

Residues of Eisenstein series and generalized Shalika models for SO_{4n}

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Abstract. We introduce the generalized Shalika model for $SO(4n)$, the split even orthogonal group of rank $2n$, and develop the local and global compatibilities with the Shalika model for $GL(2n)$, the general linear group of rank $2n$. As result, we determine the existence of poles of certain Eisenstein series on $SO(4n)$ in terms of the Shalika model on the cuspidal datum $(GL(2n), \pi)$, and give a different proof for the determination of the pole at $s = 1$ of the exterior square L-function $L(s, \pi, \Lambda^2)$ in terms of the Shalika model on π .

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1. Introduction

Let k be a number field and \mathbb{A} be the ring of adèles of k . Let π be an irreducible unitary cuspidal automorphic representation of $GL_{2n}(\mathbb{A})$. The exterior square L-function $L(s, \pi, \Lambda^2)$ has been studied extensively by the Langlands-Shahidi method ([30]) and by the Rankin-Selberg method (the work of Jacquet-Shalika in [16]). The known properties involving π and $L(s, \pi, \Lambda^2)$ are summarized in Theorem 2.2 in [18]. The main result of [16] (see also Part (5) of Theorem 2.2 in [18]) says that the existence of the pole at $s = 1$ of the (partial) exterior square L-function $L(s, \pi, \Lambda^2)$ is characterized by the nonvanishing of the Shalika period attached to π .

In this paper, we show by a different method that the nonvanishing of the Shalika period attached to π implies the existence of the pole at $s = 1$ of the exterior square L-function $L(s, \pi, \Lambda^2)$. This method was initiated by Jacquet and Rallis in [13] and developed in [17]. A more detailed account of this

method as a useful approach to determine the existence of poles of Eisenstein series or related automorphic L-functions can be found in [2, 19], and [20]. It is an interesting and important problem to find suitable period conditions which yield the existence of poles of Eisenstein series or related automorphic L-functions in general. The relations between periods of automorphic forms and the Langlands functoriality conjectures, and applications to number theory can be found in [11, 7–9, 5, 6, 22, 12, 1, 3, 4, 26, 21], for instance.

Following the idea of this approach, we find a generalized Shalika period for automorphic forms on $\mathrm{SO}_{4n}(\mathbb{A})$, where SO_{4n} is the k -split even special orthogonal group. The definition is given in §4 for the global generalized Shalika periods for automorphic forms on $\mathrm{SO}_{4n}(\mathbb{A})$, and in §2 for the local generalized Shalika functionals for irreducible admissible representations of p -adic group $\mathrm{SO}_{4n}(k_v)$, where k_v is the localization of the number field k at a finite local place v of k .

Some basic results have been established in §2 and §3 for the local generalized Shalika functionals, which classify the unitarily induced representation $I(s, \tau)$ of $\mathrm{SO}_{4n}(k_v)$ from a supercuspidal datum (M, τ) attached to the standard Siegel parabolic subgroup $P = MN = \mathrm{GL}_{2n}N$ of SO_{4n} . More precisely, it is proved in Theorem 3.1, §3, that $I(s, \tau)$ admits a non-zero local generalized Shalika functional if and only if the irreducible supercuspidal representation τ of $\mathrm{GL}_{2n}(k_v)$ (GL_{2n} is the Levi part of P) admits a non-zero Shalika functional and s must be 1. Moreover, in this case, the non-zero generalized Shalika functional on $I(1, \tau)$ must factor through the unique Langlands quotient of $I(1, \tau)$. In a forthcoming work of the both authors joint with C.-F. Nien, we will consider a general result when τ is an irreducible admissible representation of $\mathrm{GL}_{2n}(k_v)$ and the relation to the local Langlands functoriality.

The global version of Theorem 3.1 has been established in §4 for the generalized Shalika periods, which is Theorem 4.4. It shows that the cuspidal datum (M, π) has a non-zero Shalika period if and only if the residue at $s = 1$ of the Eisenstein series on $\mathrm{SO}_{4n}(\mathbb{A})$ associated to (M, π) has a nonzero generalized Shalika period. As consequence, we obtain two applications. The first is to determine the existence of the pole at $s = 1$ of the Eisenstein series in terms of the nonvanishing of the Shalika period for the cuspidal datum, from which the Eisenstein series is built. By Theorem 4.11 of [23], the relevant local intertwining operator can be normalized in terms of the Shahidi local factor. Hence we obtain as the second application that the nonvanishing of the Shalika period for π implies the existence of the pole at $s = 1$ of the exterior square L-function $L(s, \pi, \Lambda^2)$.

It is worthwhile to mention that the Fourier coefficients attached to the generalized Shalika model considered in this paper for SO_{4n} are analogues of the Fourier coefficients considered for symplectic groups Sp_{2n} by

Piatetski-Shapiro and Rallis in [27] in order to obtain a new way to study automorphic L-functions and by J.-S. Li [24] to study the distinguished cuspidal automorphic representations of $Sp_{2n}(\mathbb{A})$, and that for the quasi-split unitary groups in the more recent work ([28] and [29]). We consider the generalized Bessel models for a certain family of residual representations of $SO_{4n}(\mathbb{A})$. It is interesting to consider the generalized Shalika models for cuspidal automorphic representations of $SO_{4n}(\mathbb{A})$ as well.

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2. Generalized Shalika models

Let F be a nonarchimedean local field of characteristic zero, and ψ be a non-trivial character of F . Denote by $G = SO_{4n}$ the F -split even special orthogonal group of rank $2n$, with respect to the non-degenerate symmetric bilinear form given by $\begin{pmatrix} 1_{2n} & \\ & 1_{2n} \end{pmatrix}$. In this paper, we often identify an algebraic F -group G with its F -rational points $G(F)$ if it does not cost any confusion.

We consider the Siegel parabolic subgroup P of G given by

$$P = M \cdot V = GL_{2n} \cdot V. \quad (2.1)$$

Let \mathcal{A}_{2n} be the set of skew-symmetric matrices of degree $2n$. We may write elements of V as

$$v = \begin{pmatrix} 1_{2n} & z \\ & 1_{2n} \end{pmatrix}, \text{ with } z \in \mathcal{A}_{2n}.$$

Let $b \in \mathcal{A}_{2n}$ be a nonsingular skew-symmetric matrix, define a character ψ^b of V by

$$\psi^b(v) = \psi \left(\frac{1}{2} \text{tr}(bz) \right). \quad (2.2)$$

The stabilizer of the character ψ^b in GL_{2n} is Sp_{2n}^b , the symplectic group with respect to b ,

$$Sp_{2n}^b = \{g \in GL_{2n} \mid {}^t bg = b\}.$$

Form a group

$$H^b := Sp_{2n}^b \cdot V. \quad (2.3)$$

It is clear ψ^b can be extended to be a character of H^b by

$$\psi^b(g, v) := \psi^b(v), \quad (g, v) \in H^b. \quad (2.4)$$

Let

$$\varepsilon_l = \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{pmatrix}_{l \times l}, \quad J_{2l} = \begin{pmatrix} & \varepsilon_l \\ -\varepsilon_l & \end{pmatrix}. \quad (2.5)$$

Then $J_{2n} \in \mathcal{A}_{2n}$ is a nonsingular skew-symmetric matrix. We will drop the superscript b of ψ^b , H^b and Sp_{2n}^b if $b = J_{2n}$.

Definition 2.1. *Let (σ, V_σ) be an irreducible admissible representation of $\mathrm{SO}_{4n}(F)$. We say that σ has a local generalized Shalika functional or generalized Shalika model if $\mathrm{Hom}_{H^b(F)}(V_\sigma, \psi^b)$ is nonzero for some nonsingular b in \mathcal{A}_{2n} , i.e. there is a nonzero linear functional l_{ψ^b} on V_σ such that*

$$l_{\psi^b}(\sigma(h)v) = \psi^b(h)l_{\psi^b}(v), \text{ for } h \in H^b, v \in V_\sigma.$$

It is clear that if σ has a generalized Shalika model for some b , then it has a generalized Shalika model for every nonsingular b in \mathcal{A}_{2n} .

Let (τ, E_τ) be an irreducible admissible unitary generic representation of GL_{2n} . For $s \in \mathbb{C}$, consider the normalized induced representation

$$I(s, \tau) = \mathrm{Ind}_{P(F)}^{\mathrm{SO}_{4n}(F)}(\tau \otimes |\det_{2n}|^{\frac{s}{2}}). \quad (2.6)$$

The main result over local field F is to determine when the induced representation $I(s, \tau)$ has a generalized Shalika model, which is in §3. The corresponding global result is established in §4. To this end, we will first discuss the double coset decomposition $P \backslash \mathrm{SO}_{4n} / H$, and then consider the admissibility of the double cosets.

2.1 Double coset decomposition

In this section, we decompose the flag variety $P \backslash \mathrm{SO}_{4n}$ into H -orbits over field F . Firstly we have the following Bruhat decomposition.

