.2}$$

where ρ_2 is the second fundamental complex representation of $\mathrm{Sp}_{2n}(\mathbb{C})$, which is irreducible and has dimension $2n^2 - n - 1$, and $\mathbf{1}_{\mathrm{Sp}_{2n}}$ is the trivial representation of $\mathrm{Sp}_{2n}(\mathbb{C})$.

Let $\sigma = \otimes_v \sigma_v$ be an irreducible generic unitary cuspidal automorphic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$. Then each local component σ_v is an irreducible admissible generic representation of the local group $\mathrm{SO}_{2n+1}(k_v)$. By [24, Theorem 6.4] and [25, Theorem 6.1], each σ_v for all finite local places has a local Langlands parameter φ_v such that the correspondence between σ_v and φ_v satisfies the compatibility conditions for local L-, ϵ-, and γ-factors and its GL-twisted versions. Of course, for infinite local places, this was known through the work of Langlands [32]. We define for each local place the second fundamental local L-factor attached to σ_v by

$$L(s, \sigma_v, \rho_2) := L(s, \rho_2 \circ \varphi_v). \tag{2.3}$$

This natural definition is also compatible with the recent work of Henniart [20] on the compatibility for exterior square local factors under the local Langlands correspondence for GL over p-adic local fields. It follows from (2.2) that

$$L(s, \sigma_v, \rho_2) = \frac{L(s, \pi_v, \Lambda^2)}{\zeta_{k_v}(s)}, \tag{2.4}$$

where $L(s, \pi_v, \Lambda^2)$ is the local L-factor attached to π_v and the exterior square complex representation Λ^2 of $\mathrm{GL}_{2n}(\mathbb{C})$ via the local Langlands parameterization, and $\zeta_{k_v}(s)$ is the local Eulerian factor of the Dedekind zeta function attached to the number field k. Here π_v is the irreducible admissible representation of $\mathrm{GL}_{2n}(k_v)$ attached to the local Langlands parameter $\iota \circ \varphi_v$ through the local Langlands correspondence for GL_{2n} and is the image of σ_v under the local Langlands functorial transfer from SO_{2n+1} to GL_{2n}.

The second fundamental automorphic L-function attached to σ in the sense of Langlands is defined by the following Eulerian product:

$$L(s, \sigma, \rho_2) := \prod_v L(s, \sigma_v, \rho_2). \tag{2.5}$$

It is a theorem of Langlands ([6]) that this Eulerian product converges absolutely for the real part of s large. Then for the real part of s large, the following identity follows from (2.4):

$$L(s,\sigma,\rho_2)=\prod_v \frac{L(s,\pi_v,\Lambda^2)}{\zeta_{k_v}(s)}. \tag{2.6}$$

Let π be the image of σ under the global Langlands functorial transfer from SO_{2n+1} to GL_{2n} which is compatible with the local Langlands functorial transfer from SO_{2n+1} to GL_{2n} at all local places ([9, Theorem 7.1], [24, Theorem 6.1], and [25, Theorem 5.1]). Hence we have

$$\pi=\otimes_v\pi_v, \tag{2.7}$$

where the local components π_v are the same as the ones used in (2.4), (2.5), (2.6), and (2.7). Therefore, we obtain

$$L(s,\sigma,\rho_2)=\frac{L(s,\pi,\Lambda^2)}{\zeta_k(s)} \tag{2.8}$$

for the real part of s large and part (1) of the following theorem which is a more precise version of Theorem 1.1.

Theorem 2.1. Let σ be an irreducible generic unitary cuspidal automorphic representation of $SO_{2n+1}(\mathbb{A})$ and let ρ_2 be the second fundamental complex representation of the complex dual group $Sp_{2n}(\mathbb{C})$ of SO_{2n+1}. Then the second fundamental automorphic L-function $L(s,\sigma,\rho_2)$ enjoys the following properties.

(1) There exists an irreducible admissible unitary automorphic representation π of $GL_{2n}(\mathbb{A})$ such that

$$L(s,\sigma,\rho_2)=\frac{L(s,\pi,\Lambda^2)}{\zeta_k(s)} \tag{2.9}$$

as in (2.8) holds for the real part of s large.

(2) The Eulerian product defining the L-function $L(s,\sigma,\rho_2)$ converges absolutely for the real part of s greater than one, has meromorphic continuation to the whole complex plane, and satisfies the functional equation

$$L(s,\sigma,\rho_2)=\epsilon(s,\sigma,\rho_2)L(1-s,\sigma^\vee,\rho_2), \tag{2.10}$$

with $\epsilon(s,\sigma,\rho_2)=\epsilon(s,\pi,\Lambda^2)$, the ϵ-factor for the exterior square L-function $L(s,\pi,\Lambda^2)$.

(3) The L-function $L(s, \sigma, \rho_2)$ has possible poles at $s = 0, 1$, besides other possible poles in the open interval $(0, 1)$. □

In order to prove this theorem, we recall first the relevant results on $L(s, \pi, \Lambda^2)$ for irreducible unitary cuspidal automorphic representation π of $GL_{2n}(\mathbb{A})$.

Theorem 2.2. Let π be an irreducible, unitary, self-dual, cuspidal automorphic representation π of $GL_{2n}(\mathbb{A})$.

(1) The exterior square L-function $L(s, \pi, \Lambda^2)$ is holomorphic for the real part of s greater than one [28, Theorem 3.1].

(2) $L(s, \pi, \Lambda^2)$ has at most a simple pole at $s = 1$.

(3) $L(s, \pi, \Lambda^2)$ has a simple pole at $s = 1$ if and only if π is a Langlands functorial lifting from an irreducible generic cuspidal automorphic representation σ of $SO_{2n+1}(\mathbb{A})$.

(4) Let S be a finite set of local places of k including all Archimedean places. The complete exterior square L-function $L(s, \pi, \Lambda^2)$ has a simple pole at $s = 1$ if and only if the partial exterior square L-function $L^S(s, \pi, \Lambda^2)$ has a simple pole at $s = 1$.

(5) The partial exterior square L-function $L^S(s, \pi, \Lambda^2)$ has a simple pole at $s = 1$ if and only if π has a nontrivial Shalika model [22, the main theorem]. □

The proof for (2) follows easily from the theory of the Rankin-Selberg product L-functions $L(s, \pi \times \pi)$ [10, 35, 37]. Theorem 7.1 of [9], which is based on the theory of Rankin-Selberg convolution L-functions of SO_{2n+1} with GL_r (see [39], or [15] for instance) shows that if π of $GL_{2n}(\mathbb{A})$ is an endoscopy lifting from $SO_{2n+1}(\mathbb{A})$, then $L(s, \pi, \Lambda^2)$ has a pole at $s = 1$. Theorem A of [15] shows that if $L(s, \pi, \Lambda^2)$ has a pole at $s = 1$, then π is a weak Langlands transfer from SO_{2n+1} and [25, Theorem 7.3] shows that if $L(s, \pi, \Lambda^2)$ has a pole at $s = 1$, then every local component π_v of π is symplectic, that is, is the image of the local Langlands transfer from $SO_{2n+1}(k_v)$. This gives a complete account for (3). Part (4) is equivalent to that for irreducible unitary generic representations π_v of $GL_{2n}(k_v)$, the local exterior square L-factor $L(s, \pi_v, \Lambda^2)$ is holomorphic and nonzero at $s = 1$. When π_v is tempered, it follows from [38, Proposition 7.2] that $L(s, \pi_v, \Lambda^2)$ is holomorphic for $\mathrm{Re}(s) > 0$ (which must be nonzero), that is, Shahidi's conjecture, [38, Conjecture 7.1] holds in the case. By the classification of the unitary dual of $GL_{2n}(k_v)$ (by Vogan [41] for $k_v = \mathbb{C}$, or $\mathbb{R}$, and by Tadić [40] when k_v is nonarchimedean), the nontempered exponents α in the classification satisfies the inequalities $0 < \alpha < 1/2$, and hence the local exterior square L-factor $L(s, \pi_v, \Lambda^2)$ is holomorphic and nonzero at $s = 1$.