Proposition 2.2. *The group SO_{4n} decomposes into $2n + 1$ cosets with respect to (P, P) , and their representatives are given as follows:*

$$\omega_j = \begin{pmatrix} 1_{2n-j} & & & \\ & & & 1_j \\ & & 1_{2n-j} & \\ & 1_j & & \end{pmatrix}, \quad j = 0, \dots, 2n.$$

It reduces to calculate the decomposition

$$P \backslash P \omega_j P / H$$

for $j = 0, \dots, 2n$. In the following, we will denote by Q_{n_1, \dots, n_k} the standard parabolic subgroup of GL_{2n} with Levi part

$$GL_{n_1} \times GL_{n_2} \times \dots \times GL_{n_k}.$$

Let V_{n_1, \dots, n_k} be the unipotent radical of Q_{n_1, \dots, n_k} .

Consider the double coset decomposition

$$P \backslash P \omega_j P / H = \omega_j^{-1} P \omega_j \cap P \backslash P / H \quad (2.7)$$

One can check that

$$\omega_j^{-1} P \omega_j \cap P = Q_{2n-j, j} \cdot V_j \quad (2.8)$$

where V_j is the subgroup of V consisting of elements of the following type:

$$\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix}, \quad \text{with } z = \begin{pmatrix} x & y \\ -{}^t y & 0_j \end{pmatrix}.$$

Since P normalizes V , we obtain

$$P \backslash P \omega_j P / H \cong Q_{2n-j, j} \backslash GL_{2n} / Sp_{2n}. \quad (2.9)$$

If $0 \leq j \leq n$, the set representatives of $Q_{2n-j, j} \backslash GL_{2n} / Sp_{2n}$ can be chosen as follows:

$$\gamma_i = \begin{pmatrix} 1_i & & & \\ & 0 & 1_{2(n-j)+\frac{\varepsilon}{2}} & 0 \\ & 0 & 0 & 1_{\frac{\varepsilon}{2}} \\ & 1_{\frac{\varepsilon}{2}} & 0 & 0 \\ & & & & 1_{i+\frac{\varepsilon}{2}} \end{pmatrix} \quad (2.10)$$

with $0 \leq i \leq j$ such that $r = j - i$ is even. We notice here γ_i is chosen such that

$$\eta_i := \gamma_i J_{2n} {}^t \gamma_i = \begin{pmatrix} & & & \varepsilon_i \\ & J_{2(n-j)+r} & 0 & \\ & 0 & J_r & \\ -\varepsilon_i & & & \end{pmatrix} \quad (2.11)$$

Similarly, if $n \leq j \leq 2n$, the set of representatives of $Q_{2n-j, j} \backslash \mathrm{GL}_{2n} / \mathrm{Sp}_{2n}$ can be chosen as

$$\gamma'_i = \begin{pmatrix} 1_{j + \frac{2(n-j)-r}{2}} & & & & \\ & 0 & 0 & 1_{\frac{2(n-j)+r}{2}} & \\ & 1_{\frac{r-2(n-j)}{2}} & 0 & 0 & \\ & 0 & 1_{\frac{2(n-j)+r}{2}} & 0 & \\ & & & & 1_i \end{pmatrix} \quad (2.12)$$

for $0 \leq i \leq 2n - j$ such that $r = j - i$ is even. γ'_i is chosen so that

$$\eta_i := \gamma'_i J_{2n} {}^t \gamma'_i = \begin{pmatrix} & & & \varepsilon_i \\ & J_{2(n-j)+r} & 0 & \\ & 0 & J_r & \\ -\varepsilon_i & & & \end{pmatrix} \quad (2.13)$$

In conclusion, our computation implies the following result.

Proposition 2.3. *Notation as above.*

$$P \backslash \mathrm{SO}_{4n} / H = \bigcup_{j=0}^n \bigcup_{\substack{i=0 \\ i \equiv j \pmod{2}}}^j P \omega_j \gamma_i H \cup P \omega_{2n-j} \gamma'_i H. \quad (2.14)$$

The character ψ^{η_i} will be used in the computation of Fourier coefficients of Eisenstein series. Before we study properties of double cosets $P \backslash \mathrm{SO}_{4n} / H$, we give a formula for it. Let $z \in \mathcal{A}_{2n}$. Write z as

$$z = \begin{pmatrix} S & T \\ -{}^t T & Z \end{pmatrix} \quad (2.15)$$

with $S \in \mathrm{M}_{(2n-j) \times (2n-j)}$ and

$$S = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}, T = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}, Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}. \quad (2.16)$$

The block sizes of S, T, Z are compatible with that of (2.11) and (2.13). It is clear that

$$z \eta_i = \begin{pmatrix} -{}^t t_2 \varepsilon_i & * & * & * \\ * & s_4 J_{2(n-j)+r} & * & * \\ * & * & z_1 J_r & * \\ * & * & * & -{}^t t_2 \varepsilon_i \end{pmatrix}. \quad (2.17)$$

Hence

$$\psi^{\eta_i}(z) = \psi\left(\frac{1}{2}\mathrm{tr}(z\eta_i)\right) = \psi\left(-\mathrm{tr}(t_2\varepsilon_i) + \frac{1}{2}s_4J_{2(n-j)+r} + \frac{1}{2}z_1J_r\right). \quad (2.18)$$

Let $0 \leq j \leq 2n$, $0 \leq i \leq \min(j, 2n - j)$ so that $r = j - i$ is even. It can be checked directly that $Q_{2n-j,j} \cap \mathrm{Sp}_{2n}^{\eta_i}$ has a Levi component which consists of elements of the following form

$$\begin{pmatrix} A & & & \\ & D & & \\ & & E & \\ & & & F \end{pmatrix} \quad (2.19)$$

with $A \in \mathrm{GL}_{2i}$, $D \in \mathrm{Sp}_{2(n-j)+r}$, $E \in \mathrm{Sp}_r$, and $F = \varepsilon_i {}^t A^{-1} \varepsilon_i$. The unipotent radical of $Q_{2n-j,j} \cap \mathrm{Sp}_{2n}^{\eta_i}$ is a subgroup of $V_{i,2(n-j)+r,r,i}$, and consists of elements of the following type

$$\begin{pmatrix} 1 & X & S & T \\ & 1 & 0 & Z \\ & & 1 & Y \\ & & & 1 \end{pmatrix} \quad (2.20)$$

satisfying $X = \varepsilon_i {}^t Z J_{2(n-j)+r}$, $S = \varepsilon_i {}^t Y J_r$ and

$${}^t T \varepsilon_i + {}^t Z J_{2(n-j)+r} Z + {}^t Y J_r Y - \varepsilon_i T = 0. \quad (2.21)$$

Note that

$$P \cap \omega_j \gamma_i H \gamma_i^{-1} \omega_j^{-1} = \omega_j (P \cap H^{\eta_i}) \omega_j^{-1}. \quad (2.22)$$

Hence $P \cap \omega_j \gamma_i H \gamma_i^{-1} \omega_j^{-1}$ consists of elements of the following form

$$\begin{pmatrix} a & at & & \\ & q & & \\ & & {}^t a^{-1} & \\ & -{}^t q^{-1} & {}^t t & {}^t q^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & v & b \\ & 1 & -{}^t b & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \quad (2.23)$$

such that

$$\begin{pmatrix} a & ab \\ & {}^t q^{-1} \end{pmatrix} \in Q_{2n-j,j} \cap \mathrm{Sp}_{2n}^{\eta_i}. \quad (2.24)$$

Since the effect of $\text{Ad}(\omega_j^{-1})$ on $P \cap \omega_j \gamma_i H \gamma_i^{-1} \omega_j^{-1}$ is

$$\begin{aligned} & \begin{pmatrix} a & at & & & & \\ & q & & & & \\ & & {}^t a^{-1} & & & \\ & & -{}^t q^{-1} & {}^t t & {}^t q^{-1} & \\ & & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & v & b \\ & 1 & -{}^t b & 0 \\ & & 1 & 0 \\ & & & & & 1 \end{pmatrix} \\ & \xrightarrow{\text{Ad}(\omega_j^{-1})} \begin{pmatrix} a & ab & & & & \\ & {}^t q^{-1} & & & & \\ & & {}^t a^{-1} & & & \\ & & -q {}^t b & q & & \end{pmatrix} \begin{pmatrix} 1 & 0 & v - {}^t {}^t b + b {}^t t & t \\ & 1 & -{}^t t & 0 \\ & & 1 & 0 \\ & & & & & 1 \end{pmatrix}. \end{aligned} \quad (2.25)$$

Let L^{η_i} be the subgroup of GL_{2n} which consists of elements of the following type

$$(a, q; t) = \begin{pmatrix} a & at \\ & q \end{pmatrix}, \quad \text{with } t \in \text{M}_{(2n-j) \times j} \quad (2.26)$$

and

$$a = \begin{pmatrix} A & AX \\ & D \end{pmatrix}, \quad q = \begin{pmatrix} E & \\ \varepsilon_i A \varepsilon_i Y & \varepsilon_i A \varepsilon_i \end{pmatrix} \quad (2.27)$$

with $A \in \text{GL}_i$, $D \in \text{Sp}_{2(n-j)+r}$, $E \in \text{Sp}_r$, $X \in \text{M}_{i, 2(n-j)+r}$, $Y \in \text{M}_{i, r}$. It is clear from (2.23) and (2.24) that

$$L^{\eta_i} = Q_{2n-j, j} \cap \omega_j \gamma_i H \gamma_i^{-1} \omega_j. \quad (2.28)$$

Proposition 2.4. *Let $0 \leq j \leq 2n$. For $0 \leq i \leq \min(j, 2n - j)$ such that $r = j - i$ is even, the dimension of the algebraic variety $P\omega_j \gamma_i H$ is*

$$d(j) = n^2 - (j - n)^2. \quad (2.29)$$

In particular, open double cosets are $P\omega_n \gamma_i H$ with $0 \leq i \leq n$ so that $n - i$ is even.