We now prove Theorem 2.1. Note that the complete exterior square L-function discussed in Theorem 2.2 is given by the Langlands-Shahidi method, while the complete exterior square L-function used in Theorem 2.1 is given via the local Langlands parameterization. As mentioned before, their local components agree at all local places by Henniart's work [20]. Hence we are able to use Theorem 2.2 in order to prove Theorem 2.1.

We have already proved part (1) before stating the theorem. For parts (2) and (3), we have to use the explicit structure of the Langlands functorial transfer from SO_{2n+1} to GL_{2n} for irreducible generic unitary cuspidal automorphic representations.

Let σ be an irreducible generic unitary cuspidal automorphic representation of $SO_{2n+1}(\mathbb{A})$. By the theorem in Section 1 of [8], there exists an irreducible unitary automorphic representation $\pi = \pi(\sigma)$, which is the weak Langlands functorial transfer of σ to $GL_{2n}(\mathbb{A})$. It is proved in [24, Theorem 6.1], [25, Theorem 5.1], and [9, Theorem 7.1] that the transfer from σ to $\pi(\sigma)$ is Langlands functorial at all local places. By [15] the image of the Langlands functorial transfer from all irreducible generic cuspidal automorphic representations σ of $SO_{2n+1}(\mathbb{A})$ is completely characterized, which can be stated as follows.

Theorem 2.3 [15, Theorem A]**.** An irreducible unitary automorphic representation π of $GL_{2n}(\mathbb{A})$ is the weak Langlands functorial transfer of an irreducible generic unitary cuspidal automorphic representation σ of $SO_{2n+1}(\mathbb{A})$ if and only if π is equivalent to the following isobaric representation:

$$\pi \cong \tau_1 \boxplus \cdots \boxplus \tau_r, \tag{2.11}$$

where τ_j for $j = 1, 2, \ldots, r$, is an irreducible cuspidal automorphic representation of $GL_{2n_j}(\mathbb{A})$ with the properties that

(a) $n = \sum_{j=1}^{r} n_j$ is a partition of n with $n_j > 0$;
(b) $\tau_i \ncong \tau_j$ if $i \neq j$;
(c) the partial exterior square L-function $L^S(s, \tau_j, \Lambda^2)$ has a pole at $s = 1$. □

More explicit information about this Langlands transfer will be given in Section 3.

By Theorem 2.3, it follows that the exterior square L-function $L(s, \pi, \Lambda^2)$ can be expressed as

$$L(s, \pi, \Lambda^2) = \prod_{i=1}^{r} L(s, \tau_i, \Lambda^2) \times \prod_{1 \leq i < j \leq r} L(s, \tau_i \times \tau_j). \tag{2.12}$$

By the work of Moeglin and Waldspurger ([35], which refines the work of Shahidi and the work of Jacquet, Piatetski-Shapiro, and Shalika), the Rankin-Selberg convolution L-function $L(s, \tau_i \times \tau_j)$ converges absolutely for the real part of s greater than one, has meromorphic continuation to the whole complex plane $\mathbb{C}$, satisfies the functional equation relating s to $1 - s$, and has only possible poles at $s = 0, 1$. If $n_i \neq n_j$, then $L(s, \tau_i \times \tau_j)$ is entire. In the case when $n_i = n_j$ $(i \neq j)$, τ_i and τ_j are self-dual and not equivalent. Hence $L(s, \tau_i \times \tau_j)$ is still entire. These properties have also been established in [10, Theorem 2.4]. For the exterior square L-functions $L(s, \tau_i, \Lambda^2)$, by Theorem 2.2, it converges absolutely for the real part of s greater than one, satisfies the functional equation

$$L(s, \tau_i, \Lambda^2) = \epsilon(s, \tau_i, \Lambda^2) L(1 - s, \tau_i, \Lambda^2), \tag{2.13}$$

and has possible poles in the closed interval $[0, 1]$. Hence parts (2) and (3) now follow from (2.8) and (2.11). This completes the proof of Theorem 2.1.

Remark 2.4. We note that the other possible poles belonging to the open interval $(0, 1)$ of the second fundamental L-function $L(s, \sigma, \rho_2)$ are directly related to the zeros belonging to the open interval $(0, 1)$ of the exterior square L-functions $L(s, \tau_i, \Lambda^2)$, the Rankin-Selberg product L-functions $L(s, \tau_i \times \tau_j)$, and the Dedekind zeta function $\zeta_k(s)$!

Corollary 2.5. The global ϵ-factor $\epsilon(s, \sigma, \rho_2)$ has the following formula:

$$\epsilon(s, \sigma, \rho_2) = \prod_{i=1}^{r} \epsilon(s, \tau_i, \Lambda^2) \times \prod_{1 \le i < j \le r} \epsilon(s, \tau_i \times \tau_j). \tag{2.14}$$

□

3 Poles and endoscopy liftings

We discuss here the order of the pole at $s = 1$ of the second fundamental L-function $L(s, \sigma, \rho_2)$ and the endoscopy structure of the irreducible generic cuspidal automorphic representation σ of $SO_{2n+1}(\mathbb{A})$.

The theory of twisted endoscopy can be found in [31]. For simplicity, we first recall from [1, 2] the basic structure of all standard elliptic endoscopy groups of SO_{2n+1}. Let $n = n_1 + n_2$ with $n_1, n_2 > 0$. Take a semisimple element

$$s_{n_1,n_2} = \begin{pmatrix} -I_{n_1} & & 0 \\ & I_{2n_2} & \\ 0 & & -I_{n_1} \end{pmatrix} \in Sp_{2n}(\mathbb{C}). \tag{3.1}$$

Then the centralizer of s_{n_1,n_2} in $Sp_{2n}(\mathbb{C})$ is given by

$$H^{\vee}_{[n_1,n_2]} = \mathrm{Cent}_{Sp_{2n}(\mathbb{C})}\left(s_{n_1,n_2}\right) = Sp_{2n_1}(\mathbb{C}) \times Sp_{2n_2}(\mathbb{C}). \tag{3.2}$$

The standard elliptic endoscopy group associated to the partition $n = n_1 + n_2$ is

$$H_{[n_1,n_2]} = SO_{2n_1+1} \times SO_{2n_2+1}, \tag{3.3}$$

and the groups $H_{[n_1,n_2]}$ exhaust all standard elliptic endoscopy groups of SO_{2n+1}, in the sense of [**31**].

In general, an endoscopy transfer of automorphic representations from an endoscopy group H of G to G takes a global Arthur packet of automorphic representations of $H(\mathbb{A})$ to a global Arthur packet of automorphic representations of $G(\mathbb{A})$, which is characterized by the stability of certain distributions from the geometric side of the Arthur trace formula [**2**]. Since the automorphic representations of $H(\mathbb{A})$ and $G(\mathbb{A})$ considered in this paper are generic and cuspidal, following the Arthur conjecture on the structure of the global Arthur packets, the automorphic representations we are considering in this paper should be the distinguished representatives of the corresponding global Arthur packets. This must also take the generalized Ramanujan conjecture into account, which is the case in Arthur's formulation of his conjecture. Then by the relation between the global Arthur packets (A-packets) and the global Langlands packets (L-packets), the endoscopy transfer should take the distinguished member of a global Arthur packet to its image of the Langlands functorial lifting from H to G. In other words, the endoscopy transfer from H to G for the distinguished members of global Arthur packets should be the same as the Langlands functorial lifting from H to G. In the following, we will take this as the *definition of the endoscopy transfer* from $H_{[n_1,n_2]}$ to SO_{2n+1}. An irreducible generic unitary cuspidal automorphic representation of $SO_{2n+1}(\mathbb{A})$ is called *stable* if it is not an image of any standard endoscopy lifting.