Proof. The mapping $x \mapsto x\gamma_i^{-1}$ induces a homeomorphism

$$P \backslash P\omega_j \gamma_i H \rightarrow P \backslash P\omega_j H^{\eta_i}. \quad (2.30)$$

Since $H^{\eta_i} = \gamma_i H \gamma_i^{-1}$, $\text{Ad}\gamma_i$ induces a homeomorphism

$$P \backslash P\omega_j H \rightarrow P \backslash P\omega_j H^{\eta_i}. \quad (2.31)$$

Hence the dimension of the algebraic variety $P \backslash P\omega_j \gamma_i H$ is the same as that of $P \backslash P\omega_j H$.

Assume $i = j$. Since

$$\dim P \backslash P \omega_j \gamma_j H = \dim H - \dim P \cap \omega_j H^{\eta_j} \omega_j^{-1}. \quad (2.32)$$

By the structure of $Q_{2n-j} \cap \omega_j \mathrm{Sp}_{2n}^{\eta_j} \omega_j^{-1}$ in (2.23), (2.24), we see that

$$\dim P \cap \omega_j H^{\eta_j} \omega_j^{-1} = (j - n)^2 + 3n^2. \quad (2.33)$$

Since $\dim H = 4n^2$, the result follows. \square

2.2 Admissible double cosets

Let $0 \leq j \leq 2n$ be an even number. If $i = 0$, then L^{η_0} (cf. (2.26), (2.27)) is the subgroup of $Q_{2n-j, j}$ consists of elements of the following type:

$$\begin{pmatrix} A & X \\ & D \end{pmatrix}, \quad \text{with } A \in \mathrm{Sp}_{2n-j}, D \in \mathrm{Sp}_j, X \in M_{2n-j, j}. \quad (2.34)$$

We will write L for L^{η_0} if no confusion is caused. Let $\mathbf{1}$ be the trivial character of L .

Let γ be a representative of a double coset in $U \backslash \mathrm{GL}_{2n} / L$, where U is the standard maximal unipotent subgroup of GL_{2n} . We say that γ is admissible if for all $u \in U \cap \gamma L \gamma^{-1}$ we have

$$\psi_U(u) = 1. \quad (2.35)$$

Here ψ_U is a generic character of U .

Proposition 2.5. *Let $0 \leq j \leq 2n$ be an even number. Then every double coset in $U \backslash \mathrm{GL}_{2n} / L$ is not admissible.*

Proof. We prove the proposition for $0 \leq j \leq n$, the proof for $j > n$ is similar and we omit it.

Let ψ_U, ψ'_U be two generic characters of U . Then there is a diagonal matrix h in GL_{2n} such that

$$\psi_U(u) = \psi'_U(huh^{-1}), \quad u \in U.$$

Since h normalizes U , $U\gamma L$ is an admissible double coset for ψ'_U if and only if $Uh\gamma L$ is admissible double coset for ψ_U . Without loss of generality, we assume that the generic character ψ_U is given by

$$\psi_U(u) = \psi \left(\sum_{i=1}^{2n-1} u_{i, i+1} \right) \quad (2.36)$$

where ψ is a nontrivial character of k .

Let W be the Weyl group of GL_{2n} , define

$$W_j = W \cap Q_{2n-j, j}, \quad (2.37)$$

If we identify W with the symmetric group S_{2n} , then $W_j = S_{2n-j} \times S_j$. It is well known that

$$B \backslash \mathrm{GL}_{2n} / Q_{2n-j, j} = W / W_j. \quad (2.38)$$

and

$$Q_{2n-j, j} \backslash \mathrm{GL}_{2n} / Q_{2n-j, j} = W_j \backslash W / W_j. \quad (2.39)$$

Take a set of representatives of $Q_{2n-j, j} \backslash \mathrm{GL}_{2n} / Q_{2n-j, j}$ as follows:

$$w_m = \begin{pmatrix} 1_{2n-j-m} & 0 & 0 & 0 \\ 0 & 0 & 1_m & 0 \\ 0 & 1_m & 0 & 0 \\ 0 & 0 & 0 & 1_{j-m} \end{pmatrix}, \text{ for } 0 \leq m \leq j. \quad (2.40)$$

Then representatives of $B \backslash \mathrm{GL}_{2n} / Q_{2n-j, j}$ can be chosen to be of the following form

$$ww_m, \quad \text{for some } w \in W_j. \quad (2.41)$$

Let γ be a representative of $U \backslash \mathrm{GL}_{2n} / L$. Then γ can be chosen as

$$hww_my \quad (2.42)$$

where h is a diagonal element, $y \in \mathrm{GL}_{2n-j} \times \mathrm{GL}_j$ be a representative of $Q_{2n-j, j} / L$. We need to show that ψ_U is nontrivial on $U \cap \gamma L \gamma^{-1}$.

If $m = 0$, then we take $\gamma = hy$ with a diagonal matrix h and $y \in \mathrm{GL}_{2n-j} \times \mathrm{GL}_j / \mathrm{Sp}_{2n-j} \times \mathrm{Sp}_j$. For $z \in k$, let

$$u(z) = (u_{p,q}) \in U \quad (2.43)$$

be such that $u_{p,p} = 1$ for $1 \leq p \leq 2n$ and if $p \neq q$, $u_{p,q} = 0$ unless $p = 2n - j, q = 2n - j + 1$, and $u_{2n-j, 2n-j+1} = z$. Assume that $h = \mathrm{diag}(h_1, \dots, h_{2n})$, then

$$h^{-1}u(z)h = u \begin{pmatrix} h_{2n-j+1} & \\ & z \\ & & h_{2n-j} \end{pmatrix} \in L, \quad (2.44)$$

and $y^{-1}h^{-1}u(z)hy \in L$. Hence $u(z) \in U \cap \gamma^{-1}L\gamma$. Since we can take z sufficiently large such that $\psi(z) \neq 1$, ψ_U is nontrivial on $U \cap \gamma L \gamma^{-1}$.

Let $m > 0$, then $\mathrm{Ad}(w_m^{-1})$ acts on $Q_{2n-j, j}$ by

$$\begin{pmatrix} A & B & S & T \\ C & D & U & Z \\ & E & F & \\ & G & H & \end{pmatrix} \xrightarrow{\mathrm{Ad}(w_m^{-1})} \begin{pmatrix} A & S & B & T \\ 0 & D & 0 & F \\ C & U & C & Z \\ 0 & G & 0 & H \end{pmatrix}. \quad (2.45)$$

Here matrices in $Q_{2n-j,j}$ is written in blocks of partition

$$(2n - j - m, m, m, j - m). \quad (2.46)$$

If $m \neq 2n - j$, from (2.45), we see that the B part and Z part of $Q_{2n-j,j}$ is nonzero. Let $w = \alpha\beta$ for some $\alpha \in S_{2n-j}$, $\beta \in S_j$. If α satisfies

$$\begin{aligned} \alpha(1), \dots, \alpha(m) &\geq 2n - j - m, \\ \alpha(m + 1), \dots, \alpha(2n - j) &\leq 2n - j - m, \end{aligned} \quad (2.47)$$

and β satisfies similar condition:

$$\begin{aligned} \beta(1), \dots, \beta(m) &\geq j - m, \\ \beta(m + 1), \dots, \beta(j) &\leq j - m, \end{aligned} \quad (2.48)$$

then $\text{Ad}(w)^{-1}$ acts on $Q_{2n-j,j}$ by

$$\begin{pmatrix} A' & B' & S' & T' \\ C' & D' & U' & Z' \\ & & E' & F' \\ & & G' & H' \end{pmatrix} \xrightarrow{\text{Ad}w^{-1}} \begin{pmatrix} D^* & C^* & Z^* & U^* \\ B^* & A^* & T^* & S^* \\ & & H^* & G^* \\ & & F^* & E^* \end{pmatrix}. \quad (2.49)$$

Here the matrix on the left side is written in blocks of partition

$$(m, 2n - j - m, j - m, m), \quad (2.50)$$

while the matrix on the right is in blocks of partition

$$(2n - j - m, m, m, j - m). \quad (2.51)$$

The U' part of the matrix on the left is moved to U^* part of the matrix on the right by performing column and row permutations to ${}^tU'$. Note that ψ is nontrivial on the U' part on the left matrix of (2.49), which is move to U^* part of matrix on the right, which is invariant under $\text{Ad}(w_m)^{-1}$ as indicated in (2.45). Hence ψ is nontrivial on $U \cap \gamma L \gamma^{-1}$.