It is clear that the Langlands functorial transfers of automorphic representations can be composed. In our special cases of endoscopy transfers, we may compose endoscopy transfers and still call a composition of endoscopy transfers a (generalized) *endoscopy transfer*. In this generality, we may consider any nontrivial partition $n = n_1 + n_2 + \cdots + n_r$ with nonzero n_i's. Then the (generalized) standard elliptic endoscopy group of SO_{2n+1} associated to the partition $n = n_1+n_2+\cdots+n_r$ is $SO_{2n_1+1} \times \cdots \times SO_{2n_r+1}$, and all (generalized) standard elliptic endoscopy groups of SO_{2n+1} have this form. Set

$$H_{[n_1,\ldots,n_r]} := SO_{2n_1+1} \times SO_{2n_2+1} \times \cdots \times SO_{2n_r+1}, \tag{3.4}$$

where $n_j > 0$ and $n = \sum_{j=1}^{r} n_j$. It follows that the number r has a range from 1 to n. It is clear that the complex dual group of $H_{[n_1,\ldots,n_r]}$ is

$$H^{\vee}_{[n_1,\ldots,n_r]} = \mathrm{Sp}_{2n_1}(\mathbb{C}) \times \mathrm{Sp}_{2n_2}(\mathbb{C}) \times \cdots \times \mathrm{Sp}_{2n_r}(\mathbb{C}). \tag{3.5}$$

We now discuss the dimension of the subspace of $H^{\vee}_{[n_1,\ldots,n_r]}$-invariants in the complex representation $\Lambda^2 \circ \iota$ of $\mathrm{Sp}_{2n}(\mathbb{C})$. First the space $\mathbb{C}^{2n}$ can be decomposed into a direct sum

$$\mathbb{C}^{2n} = \mathbb{C}^{2n_1} \oplus \mathbb{C}^{2n_2} \oplus \cdots \oplus \mathbb{C}^{2n_r}. \tag{3.6}$$

Then the space of exterior square has the following decomposition:

$$\Lambda^2(\mathbb{C}^{2n}) = \big(\oplus_{l=1}^{r} \Lambda^2(\mathbb{C}^{2n_l})\big) \oplus \big(\oplus_{1\le i<j\le r} \mathbb{C}^{2n_i} \otimes \mathbb{C}^{2n_j}\big). \tag{3.7}$$

Consider the action of the group $H^{\vee}_{[n_1,\ldots,n_r]}$ on the space in (3.7) induced from (3.6). For each pair $i < j$, the space $\mathbb{C}^{2n_i} \otimes \mathbb{C}^{2n_j}$ is irreducible under $H^{\vee}_{[n_1,\ldots,n_r]}$, and for each $l = 1, 2, \ldots, r$, the space $\Lambda^2(\mathbb{C}^{2n_l})$ decomposes into a direct sum,

$$\Lambda^2(\mathbb{C}^{2n_l}) = \rho_{2,n_l} \oplus \mathbf{1}_{\mathrm{Sp}_{2n_l}}, \tag{3.8}$$

as representations of $H^{\vee}_{[n_1,\ldots,n_r]}$, where ρ_{2,n_l} is the second fundamental representation of $\mathrm{Sp}_{2n_l}(\mathbb{C})$ with the trivial action by the groups $\mathrm{Sp}_{2n_t}(\mathbb{C})$ for $t \neq l$, and as before, $\mathbf{1}_{\mathrm{Sp}_{2n_l}}$ denotes the trivial representation of $\mathrm{Sp}_{2n_l}(\mathbb{C})$. Since the other group $\mathrm{Sp}_{2n_t}(\mathbb{C})$ also acts on this one-dimensional space trivially, it follows that for each $l = 1, 2, \ldots, r$, the one-dimensional space $\mathbf{1}_{\mathrm{Sp}_{2n_l}}$ yields a subspace of $H^{\vee}_{[n_1,\ldots,n_r]}$-invariant vectors of dimension one. Hence in the restriction of the representation $\Lambda^2 \circ \iota$ to the subgroup $H^{\vee}_{[n_1,\ldots,n_r]}$, the total subspace of all $H^{\vee}_{[n_1,\ldots,n_r]}$-invariant vectors has dimension r. Therefore the restriction of the second fundamental representation ρ_2 of $\mathrm{Sp}_{2n}(\mathbb{C})$ to the subgroup $H^{\vee}_{[n_1,\ldots,n_r]}$ produces exactly an $(r-1)$-dimensional subspace of all $H^{\vee}_{[n_1,\ldots,n_r]}$-invariants. Hence we obtain the following proposition.

Proposition 3.1. For a partition of n, $n = \sum_{j=1}^{r} n_j$ with $n_j > 0$, let $H^{\vee}_{[n_1,\ldots,n_r]}$ be a subgroup of $\mathrm{Sp}_{2n}(\mathbb{C})$ defined as in (3.5). Then the multiplicity $m_{H^{\vee}_{[n_1,\ldots,n_r]}}(\rho_2)$ of the trivial representation of $H^{\vee}_{[n_1,\ldots,n]}$ occurring in the restriction of the second fundamental representation ρ_2 of $\mathrm{Sp}_{2n}(\mathbb{C})$ to the subgroup $H^{\vee}_{[n_1,\ldots,n]}$ is $r-1$. □

By part (4) of **Theorem 2.2**, for any irreducible unitary self-dual cuspidal automorphic representation τ of $GL_{2n}(\mathbb{A})$, the order of the pole at $s = 1$ of any partial exterior square L-function $L^S(s, \tau, \Lambda^2)$ is equal to the order of the pole at $s = 1$ of the completed exterior square L-function $L(s, \tau, \Lambda^2)$. By (2.4), (2.8), and (2.11), the order of the pole at $s = 1$ of any partial second fundamental L-function $L^S(s, \sigma, \rho_2)$ is equal to the order of the pole at $s = 1$ of the completed second fundamental L-function $L(s, \sigma, \rho_2)$ for any irreducible unitary cuspidal automorphic representation σ of $SO_{2n+1}(\mathbb{A})$ and any finite set S of local places including all Archimedean local places. Hence it is enough to prove **Theorem 1.2** for the partial second fundamental automorphic L-function $L^S(s, \sigma, \rho_2)$ where S is the finite set of local places such that at any finite local place ν outside S the local component σ_ν is unramified.

Theorem 3.2 (Theorem 1.2). Let σ be an irreducible generic unitary cuspidal automorphic representation of $SO_{2n+1}(\mathbb{A})$ and let ρ_2 be the second fundamental complex representation of $Sp_{2n}(\mathbb{C})$.

(1) The partial second fundamental L-function $L^S(s, \sigma, \rho_2)$ has a pole of order $r-1$ at $s = 1$ if and only if there exists a partition $n = \sum_{j=1}^{r} n_j$ with $n_j > 0$ such that σ is an endoscopy lifting from an irreducible, generic, unitary, stable, cuspidal automorphic representation $\sigma_1 \otimes \cdots \otimes \sigma_r$ of $H_{[n_1,\ldots,n_r]}(\mathbb{A})$. In particular, σ is stable if and only if the partial second fundamental L-function $L^S(s, \sigma, \rho_2)$ or equivalently, the full L-function $L(s, \sigma, \rho_2)$ is holomorphic at $s = 1$.