If one of (2.47) and (2.48) is not satisfied, say, α does not satisfy (2.47), we claim that there exists $u \in U$ such that $\psi(u) \neq 1$ and

$$w^{-1}uw = \begin{pmatrix} 1 & b & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \quad (2.52)$$

for some $b \in M_{2n-j-m,m}$. Here the matrix is written in blocks of partition $(2n - j - m, m, m, j - m)$. Obviously, if this claim is true, then ψ_U is nontrivial on $U \cap \gamma L \gamma^{-1}$.

In fact, if such u doesn't exist, then for $1 \leq \ell \leq 2n - j - 1$, either

$$\alpha(\ell) \leq 2n - j - m, \alpha(\ell + 1) \leq 2n - j - m, \quad (2.53)$$

or

$$\alpha(\ell) > 2n - j - m. \quad (2.54)$$

Then there is ℓ_0 such that

$$\begin{aligned} \alpha(1), \dots, \alpha(\ell_0) &> 2n - j - m \\ \alpha(\ell_0 + 1), \dots, \alpha(2n - j) &\leq 2n - j - m. \end{aligned} \quad (2.55)$$

Hence $\ell_0 = m$ and α satisfies (2.47). This contradicts to our assumption.

Now let $m = 2n - j$, then $j = m = n$. Let

$$y = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}, \quad \text{with } a_1, a_2 \in \text{GL}_n. \quad (2.56)$$

Denote temporarily the symplectic group for $a_i J_i {}^t a_i$ by $\text{Sp}_j(a_i)$, then

$$\text{Sp}_j(a_i) = a_i \text{Sp}_j a_i^{-1}, \quad \text{for } i = 1, 2. \quad (2.57)$$

It is clear that

$$yLy^{-1} = \left\{ \begin{pmatrix} A & U \\ & D \end{pmatrix} \mid A \in \text{Sp}_j(a_1), D \in \text{Sp}_j(a_2), U \in M_{n \times n} \right\}, \quad (2.58)$$

and $Utww_n yL$ is admissible if and only if $Utww_n(yLy^{-1})$ is admissible.

Recall the action of $\text{Ad}(w_n)^{-1}$ on $Q_{n,n}$ is

$$\begin{pmatrix} A & X \\ & D \end{pmatrix} \xrightarrow{\text{Ad}(w_n)^{-1}} \begin{pmatrix} D & \\ X & A \end{pmatrix}, \quad \text{with } A, D \in \text{GL}_n. \quad (2.59)$$

Decompose $w = w_1 w_2$, with $w_1, w_2 \in S_n$. Let u be an element of U , then

$$u = \begin{pmatrix} u_1 & u_2 \\ & u_4 \end{pmatrix} \quad (2.60)$$

with u_1, u_4 in the maximal unipotent subgroups U_0 of GL_n . Then

$$[(ww_n)^{-1}]u[ww_n] = \begin{pmatrix} w_2^{-1}u_4w_2 & \\ * & w_1^{-1}u_1w_1 \end{pmatrix} \quad (2.61)$$

In [14, Proposition 3], Jacquet and Rallis show that there is no admissible double coset of $U_0 \backslash \text{GL}_n / Sp_n$, equivalently, for every $a \in \text{GL}_n$ there exists $u' \in U_0$ such that

$$\psi_{U_0}(u) \neq 1, \quad \text{and } a^{-1}u'a \in Sp_n. \quad (2.62)$$

Here ψ_{U_0} is the restriction of ψ_U to U_0 , which is a generic character of U_0 . Now let $a = w_1 a_2^{-1}$, $u_1 \in U_0$ such that $\psi_{U_0}(u) \neq 1$ and

$$a_2 w_1^{-1} u_1 w_1 a_2^{-1} \in \mathrm{Sp}_n, \quad (2.63)$$

which is equivalent to $w_1^{-1} u_1 w_1 \in \mathrm{Sp}(a_2)$. Choose above u_1 sufficiently large, we see that $Utww_n(yLy^{-1})$ is not admissible. \square

Corollary 2.6. *Let τ be an irreducible admissible generic representation of $\mathrm{GL}_{2n}(F)$. Then we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{L(F)}(\tau, \mathbf{1}) = 0. \quad (2.64)$$

In particular, any irreducible generic cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ has no nonzero $(L(\mathbb{A}), \mathbf{1})$ -periods.

Proof. The proof is similar to the proof of Corollary 3.3 in [2], we will not give the details here. \square

When $j = 0$, we have $L = \mathrm{Sp}_{2n}$. Hence Corollary 2.6 is a generalization of the following result of Jacquet and Rallis in [14], Heumos and Rallis in [10]:

Corollary 2.7. ([14], Proposition 3; [10], Theorem 3.2.2). *Let τ be an irreducible admissible generic representation of $\mathrm{GL}_{2n}(F)$. Then we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Sp}_{2n}(F)}(\tau, \mathbf{1}) = 0. \quad (2.65)$$

In particular, any irreducible generic cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ has no nonzero $\mathrm{Sp}_{2n}(\mathbb{A})$ -periods.

3. Local functionals

Let \mathcal{S}_n be the Shalika group of GL_{2n} consisting of all elements of the following type

$$\begin{pmatrix} g & gu \\ & g \end{pmatrix}, \quad \text{with } g \in \mathrm{GL}_n, u \in M_{n \times n}. \quad (3.1)$$

Let $\psi_{\mathcal{S}_n}$ be the character of \mathcal{S}_n defined by

$$\psi_{\mathcal{S}_n} \left(\begin{pmatrix} g & gu \\ & g \end{pmatrix} \right) = \psi(\mathrm{tr}(u)). \quad (3.2)$$

Let (τ, E_τ) be an irreducible admissible representation of $\mathrm{GL}_{2n}(F)$. As in [15], we say that τ has a nonzero Shalika functional or Shalika model if

$\text{Hom}_{\mathcal{S}_n(F)}(E_\tau, \psi_{\mathcal{S}_n})$ is nonzero, i.e., there is a nonzero linear functional l on E such that

$$l(\tau(x)v) = \psi_{\mathcal{S}_n}(x)l(v), \text{ for all } x \in \mathcal{S}_n, v \in E. \quad (3.3)$$

Recall (2.1) that $P = \text{GL}_{2n} \cdot V$ is the Siegel parabolic subgroup of SO_{4n} . Let (τ, E_τ) be an irreducible admissible, generic, unitary representation of $\text{GL}_{2n}(F)$. we consider the normalized induced representation

$$\mathbb{I}(s, \tau) = \text{Ind}_P^{\text{SO}_{4n}}(\tau \otimes |\det_{2n}|^{\frac{s}{2}}), \quad (3.4)$$

as defined in (2.6).

Theorem 3.1. *The induced representation $\mathbb{I}(s, \tau)$ admits a local generalized Shalika functional if and only if τ has a Shalika functional and $s = 1$. In this case, the nontrivial Shalika functional factors through the unique Langlands quotient of $\mathbb{I}(s, \tau)$ at $s = 1$.*

Proof. Let \mathcal{V} be the space of smooth E_τ -valued functions $f : G \rightarrow E_\tau$ satisfying

$$f(pg) = \tau(p)\delta^{\frac{1}{2}}(p)f(g), \quad \text{for } p \in P, g \in \text{SO}_{4n}. \quad (3.5)$$

Here $\delta(p)$ is the modular function of P . Then $\mathbb{I}(s, \tau)$ acts on \mathcal{V} by

$$[\mathbb{I}(s, \tau)(g)]f(x) = f(xg), \text{ for } x, g \in \text{SO}_{4n}. \quad (3.6)$$

For $0 \leq j \leq n$, let \mathcal{V}_j be the P -invariant subspace of \mathcal{V} , consisting of functions compactly supported modulo P , and

$$\text{supp } f \subset \bigcup_{k \leq j} [P\omega_k P \cup P\omega_{2n-k} P]. \quad (3.7)$$

Then $\mathcal{V} = \mathcal{V}_n \supset \cdots \supset \mathcal{V}_0$ is a decreasing filtration of \mathcal{V} as P -spaces. Let J_ξ be the subspace of \mathcal{V} consisting of functions supported on double coset $P\xi H$ and compactly supported modulo P . Recall that double cosets $P \backslash \text{SO}_{4n} / H$ are computed in §2.1, more explicitly, they are

$$P\omega_j \gamma_i H, \quad P\omega_{2n-j} \gamma'_i H, \quad (3.8)$$

for $0 \leq j \leq n$ and $0 \leq i \leq n$ so that $j - i$ is even, where γ_i and γ'_i are matrices defined at (2.10) and (2.12).

Then the following sequence is exact

$$0 \rightarrow \mathcal{V}_{j-1} \rightarrow \mathcal{V}_j \rightarrow \bigoplus_i J_{\omega_j \gamma_i} \oplus J_{\omega_{2n-j} \gamma'_i} \rightarrow 0 \quad (3.9)$$

where i runs through all $0 \leq i \leq j$ so that $j - i$ is even.

Let $l : \mathbb{I}(s, \tau) \rightarrow \mathbb{C}$ be a linear functional on \mathcal{V} satisfying

$$l([\mathbb{I}(s, \tau)(h)]f) = \psi(h)l(f), \quad (3.10)$$

for all $h \in H$, $f \in \mathcal{V}$. By the localization principle of Gelfand and Kazhdan, l is generated by distributions supported on J_ξ satisfying (3.10). So without loss of generality, we assume that l is supported on $P\omega_j\gamma_i H$ for some $0 \leq j \leq n$ and $0 \leq i \leq j$ so that $j - i$ is even.