(2) The partition $[n_1, \ldots, n_r]$ is uniquely determined, up to permutation, by the irreducible generic unitary cuspidal automorphic representation σ. More precisely, the set of positive integers $\{n_1, n_2, \ldots, n_r\}$ consists of all positive integers m such that there exists an irreducible unitary cuspidal automorphic representation τ of $GL_m(\mathbb{A})$ such that the tensor product L-function $L(s, \sigma \times \tau)$ has a pole at $s = 1$.

(3) The set $\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ of irreducible generic unitary cuspidal automorphic representations of $SO_{2n_i+1}(\mathbb{A})$ is completely determined by the irreducible generic unitary cuspidal automorphic representation σ, up to equivalence. Namely, it is the set of irreducible generic unitary cuspidal automorphic representations σ' (up to equivalence) of $SO_{2l+1}(\mathbb{A})$ such that the tensor product L-function $L(s, \sigma \times \pi(\sigma'))$ has a pole at $s = 1$, where $\pi(\sigma')$ is the Langlands functorial transfer of σ' to $GL_{2l}(\mathbb{A})$ and is irreducible, unitary, and cuspidal.

(4) Let $m_\sigma(\rho_2)$ be the order of the pole at $s = 1$ of $L(s, \sigma, \rho_2)$ (or $L^S(s, \sigma, \rho_2)$ by part (4) of Theorem 2.2) and let $m_{H^\vee_{[n_1,\dots,n_r]}}(\rho_2)$ be as in Proposition 3.1. Then $m_\sigma(\rho_2) = m_{H^\vee_{[n_1,\dots,n_r]}}(\rho_2)$. □

Part (4) of Theorem 3.2 follows from part (1) of Theorem 3.2 and Proposition 3.1. The proof of the rest of Theorem 3.2 uses explicit results about the Langlands functorial transfer from irreducible generic cuspidal automorphic representations of $SO_{2n+1}(\mathbb{A})$ to $GL_{2n}(\mathbb{A})$. Basic results about this Langlands functorial transfer are given in Theorem 2.3. Furthermore, by the local converse theorem for SO_{2n+1} [24, Theorem 5.1], the Langlands functorial transfer from irreducible generic cuspidal automorphic representations of $SO_{2n+1}(\mathbb{A})$ to $GL_{2n}(\mathbb{A})$ is injective [24, Theorem 5.2]. More precise properties of this Langlands functorial transfer can be summarized as follows.

(1) If $\pi = \pi(\sigma)$ is the Langlands functorial transfer of σ, then σ is uniquely determined by π and for every irreducible cuspidal automorphic representation τ' of $GL_l(\mathbb{A})$ (any $l \geq 1$), one has

$$L(s, \sigma \times \tau') = L\big(s, \pi(\sigma) \times \tau'\big). \tag{3.9}$$

(2) The functorial transfer from σ to $\pi(\sigma)$ is compatible with the local Langlands functorial transfer at every local place.

(3) The set $\{\tau_1, \dots, \tau_r\}$ consists of all irreducible unitary cuspidal automorphic representations τ' of $GL_l(\mathbb{A})$ with $l = 1, 2, \dots$, such that the tensor product L-function $L(s, \sigma \times \tau')$ has a pole (of order one) at $s = 1$.

(4) For each τ_j of $GL_{2n_j}(\mathbb{A})$ with $j = 1, 2, \dots, r$, there exists a unique up to equivalence irreducible generic unitary cuspidal automorphic representation σ_j of $SO_{2n_j+1}(\mathbb{A})$ such that the Langlands functorial transfer of σ_j from $SO_{2n_j+1}(\mathbb{A})$ to $GL_{2n_j}(\mathbb{A})$ is τ_j.

The proof to establish these precise properties follows from the work in [9, 15, 24, 25]. For other classical groups, it is crucial to extend the results in [24, 25] to other classical groups.

Let σ be an irreducible generic cuspidal automorphic representation of $SO_{2n+1}(\mathbb{A})$, and let m be the order of the pole at $s = 1$ of the partial second fundamental automorphic L-function $L^S(s, \sigma, \rho_2)$. From (3.11) above, the Langlands functorial transfer $\pi(\sigma)$ is of type $\pi(\sigma) \cong \tau_1 \boxplus \cdots \boxplus \tau_r$ as automorphic representations of $GL_{2n}(\mathbb{A})$. As in (2.8), we have

$$L^S\big(s, \sigma, \rho_2\big) = \frac{L^S\big(s, \sigma, \Lambda^2 \circ \iota\big)}{\zeta_k^S(s)} = \frac{L^S\big(s, \pi(\sigma), \Lambda^2\big)}{\zeta_k^S(s)}. \tag{3.10}$$

As in (2.11), the exterior square L-function $L^S(s, \pi(\sigma), \Lambda^2)$ can be expressed as

$$L^S\big(s, \pi(\sigma), \Lambda^2\big) = \prod_{j=1}^{r} L^S\big(s, \tau_j, \Lambda^2_{2n_j}\big) \cdot \prod_{1 \le i < j \le r} L^S\big(s, \tau_i \times \tau_j\big), \tag{3.11}$$

where $\Lambda^2_{2n_j} = \Lambda^2(\mathbb{C}^{2n_j})$. For each $j = 1, 2, \dots, r$, the partial exterior square L-function $L^S(s, \tau_j, \Lambda^2_{2n_j})$ has a simple pole at $s = 1$ by property (1) above. For each pair $1 \le i < j \le r$, the partial Rankin-Selberg product L-function $L(s, \pi_i \times \pi_j)$ is holomorphic and nonzero at $s = 1$ since $\tau_i \not\cong \tau_j$ by Theorem 2.3. Hence the order of the pole at $s = 1$ of $L^S(s, \pi(\sigma), \Lambda^2)$ is r. It follows that the order of the pole at $s = 1$ of the partial second fundamental L-function $L^S(s, \sigma, \rho_2)$ is $m = r - 1$ if and only if the Langlands functorial transfer $\pi(\sigma)$ is of type (2.12). Now by property (4), for each $i = 1, 2, \dots, r$, there exists a unique up to equivalent irreducible, generic, cuspidal automorphic representation σ_i of $SO_{2n_i+1}(\mathbb{A})$ which lifts to τ_i. It is clear that σ_i's are stable in the sense of our definition in the beginning of this section. This proves part (1) of Theorem 3.2.

Parts (2) and (3) follows essentially from properties (1), (3), and (4). This completes the proof of Theorem 3.2, and hence that of Theorem 1.2.

Remark 3.3. In a special case when $r = n$, Theorem 3.2 was part of [13, Conjecture 1]. It is worthwhile to mention that when $r = n$, the subgroup $H^{\vee}_{[1^n]} = Sp_2(\mathbb{C}) \times \cdots \times Sp_2(\mathbb{C})$ is *visible* in the sense of Kac [27]. One notices that [13, Conjecture 1] asserts a period condition characterizing the endoscopy structure of the irreducible generic cuspidal automorphic representation σ of $SO_{2n+1}(\mathbb{A})$, just as part (5) of Theorem 2.2. It will be a very interesting problem to find a period condition in general.

4 On the Langlands problem

In [33], Problem (II) addresses the possible relation between poles of automorphic L-functions and the arithmetic structure of automorphic representations. The arithmetic structure is carried by a mysterious algebraic subgroup of the Langlands dual group ${}^LG$ of G, which is denoted by $\mathcal{H}_\pi$. It is not known how to define $\mathcal{H}_\pi$ precisely for a given π. In general, the group $\mathcal{H}_\pi$ may not be the Langlands dual group of a reductive algebraic group over k.