As in the proof of Proposition 2.4, $P \cap \omega_j H^{\eta_i} \omega_j^{-1}$ consists of elements of the following form

$$x = \begin{pmatrix} g & \\ & {}_t g^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \in P, \quad (3.11)$$

here

$$g = \begin{pmatrix} a & at \\ & q \end{pmatrix} \in \mathcal{Q}_{2n-j, j}, \quad u = \begin{pmatrix} v & b \\ -tb & 0 \end{pmatrix} \quad (3.12)$$

such that

$$\begin{pmatrix} a & ab \\ & {}_t q^{-1} \end{pmatrix} \in \mathcal{Q}_{2n-j, j} \cap \mathrm{Sp}^{\eta_i}. \quad (3.13)$$

Set

$$H_0 = \gamma_i^{-1} \omega_j^{-1} [P \cap \omega_j H^{\eta_i} \omega_j^{-1}] \omega_j \gamma_i. \quad (3.14)$$

Then H_0 is a subgroup of H . Denote the restriction of the character ψ of H (cf. (2.4)) on H_0 by ψ again. Let $C_\psi^\infty(H, H_0)$ be the set of smooth functions on H compactly supported modulo H_0 and left ψ -invariant. Let H_c be the unipotent group of H which consists of elements of the following type

$$\begin{pmatrix} g & gu \\ & {}_t g^{-1} \end{pmatrix}, \quad \text{with } g = \gamma_i^{-1} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \gamma_i \in \mathrm{Sp}_{2n}, \quad u = \gamma_i^{-1} \begin{pmatrix} 0 & 0 \\ 0 & z' \end{pmatrix} \gamma_i \in \mathcal{A}_{2n}. \quad (3.15)$$

Here $z \in \mathbb{M}_{j \times (2n-j)}$, $z' \in \mathbb{M}_{j \times j}$. Then $H_0 \cap H_c = \{1\}$ and $H = H_0 \cdot H_c$. Let $C_c^\infty(H_c)$ be the space of smooth function on H_c with compact support. Then the restriction induces a homeomorphism

$$C_\psi^\infty(H, H_0) \cong C_c^\infty(H_c) \quad (3.16)$$

Let dh be the Haar measure on H_c . Define a surjective mapping $\mathcal{P}_{\omega_j \gamma_i} : J_{\omega_j \gamma_i} \rightarrow E_\tau$ by

$$\mathcal{P}_{\omega_j \gamma_i}(f) = \int_{H_c(F)} f(h) dh. \quad (3.17)$$

Since l is supported on $P\omega_j\gamma_i H$ and satisfies (3.10), there is a linear functional $l' : E_\tau \rightarrow \mathbb{C}$ such that the following diagram is commutative

$$\begin{array}{ccc} J_{\omega_j\gamma_i} & \xrightarrow{\mathcal{P}_{\omega_j\gamma_i}} & E_\tau \\ l \downarrow & & \downarrow l' \\ \mathbb{C} & \xrightarrow{=} & \mathbb{C} \end{array}$$

and satisfies

$$\begin{aligned} l'(\tau(x)|\det_{2n}(x)|^{\frac{\alpha}{2}}\delta_{\mathbb{P}}^{\frac{1}{2}}(x)v) \\ = \psi(\gamma_i^{-1}\omega_j^{-1}x\omega_j\gamma_i)|\det \text{Ad}(\gamma_i^{-1}\omega_j^{-1}x\omega_j\gamma_i)|_{H_c}|l'(v), \end{aligned} \quad (3.18)$$

for $x \in L^{\eta_i}$, $v \in E_\tau$. We note here that

$$\det(\gamma_i^{-1}\omega_j^{-1}x\omega_j\gamma_i)|_{H_c} = \det(a)^{2i}, \text{ for } x \in L^{\eta_i}. \quad (3.19)$$

If $i \neq 0$ and $2(n-j) + r \neq 0$, for $X \in M_{i \times 2(n-j)+r}$, $Y \in M_{r \times i}$, define

$$a = \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & \\ Y & 1 \end{pmatrix}. \quad (3.20)$$

We choose $0 \neq X$ such that there is $b \in M_{(2n-j) \times j}$ such that

$$\begin{pmatrix} a & ab \\ & {}_t q^{-1} \end{pmatrix} \in P \cap \omega_j H^{\eta_i} \omega_j^{-1}. \quad (3.21)$$

For $t = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} \in M_{2n-j, j}$, define

$$x = \begin{pmatrix} 1 & X & t_1 & t_2 \\ & 1 & t_3 & t_4 \\ & & 1 & 0 \\ & & & Y & 1 \end{pmatrix} \in L^{\eta_i}. \quad (3.22)$$

Then simple computation shows that

$$\psi(\gamma_i^{-1}\omega_j^{-1}x\omega_j\gamma_i) = \psi(-\text{tr}(t_2\varepsilon_i)). \quad (3.23)$$

Let

$$t' = \begin{pmatrix} t'_1 & t'_2 \\ t'_3 & t'_4 \end{pmatrix} \in M_{2n-j, j}$$

be such that $\psi(-\text{tr}(Xt'_4\varepsilon_i)) \neq 1$. If we let

$$x' = \begin{pmatrix} 1 & 0 & t'_1 & t'_2 \\ & 1 & t'_3 & t'_4 \\ & & 1 & \\ & & & 0 & 1 \end{pmatrix} \quad (3.24)$$

then

$$\begin{aligned} \psi(\gamma_i^{-1}\omega_j^{-1}xx'\omega_j\gamma_i) &= \psi(-\mathrm{tr}[(t_2 + t'_2)\varepsilon_i])\psi(-\mathrm{tr}(Xt'_4\varepsilon_i)) \\ &\neq \psi(\gamma_i^{-1}\omega_j^{-1}x\omega_j\gamma_i) \cdot \psi(\gamma_i^{-1}\omega_j^{-1}x'\omega_j\gamma_i). \end{aligned} \quad (3.25)$$

This contradicts (3.18), which states that the function

$$\psi(\gamma_i^{-1}\omega_j^{-1}x\omega_j\gamma_i), \quad x \in L^{\eta_i}$$

is a character, hence in this case $l' = 0$.

If $i = 0$, then (3.18) becomes

$$l'(\tau(x)v) = l'(v), \quad \text{for } x \in L^{\eta_0}. \quad (3.26)$$

This is impossible by Proposition 2.5. So in this case, $l' = 0$.

If $2(n - j) + r = 0$, then $r = 0$ and $n = j = i$. Denote the group L^{η_n} defined at (2.28) simply by L . Then L is the subgroup of GL_{2n} consisting of elements of the following type

$$x = \begin{pmatrix} g & gu \\ \varepsilon_n g \varepsilon_n & \end{pmatrix}, \quad \text{with } g \in \mathrm{GL}_n, u \in M_{n \times n}. \quad (3.27)$$

Hence $|\det_{2n}(x)|^{\frac{1}{2}} = |\det_n(g)|$. By (2.18),

$$\psi(\gamma_n^{-1}\omega_n^{-1}x\omega_n\gamma_n) = \psi(-u\varepsilon_n), \quad (3.28)$$

The equation (3.18) of l' is

$$l'(\tau(x)\alpha^{s-1}) = \psi(-u\varepsilon_n)l'(v), \quad v \in E. \quad (3.29)$$

Let

$$x_0 = \begin{pmatrix} 1 & \\ & -\varepsilon_n \end{pmatrix}. \quad (3.30)$$

Then $L \cong \mathrm{Ad}(x_0)(\mathcal{S}_n)$, where \mathcal{S}_n is the Shalika subgroup of GL_{2n} , and

$$\psi(\gamma_n^{-1}\omega_n^{-1}x\omega_n\gamma_n) = \psi_{\mathcal{S}_n} \mathrm{Ad}(x_0^{-1})(x). \quad (3.31)$$

Here $\psi_{\mathcal{S}_n}$ is the character of \mathcal{S}_n defined at (3.2). Hence if l' is nonzero and supported on $P\omega_n\gamma_n H$, then the following representation

$$\tau \otimes |\det_{2n}|^{\frac{s-1}{2}}$$

must have a nonzero Shalika functional. In [15], Jacquet and Rallis shows that if an irreducible representation of GL_{2n} has a Shalika functional, then it is self-dual. So $s = 1$ and τ is self-dual.