Problem 4.1 (Langlands [33]). Let G be a reductive algebraic group defined over a number field k and let π be an irreducible (unitary) automorphic representation of $G(\mathbb{A})$. There exists an algebraic subgroup $\mathcal{H}_\pi$ of the Langlands dual group ${}^LG$ of G such that for all finite-dimensional complex representation ρ of ${}^LG$, the multiplicity $m_{\mathcal{H}_\pi}(\rho)$ of the trivial representations of $\mathcal{H}_\pi$ occurring in the restriction of ρ to $\mathcal{H}_\pi$ is equal to the order of $m_\pi(\rho)$ of the pole at $s = 1$ of the Langlands automorphic L-function $L(s, \pi, \rho)$ associated to the pair (π, ρ).

If one assumes the existence of the hypothetical Langlands group $\mathcal{L}_k$ such that the global Langlands parameters from $\mathcal{L}_k \times SL_2(\mathbb{C})$ to ${}^LG$ classify the automorphic representations of $G(\mathbb{A})$ up to L-packets, then it is natural to expect that the mysterious algebraic subgroup $\mathcal{H}_\pi$ is the image of $\phi(\mathcal{L}_k \times SL_2(\mathbb{C}))$ in ${}^LG$ if $\pi \in \Pi(\phi)$, the L-packet attached to the global Langlands parameter ϕ. In this case, if $\mathcal{H}_\pi$ is a connected reductive algebraic subgroup of $G^\vee$, then by [34, Theorem 1], $\mathcal{H}_\pi$ is completely determined by the numbers $m_{\mathcal{H}_\pi}(\rho)$ for all finite dimensional complex representations ρ of $G^\vee$, which is called the *dimension data* of $\mathcal{H}_\pi$ in [34].

We study the relation between the Langlands problem and the Langlands functoriality. To this end, we make the following assumption, which has been verified for many cases, but is still not known in general [7, 12].

Assumption 4.2 (Langlands conjecture). All Langlands automorphic L-functions $L(s, \pi, \rho)$ have meromorphic continuation to $\mathbb{C}$ and satisfy the standard functional equation.

As explained in [33], Problem 4.1 seems to make sense if π is of Ramanujan-type, that is, it is locally tempered at all local places. In this case, it is expected that $L(s, \pi, \rho)$ should be holomorphic for the real part of s greater than one. We assume that π is of Ramanujan-type, or simply take π to be generic, and assume that the reductive group G is k-split, so that we may replace the Langlands dual group ${}^LG$ by the complex dual group $G^\vee$ in our discussion below.

Let $m_\pi(\rho)$ be the order of the pole at $s = 1$ of the automorphic L-function $L(s, \pi, \rho)$. Define

$$\mathcal{R}_\pi =: \left\{\rho \mid m_\pi(\rho) \geq 1\right\}, \tag{4.1}$$

where ρ are finite-dimensional irreducible complex representations of $G^\vee$ and

$$\mathcal{N}_\pi =: \left\{m_\pi(\rho) \mid \rho \in \mathcal{R}_\pi\right\}, \tag{4.2}$$

which is closely related to the dimension data in the sense of [34].

4.1 Observable subgroups

Assume that for a given π as above, such an algebraic subgroup $\mathcal{H}_\pi$ of $G^\vee$ exists, so that Problem 4.1 has an affirmative solution, that is, one has

$$m_{\mathcal{H}_\pi}(\rho) = m_\pi(\rho) \tag{4.3}$$

for all finite-dimensional irreducible representations ρ of $G^\vee$. For any $\rho \in \mathcal{R}_\pi$, denote by V_ρ the space of the representation ρ of $G^\vee$. By (4.3), we have

$$m_\pi(\rho) = m_{\mathcal{H}_\pi}(\rho) = \dim_{\mathbb{C}} V_\rho^{\mathcal{H}_\pi} \in \mathcal{N}_\pi. \tag{4.4}$$

Then there exists linearly independent vectors $v_1, \ldots, v_{m_\pi(\rho)} \in V_\rho^{\mathcal{H}_\pi}$. We denote by $G^\vee_{v_j}$ the stabilizer of v_j, that is,

$$G^\vee_{v_j} = \{g \in G^\vee \mid \rho(g)(v_j) = v_j\}. \tag{4.5}$$

By [17, Theorem 1.2], these groups are observable subgroups of $G^\vee$. In general, one may call an algebraic subgroup $\mathcal{A}$ of $G^\vee$ *observable* if $G^\vee/\mathcal{A}$ is quasiaffine following [5] (or [36, page 172]). Following [17, Chapter One], one has the following definition.

Definition 4.3. Let $\mathcal{A}$ be an algebraic subgroup of $G^\vee$. Define

$$\mathcal{A}'' = \left\{g \in G^\vee \mid f(xg) = f(x)\ \forall f \in \mathbb{C}\left[G^\vee\right]^{\mathcal{A}}\right\}, \tag{4.6}$$

where $\mathbb{C}[G^\vee]^{\mathcal{A}}$ denotes the $\mathcal{A}$-invariants in $\mathbb{C}[G^\vee]$. Then the algebraic subgroup $\mathcal{A}''$ containing $\mathcal{A}$ is called the *observable hull* of $\mathcal{A}$. If $\mathcal{A} = \mathcal{A}''$, then $\mathcal{A}$ is called *observable*.

By [17, Theorem 2.1], an algebraic subgroup $\mathcal{A}$ of $G^\vee$ is observable if and only if there is a finite-dimensional complex representation ρ of $G^\vee$ and a vector $v \in V_\rho$ such that $\mathcal{A}$ is the stabilizer of v in $G^\vee$, and the above two definitions are equivalent.

Let $\mathcal{A}_\rho = \cap_{j=1}^{m_\pi(\rho)} G^\vee_{v_j}$. Since $G^\vee_{v_j}$ are observable, it follows that $\mathcal{A}_\rho$ is observable. It is clear that $\mathcal{H}_\pi \subset \mathcal{A}_\rho$ for all $\rho \in \mathcal{R}_\pi$, and hence we have $\mathcal{H}_\pi \subset \cap_{\rho\in\mathcal{R}_\pi}\mathcal{A}_\rho$. Moreover, we prove the following proposition.

Proposition 4.4. With notations as above, $\cap_{\rho\in\mathcal{R}_\pi}\mathcal{A}_\rho$ is the observable hull of $\mathcal{H}_\pi$. □

Proof. We first show that any observable subgroup $\mathcal{A}$ of $G^\vee$ containing $\mathcal{H}_\pi$ also contains $\cap_{\rho\in\mathcal{R}_\pi}\mathcal{A}_\rho$. Let $\mathcal{A}$ be an observable subgroup containing $\mathcal{H}_\pi$. By [17, Theorem 2.1], there exists an irreducible finite-dimensional complex representation ρ of $G^\vee$ and a vector $v \in V_\rho$ such that $\mathcal{A} = \mathrm{Stab}_{G^\vee}(v)$. Since $\mathcal{H}_\pi$ is a subgroup of $\mathcal{A}$, we know that

$$m_\pi(\rho) = m_{\mathcal{H}_\pi}(\rho) \geq 1. \tag{4.7}$$

Hence $\rho \in \mathcal{R}_\pi$ and $\mathcal{A}_\rho$ is a subgroup of $\mathcal{A}$. Therefore $\mathcal{A}$ contains $\cap_{\rho\in\mathcal{R}_\pi}\mathcal{A}_\rho$.

On the other hand, by [17, Lemma 1.1], $(\mathcal{H}_\pi)''$ is the smallest observable subgroup of $G^\vee$ containing $\mathcal{H}_\pi$. If we take $\mathcal{A} = (\mathcal{H}_\pi)''$ in the above argument, we have

$$\cap_{\rho\in\mathcal{R}_\pi}\mathcal{A}_\rho \subset \left(\mathcal{H}_\pi\right)''. \tag{4.8}$$

By the definition of $(\mathcal{H}_\pi)''$, we have for any $\rho \in \mathcal{R}_\pi$, $(\mathcal{H}_\pi)'' \subset \mathcal{A}_\rho$, and hence

$$\cap_{\rho\in\mathcal{R}_\pi}\mathcal{A}_\rho \supset \left(\mathcal{H}_\pi\right)''. \tag{4.9}$$

Therefore, $\cap_{\rho\in\mathcal{R}_\pi}\mathcal{A}_\rho = (\mathcal{H}_\pi)''$ is the observable hull of $\mathcal{H}_\pi$. ■

In general, the algebraic subgroup $\mathcal{H}_\pi$ may not be observable, that is, $\mathcal{H}_\pi$ is a proper subgroup of $\cap_{\rho\in\mathcal{R}_\pi}\mathcal{A}_\rho$. Hence, for any irreducible finite-dimensional complex representation ρ of $G^\vee$, we have

$$V_\rho^{\mathcal{A}_\pi} \subset V_\rho^{\mathcal{H}_\pi}, \tag{4.10}$$

where $\mathcal{A}_\pi = \cap_{\rho\in\mathcal{R}_\pi}\mathcal{A}_\rho$. In fact, we prove the following proposition.

Proposition 4.5. For any irreducible finite-dimensional complex representation ρ of $G^\vee$,

$$V_\rho^{\mathcal{A}_\pi} = V_\rho^{\mathcal{H}_\pi}, \tag{4.11}$$

in particular, $m_{\mathcal{A}_\pi}(\rho) = m_{\mathcal{H}_\pi}(\rho)$. □

Proof. Let ρ be an irreducible finite-dimensional complex representation of $G^\vee$. If $\rho \notin \mathcal{R}_\pi$, then $V_\rho^{\mathcal{H}_\pi} = \{0\}$, and hence we must have

$$V_\rho^{\mathcal{A}_\pi} = V_\rho^{\mathcal{H}_\pi}. \tag{4.12}$$

Now we assume that $\rho \in \mathcal{R}_\pi$. Then we have $m_\pi(\rho) = m_{\mathcal{H}_\pi}(\rho) \geq 1$ and hence $V_\rho^{\mathcal{H}_\pi} \neq \{0\}$. For any $0 \neq v \in V_\rho^{\mathcal{H}_\pi}$, we have $\mathcal{A}_\pi \subset \mathcal{A}_\rho \subset \mathrm{Stab}_{G^\vee}(v)$, since $\rho \in \mathcal{R}_\pi$. Therefore, we have $v \in V_\rho^{\mathcal{A}_\pi}$. We are done. ■

4.2 Relation with functorial transfers

We discuss the relation between the subgroup $\mathcal{A}_\pi$ introduced in the previous section and the functoriality structure of π.

For a given $\rho \in \mathcal{R}_\pi$, if there is a nonzero vector $v \in V_\rho^{\mathcal{H}_\pi}$ such that $\mathrm{Stab}_{G^\vee}(v)$ is a reductive subgroup, then the Langlands functoriality conjecture predicts that π should be a Langlands functorial lifting from H, whose complex dual group is $\mathrm{Stab}_{G^\vee}(v)$. Since the irreducible unitary cuspidal automorphic representations are assumed to be of Ramanujan type, it is natural to assume that the observable hull $\mathcal{A}_\pi = (\mathcal{H}_\pi)''$ is an elliptic algebraic subgroup of $G^\vee$ ([1, 2], or [26]). In the following we give more detailed discussion for the case when $G = SO_{2n+1}$.

For any irreducible generic unitary cuspidal automorphic representation σ of $SO_{2n+1}(\mathbb{A})$, the integer $m_\sigma(\rho_1)$ is zero since the standard L-function of σ is holomorphic at $s = 1$. Hence $\rho_1 \notin \mathcal{R}_\sigma$. By Theorem 1.2 (Theorem 3.2), the second fundamental representation ρ_2 may belong to $\mathcal{R}_\sigma$. Then by part (1) of Theorem 3.2, the integer $m_\sigma(\rho_2)$ gives a partition of $n = \sum_{i=1}^r n_i$ with $r = m_\sigma(\rho_2) + 1$, but without knowing what n_i's should be. This partition yields a standard elliptic endoscopy group $H_{[n_1,\dots,n_r]} = SO_{2n_1+1} \times \cdots \times SO_{2n_r+1}$ of SO_{2n+1}, similarly without knowing the size of the special orthogonal groups. If we assume that the observable subgroup $\mathcal{A}_{\rho_2}$ containing $\mathcal{H}_\pi$ (as defined in Section 4.1) is elliptic in $G^\vee$, then from the calculation of the standard elliptic endoscopy in Section 3, we must have

$$\mathcal{A}_{\rho_2} = H^\vee_{[n_1,\dots,n_r]} = Sp_{2n_1}(\mathbb{C}) \times \cdots \times Sp_{2n_r}(\mathbb{C}) \tag{4.13}$$

corresponding to the above-given partition $n = \sum_{i=1}^r n_i$ with $r = m_\sigma(\rho_2) + 1$. In order to determine the structure of $H_{[n_1,\dots,n_r]}$, that is, the partition $n = \sum_{i=1}^r n_i$, completely in terms of σ, we have to consider more irreducible representations $\rho \in \mathcal{R}_\sigma$. Note that the partition $n = \sum_{i=1}^r n_i$ is completely determined by σ in Theorem 3.2 by means of the order of poles of the tensor product L-functions $L(s, \sigma \times \tau)$. However, this seems not compatible with the Langlands problem (**Problem 4.1**).

Let $\rho_1, \rho_2, \ldots, \rho_n$ be the fundamental representations of $\mathrm{Sp}_{2n}(\mathbb{C})$. By (**2.1**), we have the following split exact sequence:

$$0 \longrightarrow V_{\rho_a}^{(2n)} \longrightarrow \Lambda^a\left(\mathbb{C}^{2n}\right) \longrightarrow \Lambda^{a-2}\left(\mathbb{C}^{2n}\right) \longrightarrow 0, \tag{4.14}$$

where $V_{\rho_a}^{(2n)}$ is the space of the representation ρ_a of $\mathrm{Sp}_{2n}(\mathbb{C})$. We define here that $\Lambda^{-1}(\mathbb{C}^{2n}) = 0$.

We want to first investigate the pole at $s = 1$ of the ath fundamental L-function $L(s, \sigma, \rho_a)$ for $a = 3, 4, \ldots, n$. For the real part of s large, we have

$$L\left(s, \sigma, \rho_a\right) = \frac{L\left(s, \pi(\sigma), \Lambda^a\right)}{L\left(s, \pi(\sigma), \Lambda^{a-2}\right)}, \tag{4.15}$$

where $\pi(\sigma)$ is the image of the Langlands functorial transfer of σ to $\mathrm{GL}_{2n}(\mathbb{A})$. By **Assumption 4.2**, the above identity is valid for all complex value s. Hence we have the following formula for the order of the poles at $s = 1$:

$$m_\sigma\left(\rho_a\right) = m_{\pi(\sigma)}\left(\Lambda^a\right) - m_{\pi(\sigma)}\left(\Lambda^{a-2}\right). \tag{4.16}$$

Assume that the image $\pi(\sigma)$ is given by the partition $n = \sum_{i=1}^r n_i$. Then the order of the pole at $s = 1$ of $L(s, \pi(\sigma), \Lambda^a)$ is given by

$$m_{\pi(\sigma)}\left(\Lambda^a\right) \geq \dim \mathrm{Hom}_{H^\vee_{[n_1,\ldots,n_r]}}\left(\Lambda^a, 1\right). \tag{4.17}$$

When $a = 1$, by the theory of principal L-functions in [**21**], we know that $m_{\pi(\sigma)}(\Lambda^1) = 0$. One checks easily that $\mathrm{Hom}_{H^\vee_{[n_1,\cdots,n_r]}}(\Lambda^1, 1) = 0$. Hence, in (**4.17**) equality holds for $a = 1$. When $a = 2$, **Proposition 3.1** and **Theorem 2.3** show that in (**4.17**) equality holds again. For $a = 3$ and $n = 3$, the exterior cube L-function $L(s, \pi, \Lambda^3)$ is holomorphic at $s = 1$ when π is self-dual following [**14**, **30**]. We make the following assumption which might not hold and depends on the nature of σ in general. Since it is deeply involved, we will not offer any further comments on this assumption here.