Conversely, if τ is an irreducible admissible, self-dual, generic, unitary representation of $\mathrm{GL}_{2n}(F)$ with a nonzero Shalika functional l' , we can construct a generalized Shalika functional l of $\mathrm{I}(1, \tau)$ by defining it to be zero on

$$\mathcal{V}_{n-1} \bigoplus_{\substack{0 \leq i < n \\ i \equiv n \pmod{2}}} J_{\omega_n \gamma_i}. \quad (3.32)$$

and on $J_{\omega_n \gamma_n}$ by

$$l(f) := l'(\mathcal{P}_{\omega_n \gamma_n}(f)), \text{ for } f \in J_{\omega_n \gamma_n}. \quad (3.33)$$

From the argument above, we see that if l is a generalized Shalika functional of $\mathrm{I}(1, \tau)$, then it factors through \mathcal{V}_{n-1} which is the maximal P -invariant subspace of \mathcal{V} . As the Jacquet functor is exact, l factors through the Langlands quotient of $\mathrm{I}(1, \tau)$. \square

4. Global periods

Let k be a number field and \mathbb{A} be the ring of adèles of k . Let SO_{4n} be the k -split even special orthogonal group. Recall from (2.3) and (2.4) the definition of the subgroup H of SO_{4n} and the character ψ_H of H . Note that it is denoted by ψ^b in (2.4). Let φ be an automorphic form on $\mathrm{SO}_{4n}(\mathbb{A})$. We define the generalized Shalika period of φ by

$$\mathcal{P}_{H, \psi}(\varphi) := \int_{H(k) \backslash H(\mathbb{A})} \varphi(h) \psi_H^{-1}(h) dh. \quad (4.1)$$

Note that if φ is not cuspidal, the integral may diverge, and will be defined via regularization.

We shall consider Eisenstein series on $\mathrm{SO}_{4n}(\mathbb{A})$ associated to the maximal parabolic subgroup $P_{2n} = M_{2n}U_{2n}$ and irreducible cuspidal automorphic representation π of the Levi part $M_{2n}(\mathbb{A}) \cong \mathrm{GL}_{2n}(\mathbb{A})$. The location of the pole of the Eisenstein series is expected to be determined in terms of the location of poles of the exterior square L -function $L(s, \pi, \Lambda^2)$ by the Langlands theory of the constant terms of Eisenstein series. We shall use (generalized) Shalika model to realize the residue of the Eisenstein series, and as a by-product, we determine the location of the poles of the Eisenstein series by means of the Shalika model of the cuspidal data.

Let $K = \prod_v K_v$ be a maximal compact subgroup of $\mathrm{SO}_{4n}(\mathbb{A})$ such that $\mathrm{SO}_{4n}(\mathbb{A})$ has the Iwasawa decomposition

$$\mathrm{SO}_{4n}(\mathbb{A}) = P_{2n}(\mathbb{A})K.$$

In particular, for each finite local place v , $K_v = \mathrm{SO}_{4n}(\mathcal{O}_v)$, where \mathcal{O}_v is the ring of integers in the local field k_v . Then the Langlands decomposition of $\mathrm{SO}_{4n}(\mathbb{A})$ is

$$\mathrm{SO}_{4n}(\mathbb{A}) = U_{2n}(\mathbb{A})M_{2n}^1 A_{2n}^+ K.$$

Let A_{2n} be the (split) center of M_{2n} , the unique reduced root in $\Phi^+(P_{2n}, A_{2n})$ can be identified with simple root α_{2n} . As normalized in [30], we denote $\tilde{\alpha}_{2n} := \langle \rho_{P_{2n}}, \alpha_{2n} \rangle^{-1} \rho_{P_{2n}}$, where $\rho_{P_{2n}}$ is half of the sum of all positive root in U_{2n} and $\langle \cdot, \cdot \rangle$ is the usual Killing-Cartan form for the root system of SO_{4n} . We let

$$\mathfrak{a}_{M_{2n}} = \mathrm{Hom}_{\mathbb{R}}(X(M_{2n}), \mathbb{R}), \quad \mathfrak{a}_{M_{2n}}^* = X(M_{2n}) \otimes \mathbb{R}, \quad (4.2)$$

where $X(M_{2n})$ denotes the group of all rational characters of M_{2n} . Since P_{2n} is maximal, \mathfrak{a}_{2n}^* is of dimension one. We identify \mathbb{C} with $\mathfrak{a}_{2n, \mathbb{C}}^*$ via $s \mapsto s\tilde{\alpha}_{2n}$.

For simplicity, we use the notation without the indication $2n$. Let $H_P : M \rightarrow \mathfrak{a}_M$ be the map defined as follows, for any $\chi \in \mathfrak{a}_M^*$,

$$H_P(m)(\chi) = \prod_v |\chi(m_v)|_v \quad (4.3)$$

for $m \in M(\mathbb{A})$. It follows that H_P is trivial on M^1 . This map H_P can be extended as a function over $\mathrm{SO}_{4n}(\mathbb{A})$ via the Iwasawa decomposition or the Langlands decomposition above. By direct computation, we know that

$$H_P(m)(s) = |\det_{2n} a|^{\frac{s}{2}}, \quad H_P(m)(\rho_P) = |\det_{2n} a|^{\frac{2n-1}{2}} \quad (4.4)$$

where $s \in \mathbb{C}$ and $m = m(a) = \mathrm{diag}(a, {}^t a^{-1}) \in M$ with $a \in \mathrm{GL}_{2n}$.

Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ with trivial central character. Then (M, π) is a cuspidal datum attached the Levi subgroup M . Let $\phi(g)$ be a complex-valued smooth function on $\mathrm{SO}_{4n}(\mathbb{A})$ which is left $U(\mathbb{A})M(k)$ -invariant and right K -finite. Writing by the Langlands decomposition

$$g = um^1 a \kappa \in U(\mathbb{A})M^1 A_M^+ K,$$

we assume that

$$\phi(g) = \phi(m^1 \kappa). \quad (4.5)$$

If we fix a $\kappa \in K$, the map

$$m^1 \mapsto \phi(m^1 \kappa)$$

defines a $K \cap M^1$ -finite vector in the space of cuspidal representation π of $M(\mathbb{A})$. We set

$$F(g; \phi, s) := H_P(g)(s + \rho_P)\phi(g).$$

Attached to such a function $F(g; \phi, s)$, we define an Eisenstein series

$$E(g; \phi, s) := \sum_{\gamma \in P(k) \backslash \mathrm{SO}_{4n}(k)} F(\gamma g; \phi, s) \quad (4.6)$$

By the general theory of Eisenstein series [25], this Eisenstein series converges absolutely for the real part $\mathrm{Re}(s) > \frac{2n-1}{2}$ and has a meromorphic continuation to the whole s -plane with finitely many possible simple poles for $\mathrm{Re}(s) > 0$. By the Langlands theory of the constant terms of Eisenstein series, the existence of the poles on the positive half plane of this Eisenstein series should be detected in general by means of that of the constant terms of the Eisenstein series.

In the case under consideration, the only nonzero constant term of the Eisenstein series $E(g; \phi, s)$ is the one along the maximal parabolic subgroup P , i.e.

$$E_P(g; \phi, s) := \int_{U(k) \backslash U(\mathbb{A})} E(ug; \phi, s) du \quad (4.7)$$

where du is the Haar measure on $U(k) \backslash U(\mathbb{A})$, which is normalized so that the total volume equals one. By assuming the real part of s large, we have

$$E_P(g; \phi, s) = F(g; \phi, s) + F(g; M(s, w_{2n})(\phi), -s) \quad (4.8)$$

where $M(s, w_{2n})$ is the standard (global) intertwining operator attached to the maximal parabolic subgroup P and the Weyl element w_{2n} , which has the property that $w_{2n} M w_{2n}^{-1} = M$ and $w_{2n} U w_{2n}^{-1} = U^-$ (the opposite of U). This equation holds for all $s \in \mathbb{C}$ by meromorphic continuation. The intertwining operator $M(s, w_{2n})$ can be expressed as

$$M(s, w_{2n}) = \otimes_v A(s, \pi_v, w_{2n}) \quad (4.9)$$

by following the notations in [31], where $A(s, \pi_v, w_{2n})$ is the local intertwining operator

$$\mathrm{Ind}_{P(k_v)}^{\mathrm{SO}_{4n}(k_v)}(\pi_v \otimes |\det_{2n} a|^{\frac{s}{2}}) \rightarrow \mathrm{Ind}_{P(k_v)}^{\mathrm{SO}_{4n}(k_v)}(w_{2n}(\pi_v) \otimes |\det_{2n} a|^{-\frac{s}{2}}).$$

Then, one can write

$$A(s, \pi_v, w_{2n}) = \frac{L(s, \pi_v, \Lambda^2)}{L(1+s, \pi_v, \Lambda^2) \epsilon(s, \pi_v, \Lambda^2, \psi_v)} \cdot N(s, \pi_v, w_{2n}).$$

Proposition 4.1. (Theorem 4.11 [23]). *Assume that π_v is the local v -component of an irreducible unitary cuspidal automorphic representation π of $M(\mathbb{A}) = \mathrm{GL}_{2n}(\mathbb{A})$. Then the normalized local intertwining operator $N(s, \pi_v, w_{2n})$ is holomorphic and nonzero for the real part $\mathrm{Re}(s) \geq \frac{1}{2}$.*

By Proposition 4.1, we express the intertwining operator $M(s, w_{2n})$ as

$$M(s, w_{2n}) = \frac{L(s, \pi, \Lambda^2)}{L(1+s, \pi, \Lambda^2)\epsilon(s, \pi, \Lambda^2)} \cdot N(s, \pi, w_{2n}). \quad (4.10)$$

The constant term in (4.8) can be expressed as

$$E_P(g; \phi, s) = F(g; \phi, s) \quad (4.11)$$

$$+ \frac{L(s, \pi, \Lambda^2)}{L(1+s, \pi, \Lambda^2)\epsilon(s, \pi, \Lambda^2)} \cdot F(g; N(s, \pi, w_{2n})(\phi), -s).$$

The analytic properties of the exterior square L -functions $L(s, \pi, \Lambda^2)$ are summarized in [18], which in particular says that $L(s, \pi, \Lambda^2)$ has meromorphic continuation to the whole complex plane and has a possible simple pole at $s = 1$. Hence the Eisenstein series $E(g; \phi, s)$ is holomorphic for the real part of s greater than $\frac{1}{2}$, except possibly at $s = 1$ where it may have a simple pole. $E(g; \phi, s)$ has a simple pole at $s = 1$ if and only if the exterior square L -functions $L(s, \pi, \Lambda^2)$ has a simple pole at $s = 1$. This is the method that determines the location of poles of Eisenstein series in terms of the location of the poles of the relevant L -functions.