Assumption 4.6. For all a, $m_{\pi(\sigma)}(\Lambda^a) = \dim \mathrm{Hom}_{H^\vee_{[n_1,\ldots,n_r]}}(\Lambda^a, 1)$, that is, the equality in (**4.17**) holds for all a.

In the following we use Assumption 4.6 in our argument.

For the given partition $n = \sum_{j=1}^{r} n_j$ with all $n_j > 0$ determined by $\pi(\sigma)$. we write $\mathbb{C}^{2n} = \mathbb{C}^{2n_1} \oplus \cdots \oplus \mathbb{C}^{2n_r}$. It follows that

$$\Lambda^a\left(\mathbb{C}^{2n}\right) = \oplus_{a_1+\cdots+a_r=a}\Lambda^{a_1}\left(\mathbb{C}^{2n_1}\right) \otimes \cdots \otimes \Lambda^{a_r}\left(\mathbb{C}^{2n_r}\right). \tag{4.18}$$

This is a decomposition as $H^{\vee}_{[n_1,\ldots,n_r]}$-modules.

For each $j \in \{1, 2, \ldots, r\}$, as a representation of $\mathrm{Sp}_{2n_j}(\mathbb{C})$, one has

$$0 \longrightarrow V^{(2n_j)}_{\rho_{a_j}} \longrightarrow \Lambda^{a_j}\left(\mathbb{C}^{2n_j}\right) \longrightarrow \Lambda^{a_j-2}\left(\mathbb{C}^{2n_j}\right) \longrightarrow 0 \tag{4.19}$$

if $a_j \le 2n_j$, where $V^{(2n_j)}_{\rho_{a_j}}$ denotes the space of the representation ρ_{a_j} of $\mathrm{Sp}_{2n_j}(\mathbb{C})$. Otherwise, we know that $\Lambda^{a_j}(\mathbb{C}^{2n_j}) = 0$. When $0 < a_j \le 2n_j$, since $V^{(2n_j)}_{\rho_{a_j}}$ is irreducible as a representation of $\mathrm{Sp}_{2n_j}(\mathbb{C})$, one obtains that

$$\dim_{\mathbb{C}} \Lambda^{a_j}\left(\mathbb{C}^{2n_j}\right)^{\mathrm{Sp}_{2n_j}(\mathbb{C})} = \dim_{\mathbb{C}} \Lambda^{a_j-2}\left(\mathbb{C}^{2n_j}\right)^{\mathrm{Sp}_{2n_j}(\mathbb{C})}. \tag{4.20}$$

If $a_j = 0$, then $\dim_{\mathbb{C}} \Lambda^{a_j}(\mathbb{C}^{2n_j})^{\mathrm{Sp}_{2n_j}(\mathbb{C})} = 1$. Hence if a_j is odd, then we have $\dim_{\mathbb{C}} \Lambda^{a_j}(\mathbb{C}^{2n_j})^{\mathrm{Sp}_{2n_j}(\mathbb{C})} = 0$; and if a_j is even or zero, then we have $\dim_{\mathbb{C}} \Lambda^{a_j}(\mathbb{C}^{2n_j})^{\mathrm{Sp}_{2n_j}(\mathbb{C})} = 1$. Therefore we obtain the following formula for the dimension of $H^{\vee}_{[n_1,\ldots,n_r]}$-invariants in the space $\Lambda^a(\mathbb{C}^{2n})$.

Proposition 4.7. The dimension of the subspace of the $H^{\vee}_{[n_1,\ldots,n_r]}$-invariants in the space $\Lambda^{2l+1}(\mathbb{C}^{2n})$ is zero for $l = 0, 1, \ldots, [(n-1)/2]$; and the dimension

$$\dim_{\mathbb{C}} \left(\Lambda^{2l}\left(\mathbb{C}^{2n}\right)\right)^{H^{\vee}_{[n_1,\ldots,n_r]}} := m_{H^{\vee}_{[n_1,\ldots,n_r]}}\left(\Lambda^{2l}\left(\mathbb{C}^{2n}\right)\right) \tag{4.21}$$

of the subspace of the $H^{\vee}_{[n_1,\ldots,n_r]}$-invariants in the space $\Lambda^{2l}(\mathbb{C}^{2n})$, with $l = 1, 2, \ldots, [n/2]$, is equal to the number of all ordered r-partitions $(l_1, \ldots, l_r)$ of l satisfying the following conditions:

(1) $l = l_1 + l_2 + \cdots + l_r$ with $0 \le l_j \le n_j$ for $j = 1, 2, \ldots, r$,

(2) $(l_1, l_2, \ldots, l_r)$ is an ordered r-tuple. □

It is now easy to deduce the following formula.

Corollary 4.8. The dimension of the $H^{\vee}_{[n_1,\dots,n_r]} = Sp_{2n_1}(\mathbb{C}) \times \cdots \times Sp_{2n_r}(\mathbb{C})$-invariants in $V^{(2n)}_{\rho_{2l}}$ is given by the following formula:

$$m_{H^{\vee}_{[n_1,\dots,n_r]}}(V^{(2n)}_{\rho_{2l}}) = m_{H^{\vee}_{[n_1,\dots,n_r]}}(\Lambda^{2l}(\mathbb{C}^{2n})) - m_{H^{\vee}_{[n_1,\dots,n_r]}}(\Lambda^{2l-2}(\mathbb{C}^{2n})), \tag{4.22}$$

for $l = 1, 2, \dots, [n/2]$, and $m_{H^{\vee}_{[n_1,\dots,n_r]}}(V^{(2n)}_{\rho_{2l+1}})$ is zero for $l = 0, 1, \dots, [(n-1)/2]$. □

In the following we give a formula for $m_{H^{\vee}_{[n_1,\dots,n_r]}}(\Lambda^{2l}(\mathbb{C}^{2n}))$, which is the dimension of the subspace of the $H^{\vee}_{[n_1,\dots,n_r]}$-invariants in the space $\Lambda^{2l}(\mathbb{C}^{2n})$, with $l = 1, 2, \dots, n$. Basically we want to give a more explicit formula for the result stated in Proposition 4.7.

For a given partition $n = n_1 + n_2 + \cdots + n_r$, we may assume that

$$n_1 \geq n_2 \geq \cdots \geq n_r \geq 1. \tag{4.23}$$

For $l = 1, 2, \dots, [n/2]$, we define $\underline{n}^t(l) := [n^t_1, \dots, n^t_l]$, called the *truncated transpose partition* of $\underline{n} = [n_1, \dots, n_r]$, where $n^t_i = |\{j \mid n_j \geq i\}|$. For the partition $l = l_1 + \cdots + l_r$ with $0 \leq l_i \leq n_i$, we write it as a *normalized partition* of l:

$$\underline{l} = [f_1^{s_1}, f_2^{s_2}, \dots, f_\alpha^{s_\alpha}], \quad f_1 > f_2 > \cdots > f_\alpha > 0,\ r = s_1 + \cdots + s_\alpha. \tag{4.24}$$

Then we have $l = s_1 f_1 + \cdots + s_\alpha f_\alpha$. Now the following formula for the dimension $m_{H^{\vee}_{[n_1,\dots,n_r]}}(\Lambda^{2l}(\mathbb{C}^{2n}))$ follows easily from a direct counting of partitions of $\underline{l} = [l_1, l_2, \dots, l_r]$ subject to the condition $0 \leq l_i \leq n_i$.

Corollary 4.9. For a given partition $n = n_1 + n_2 + \cdots + n_r$, and for $l = 1, 2, \dots, [n/2]$,

$$m_{H^{\vee}_{[n_1,\dots,n_r]}}(\Lambda^{2l}(\mathbb{C}^{2n})) = \sum_{[f_1^{s_1}, f_2^{s_2}, \dots, f_\alpha^{s_\alpha}]} \prod_{i=1}^{\alpha} \binom{n^t_{f_i} - \sum_{j=1}^{i-1} s_j}{s_i}, \tag{4.25}$$

where $\underline{l} = [f_1^{s_1}, f_2^{s_2}, \dots, f_\alpha^{s_\alpha}]$ goes over all partitions of l as above subject to the condition $0 \leq l_i \leq n_i$, and for $a \geq b$,

$$\binom{a}{b} = \frac{a!}{b!(a-b)!}. \tag{4.26}$$

□

Proof. First, any partition $l = l_1 + \cdots + l_r$ with $0 \le l_i \le n_i$ can be normalized to a unique partition as in (4.24):

$$\underline{l} = \left[f_1^{s_1}, f_2^{s_2}, \ldots, f_\alpha^{s_\alpha}\right]. \tag{4.27}$$

Then we count how many partitions of l can be normalized to a given normalized partition as (4.24). It is clear that the formula follows from direct counting. ■

Under Assumption 4.2 for the L-functions attached to the fundamental representations $\rho_3, \ldots, \rho_n$, and under Assumption 4.6, it is clear that if an irreducible generic unitary cuspidal automorphic representation σ of SO_{2n+1} is an endoscopy transfer from an irreducible generic unitary cuspidal automorphic representation $\sigma_1 \otimes \cdots \otimes \sigma_r$ of $H_{[n_1,\ldots,n_r]}(\mathbb{A}) = SO_{2n_1+1}(\mathbb{A}) \times \cdots \times SO_{2n_r+1}(\mathbb{A})$, then the order of the pole at $s = 1$ of the $2l$th fundamental L-function $L(s, \sigma, \rho_{2l})$ is completely determined by the partition $n = \sum_{i=1}^r n_i$ by Corollaries 4.8 and 4.9.

From the proof of Theorem 1.2, it is easy to see that the number r of the factors $Sp_{2n_i}(\mathbb{C})$ is completely determined by the second fundamental representation ρ_2. However, the sizes of the factors, namely, $2n_i$'s are determined by the existence of the pole at $s = 1$ of tensor product L-functions for $SO_{2n+1} \times GL_{2m}$ for all possible positive integers m. This seems not to quite fit into the framework of the Langlands problem. We want to show that in terms of the order the pole at $s = 1$ of the $2l$th fundamental L-function $L(s, \sigma, \rho_{2l})$ for $l = 1, 2, \ldots, [n/2]$, the partition $n = \sum_{i=1}^r n_i$ can be uniquely determined.

Assume that the pole at $s = 1$ of the $2l$th fundamental L-function $L(s, \sigma, \rho_{2l})$ has order m_l. Then for $l = 1$, we know that $m_1 = r - 1 = n_1^t - 1$, or equivalently,

$$n_1^t = m_1 + 1. \tag{4.28}$$

For $l = 2$, l has the normalized partitions $[2]$ and $[1^2]$ (as (4.24)). Then we have

$$\begin{aligned} m_2 &= m_{H^\vee_{[n_1,\ldots,n_r]}}\left(\Lambda^4\left(\mathbb{C}^{2n}\right)\right) - m_{H^\vee_{[n_1,\ldots,n_r]}}\left(\Lambda^2\left(\mathbb{C}^{2n}\right)\right) \\ &= \binom{n_2^t}{1} + \binom{n_1^t}{2} - \binom{n_1^t}{1}. \end{aligned} \tag{4.29}$$

It simplifies to

$$\begin{aligned} m_2 &= n_2^t + \frac{\left(n_1^t - 1\right)\left(n_1^t - 2\right)}{2} - n_1^t \\ &= n_2^t + \frac{m_1\left(m_1 - 1\right)}{2} - \left(m_1 + 1\right) \\ &= n_2^t + \frac{m_1^2 - 3m_1 - 2}{2}. \end{aligned} \tag{4.30}$$

Hence we have

$$n_2^t = m_2 - \frac{m_1^2 - 3m_1 - 2}{2}. \tag{4.31}$$

This means that in the partition $n = \sum_{i=1}^r n_i$ with $n_1 \geq \cdots \geq n_r \geq 1$, the 1-part is given by $1^{n_1^t - n_2^t}$ with

$$n_1^t - n_2^t = -m_2 + \frac{m_1(m_1 - 1)}{2}. \tag{4.32}$$

For a general l, there is a unique normalized partition $[l]$ (as (4.24)), and all other normalized partitions have parts all less than l. By an induction argument, we know that the number n_l^t is uniquely expressed in terms of $m_l, m_{l-1}, \ldots, m_1$. Hence in the partition $n = \sum_{i=1}^r n_i$ with $n_1 \geq \cdots \geq n_r \geq 1$, the $(l-1)$-part is given by $(l-1)^{n_{l-1}^t - n_l^t}$, which is now completely determined by the orders $m_1, m_2, \ldots, m_l$ of the poles at $s = 1$ of the fundamental L-functions $L(s, \sigma, \rho_{2t})$ for $t = 1, 2, \ldots, l$, respectively. This proves the following theorem.

Theorem 4.10 (Theorem 1.3)**.** Assume the validity of Assumption 4.2 for the L-functions attached to the fundamental representations $\rho_3, \rho_4, \ldots, \rho_n$ and Assumption 4.6. For a given irreducible generic unitary cuspidal automorphic representation σ of $SO_{2n+1}(\mathbb{A})$, the structure of the algebraic subgroup $H^\vee_{[n_1,\ldots,n_r]}$, that is, the partition $n = \sum_{j=1}^r n_j$, is completely determined by the order of the pole at $s = 1$ of the L-function $L(s, \sigma, \rho)$ for

$$\rho \in \left\{\rho_2, \rho_4, \ldots, \rho_{2[n/2]}\right\} \tag{4.33}$$

of the complex dual group $Sp_{2n}(\mathbb{C})$. Moreover, the L-function $L(s, \sigma, \rho)$ is holomorphic at $s = 1$ for $\rho \in \{\rho_1, \rho_3, \ldots, \rho_{2[n/2]+1}\}$. □

Finally, we remark that for a given irreducible generic cuspidal automorphic representation σ of $SO_{2n+1}(\mathbb{A})$, there should be a smallest connected algebraic subgroup of $Sp_{2n}(\mathbb{C})$ of type $H^\vee_{[n_1,\ldots,n_r]}$ containing the mysterious algebraic subgroup $\mathcal{H}_\sigma$. It is an interesting problem to investigate the poles of automorphic L-functions attached to σ which essentially catch the subgroup $\mathcal{H}_\sigma$. A local version of global theory in this paper is considered in the author's work in progress.

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