4.1 Residues of Eisenstein Series

In this section, we show that the location of the possible simple pole of the Eisenstein series may be determined by means of the generalized Shalika model of the residue. Let s_0 be a real number greater than $\frac{1}{2}$ and define

$$E_{s_0}(g; \phi) := \mathrm{Res}_{s=s_0} E(g; \phi, s).$$

We want to calculate the following period integral

$$\mathcal{P}_{H, \psi}(E_{s_0}(\cdot; \phi)) = \int_{H(k) \backslash H(\mathbb{A})} E_{s_0}(r; \phi) \psi^{-1}(r) dr. \quad (4.12)$$

It is clear that one might write the period as

$$\begin{aligned} & \mathcal{P}_{H, \psi}(E_{s_0}(\cdot, \phi)) \\ &= \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{V(k) \backslash V(\mathbb{A})} E_{s_0}(vx; \phi) \psi_V(v) dv dx \\ &= \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \mathrm{Res}_{s=s_0} \left[\int_{V(k) \backslash V(\mathbb{A})} E(vx; \phi, s) \psi_V(v) dv \right] dx. \end{aligned} \quad (4.13)$$

The convergence of the integration along the variable x needs to be justified via the Arthur's truncation method. We are going to calculate the inner integration first, which is the Fourier coefficient of the Eisenstein series.

4.2 Fourier coefficients of Eisenstein series

We shall study in this section the Fourier coefficient of the Eisenstein series

$$\int_{V(k)\backslash V(\mathbb{A})} E(vh; \phi, s)\psi_V(v)dv, \quad (4.14)$$

which occurs as an inner integration of (4.13). In the following we assume the real part of s be large, so that the Eisenstein series converges absolutely and uniformly on every compact subset in $\mathrm{SO}_{4n}(k)\backslash\mathrm{SO}_{4n}(\mathbb{A})$. By Proposition 2.3, we have

$$\begin{aligned} (4.14) &= \int_{V(k)\backslash V(\mathbb{A})} \sum_{\gamma \in P(k)\backslash\mathrm{SO}_{4n}(k)} F(\gamma vh; \phi, s)\psi_V(v)dv \\ &= \sum_{j=0}^{2n} \sum_{\gamma \in P^{\omega_j}(k)\backslash P(k)/V(k)} \int_{V^{\omega_j, \gamma}(k)\backslash V(\mathbb{A})} F(\omega_j \gamma vh; \phi, s)\psi_V(v)dv \end{aligned} \quad (4.15)$$

where $P^{\omega_j} = \omega_j^{-1}P\omega_j \cap P$ and $V^{\omega_j, \gamma} = (\omega_j \gamma)^{-1}P\omega_j \gamma \cap V$. In the discussion in §2 and Proposition 2.3, we write $P^{\omega_j} \backslash P/H$ as a disjoint union

$$P^{\omega_j} \backslash P/H = \cup_i P^{\omega_j} \gamma_i \mathrm{Sp}_{2n} V, \quad (4.16)$$

where i runs through $0 \leq i \leq \min(j, 2n-j)$ so that $r = j-i$ be even. Here for $j > n$, we write γ_i for γ'_i there. For each given ω_j , we set

$$\mathrm{Sp}_{2n, \gamma_i} := \gamma_i^{-1} P^{\omega_j} \gamma_i \cap \mathrm{Sp}_{2n}. \quad (4.17)$$

Then we obtain

$$\begin{aligned} (4.15) &= \sum_{j=0}^{2n} \sum_{\gamma_i} \sum_{\epsilon_i \in \mathrm{Sp}_{2n, \gamma_i}(k)\backslash\mathrm{Sp}_{2n}(k)} \\ &\quad \times \int_{V^{\gamma_i, \epsilon_i}(k)\backslash V(\mathbb{A})} f(\omega_j \gamma_i \epsilon_i vh; \phi, s)\psi_V(v)dv. \end{aligned} \quad (4.18)$$

where $V^{\gamma_i, \epsilon_i} = (\gamma_i \epsilon_i)^{-1} P^{\omega_j} (\gamma_i \epsilon_i) \cap V$. Since Sp_{2n} normalizes V and stabilizes the character ψ_V , we have $V^{\gamma_i, \epsilon_i} = V^{\omega_j, \gamma_i}$ and we can express (4.18) as

$$(4.18) = \sum_{j=0}^{2n} \sum_{\gamma_i} \sum_{\epsilon_i \in \mathrm{Sp}_{2n, \gamma_i}(k)\backslash\mathrm{Sp}_{2n}(k)} \int_{V^{\gamma_i}(k)\backslash V(\mathbb{A})} f(\omega_j \gamma_i v \epsilon_i h; \phi, s)\psi_V(v)dv. \quad (4.19)$$

In the following we will discuss (4.19) for each ω_j case by case.

If $j = 0$, then $\omega_0 = e$, $\gamma_i = \epsilon_i = e$. Hence

$$(4.19)_{j=0} = \int_{V(k)\backslash V(\mathbb{A})} F(vh; \phi, s)\psi_V(v)dv \quad (4.20)$$

$$= F(h; \phi, s) \int_{V(k)\backslash V(\mathbb{A})} \psi_V(v)dv = 0, \quad (4.21)$$

since ψ_V is nontrivial on $V(k)\backslash V(\mathbb{A})$.

Let $0 \leq j \leq n$. The double coset representatives γ_i of $P^{\omega_j}\backslash P/H$ are given in (2.10), for $0 \leq i \leq j$ such that $r = j - i$ is even. Let $V_0 \subset V$ be the subgroup consists of elements of the following form

$$u = \gamma_i^{-1} \begin{pmatrix} 1 & 0 & v & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \gamma_i, \text{ with } v \in M_{2n-j, j}. \quad (4.22)$$

It is clear that

$$\omega_j \gamma_i u \gamma_i^{-1} \omega_j = \begin{pmatrix} 1 & 0 & v & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \in V(\mathbb{A}), \quad (4.23)$$

hence $V_0 \subset V^{\omega_j, \gamma_i}$. We notice that

$$\psi_V(u) = \psi^{\eta_i} \begin{pmatrix} 1 & 0 & v & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}. \quad (4.24)$$

By the formula of ψ^{η_i} in (2.18), if $2(n - j) + r \neq 0$, then ψ_V is nontrivial on $V_0(\mathbb{A})$, we then obtain

$$\int_{V_0(k)\backslash V_0(\mathbb{A})} F(\omega_j \gamma_i v \epsilon_i h; \phi, s)\psi_V(v)dv \quad (4.25)$$

$$= F(\omega_j \gamma_i \epsilon_i h; \phi, s) \int_{V_0(k)\backslash V_0(\mathbb{A})} \psi_V(v)dv = 0 \quad (4.26)$$

This proves the integral in (4.19) vanishes when $0 \leq j \leq n$, $i \neq n$, i.e.

$$(4.19)_{j < n} = \sum_{\gamma_i} \sum_{\epsilon_i} \int_{V^{\omega_j, \gamma_i(k)} \setminus V(\mathbb{A})} F(\omega_j \gamma_i v \epsilon_i h; \phi, s) \psi_V(v) dv = 0 \quad (4.27)$$

$$\begin{aligned} (4.19)_{j=n} &= \sum_{\gamma_i \neq \gamma_n} \sum_{\epsilon_i} \int_{V^{\omega_n, \gamma_i(k)} \setminus V(\mathbb{A})} F(\omega_j \gamma_i v \epsilon_i h; \phi, s) \psi_V(v) dv \\ &\quad + \sum_{\epsilon_n \in \mathrm{Sp}_{2n, \gamma_n}(k) \setminus \mathrm{Sp}_{2n}(k)} \\ &\quad \times \int_{V^{\omega_n, \gamma_n(k)} \setminus V(\mathbb{A})} F(\omega_n \gamma_n v \epsilon_n h; \phi, s) \psi_V(v) dv \\ &= \sum_{\epsilon_n} \int_{V^{\omega_n, \gamma_n(k)} \setminus V(\mathbb{A})} F(\omega_n \gamma_n v \epsilon_n h; \phi, s) \psi_V(v) dv \end{aligned} \quad (4.28)$$

Since $\gamma_n = 1_{2n}$, simple computation shows that V^{ω_n, γ_n} is the subgroup of V consists of elements of V of the following type:

$$\begin{pmatrix} 1 & 0 & v & b \\ & 1 & -b^t & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \quad (4.29)$$

and

$$\mathrm{Sp}_{2n, \gamma_n} = \left\{ \begin{pmatrix} g & gu \\ 0 & \epsilon_n {}^t g^{-1} \epsilon_n \end{pmatrix} \in \mathrm{Sp}_{2n} \mid g \in \mathrm{GL}_n, u \in M_{n \times n} \right\} \quad (4.30)$$

is the Siegel parabolic of Sp_{2n} , we change notation now and denote it by P_0 . Hence

$$(4.19)_{j=n} = \sum_{\epsilon_n \in P_0(k) \setminus \mathrm{Sp}_{2n}(k)} \int_{V^{\omega_n, \gamma_n(k)} \setminus V(\mathbb{A})} F(\omega_n v \epsilon_n h; \phi, s) \psi_V(v) dv \quad (4.31)$$

The proof of vanishing of (4.19) for $j > n$ is similar to that of $j < n$, hence,

$$(4.19)_{j > n} = \sum_{\gamma_i} \sum_{\epsilon_i} \int_{V^{\omega_j, \gamma_i} \setminus V(\mathbb{A})} F(\omega_j \gamma_i v \epsilon_i h; \phi, s) \psi_V(v) dv = 0 \quad (4.32)$$

In other words, we have proved that

Proposition 4.2. *The ψ_V -Fourier coefficient of the Eisenstein series $E(vh; \phi, s)$ has the following expression:*

$$\begin{aligned} & \int_{V(k) \backslash V(\mathbb{A})} E(vh; \phi, s) \psi_V(v) dv \\ &= \sum_{\epsilon_n \in P_0(k) \backslash \mathrm{SP}_{2n}(k)} \int_{V^{\omega_n, \gamma_n}(k) \backslash V(\mathbb{A})} F(\omega_n v \epsilon_n h; \phi, s) \psi_V(v) dv. \end{aligned} \quad (4.33)$$

Remark 4.3. It may be important to point out that both the global functional and the local functional are supported in the same (P, H) double coset. It is this particular double coset which relates the Shalika subgroup of GL_{2n} to the generalized Shalika subgroup of SO_{4n} .

Since V is abelian,

$$V = V^{\omega_n, \gamma_n} \times V_{\gamma_n} \quad (4.34)$$

where V_{γ_n} is the subgroup of V , which consists of elements of the following form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & u \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}. \quad (4.35)$$

Define

$$V_{n,n} = \omega_n V^{\omega_n, \gamma_n} \omega_n^{-1} \cap P, \quad (4.36)$$

which is the unipotent radical of the parabolic $P_{n,n}$ of GL_{2n} . The character $\psi_{n,n}$ of $V_{n,n}$ defined by

$$\psi_{n,n} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \psi(-x \varepsilon_n) \quad (4.37)$$

is the character induced from ψ_V by means of conjugation by ω_n . Hence the integral in (4.33) can be expressed as

$$\begin{aligned} \mathcal{F}(g; \phi, s) &:= \int_{V^{\omega_n, \gamma_n}(k) \backslash V(\mathbb{A})} F(\omega_n v g; \phi, s) \psi_V(v) dv \\ &= \int_{V_{\gamma_n}(\mathbb{A})} \int_{V_{n,n}(k) \backslash V_{n,n}(\mathbb{A})} F(v \omega_n v_2 g; \phi, s) \psi_{n,n}(v) dv dv_2. \end{aligned} \quad (4.38)$$

It is clear the first integral over $V_{n,n}(k) \backslash V_{n,n}(\mathbb{A})$ is the $\psi_{n,n}$ -Fourier coefficient of the irreducible cuspidal automorphic representation π and the second integration over $V_{\gamma_n}(\mathbb{A})$ is the intertwining operator associated to the Weyl element ω_n in this case.

For the given irreducible cuspidal representation π of $\mathrm{GL}_{2n}(\mathbb{A})$, the $\psi_{n,n}$ -Fourier coefficient

$$\mathcal{W}^{\psi_{n,n}}(h, \varphi_\pi) := \int_{V_{n,n}(k) \backslash V_{n,n}(\mathbb{A})} \varphi_\pi(vh) \psi_{n,n}(v) dv \quad (4.39)$$

generates an automorphic representation of the centralizer GL_n of the character $\psi_{n,n}$, which is denoted by $\mathcal{W}(\pi, \psi_{n,n})$.

It is easy to see that the function $\mathcal{F}(g; \phi, s)$ is invariant under the left translation by the unipotent radical N_0 of P_0 . If $m(h)$ denotes the element of the Levi subgroup of P_0 , then the function

$$m(h) \mapsto \mathcal{F}(m(h)g; \phi, s) \quad (4.40)$$

belongs to the space of automorphic representation

$$|\det(h)|^{s+n} \mathcal{W}(\pi, \psi_{n,n}) \quad (4.41)$$

of GL_n . In fact, assume $h \in \mathrm{GL}_n$, then

$$m(h) = \begin{pmatrix} h & & & \\ & \varepsilon_n {}^t h^{-1} \varepsilon_n & & \\ & & {}^t h^{-1} & \\ & & & \varepsilon_n h \varepsilon_n \end{pmatrix} \in P_0.$$

Here elements in GL_{2n} are identified as those in M . Note that the Jacobian of $m(h)$ on V_{γ_n} is $|\det h|^{-(n-1)}$. Hence (4.38) becomes

$$\begin{aligned} \mathcal{F}(m(h)g; \phi, s) &= |\det h|^{-(n-1)} \int_{V_{\gamma_n}(\mathbb{A})} \int_{V_{n,n}(k) \backslash V_{n,n}(\mathbb{A})} \\ &\quad \times F(v\omega_n m(h)v_2g; \phi, s) \psi_{n,n}(v) dv dv_2 \end{aligned}$$

By (2.25),

$$\omega_n m(h) \omega_n^{-1} = \begin{pmatrix} h & & & \\ & \varepsilon_n h \varepsilon_n & & \\ & & {}^t h^{-1} & \\ & & & \varepsilon_n {}^t h^{-1} \varepsilon_n \end{pmatrix},$$

which is a centralizer of $\psi_{n,n}$. Since its Jacobian on $V_{n,n}$ is 1, if we define $\phi'(x) = \phi(m(h)x)$, then by (4.4)

$$\begin{aligned} \mathcal{F}(m(h)g; \phi, s) &= |\det h|^{-(n-1)} H_P(m(h))(s + \rho_P) \mathcal{F}(g; \phi', s) \\ &= |\det h|^{-(n-1)} |\det h^2|^{\frac{s}{2}} |\det h^2|^{\frac{2n-1}{2}} \mathcal{F}(g; \phi', s) \\ &= |\det h|^{s+n} \mathcal{F}(g; \phi', s). \end{aligned}$$

It follows that the function in (4.33) is an Eisenstein series of $\mathrm{Sp}_{2n}(\mathbb{A})$ of form

$$\sum_{\epsilon \in P_0(k) \backslash \mathrm{Sp}_{2n}(k)} \mathcal{F}(\epsilon h; \phi, s). \quad (4.42)$$

where $\mathcal{F}(g; \phi, s)$ is defined as in (4.38).

4.3 Period identity

We continue here the explicit calculation of the period $\mathcal{P}_{H,\phi}(E_{s_0}(\cdot; \phi))$ of the residue $E_{s_0}(\cdot; \phi)$ defined in (4.12). By (4.13), (4.33) and (4.42), we have

$$\mathcal{P}_{H,\phi}(E_{s_0}(\cdot; \phi)) = \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \mathrm{Res}_{s=s_0} \sum_{\epsilon \in P_0(k) \backslash \mathrm{Sp}_{2n}(k)} \mathcal{F}(\epsilon h; \phi, s) dh. \quad (4.43)$$

It is clear that integral (4.43) may not be convergent, so we use the Arthur truncation method to regularize the integral. By §I.2.13 of [25], we apply the truncation method to the automorphic function

$$I_{n,s}(h) := \sum_{\epsilon \in P_0(k) \backslash \mathrm{Sp}_{2n}(k)} \mathcal{F}(\epsilon h; \phi, s)$$

and its residue at $s = s_0$

$$I_{n,s_0}(h) := \mathrm{Res}_{s=s_0} \sum_{\epsilon \in P_0(k) \backslash \mathrm{Sp}_{2n}(k)} \mathcal{F}(\epsilon h; \phi, s)$$

Recall from §I.2.13 of [25], for any locally L^1 -function ϕ on the quotient $\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})$, the truncated function, which is also a locally L^1 -function, is given by

$$\begin{aligned} \Lambda^T \phi(g) &:= \sum_{P \subset \mathrm{Sp}_{2n}} (-1)^{r(G)-r(M)} \\ &\times \sum_{\gamma \in P(k) \backslash \mathrm{Sp}_{2n}(k)} \phi_P(\gamma g) \hat{\tau}_P(\log_M(m_P(\gamma g)) - T_M) \end{aligned} \quad (4.44)$$

where $B \subset P = MN$ are standard parabolic subgroups of G , which may be equal to G , and ϕ_P is the constant term of ϕ along N , and the other notations are the same as in [25].

Applying (4.44) to the automorphic function $I_{n,s}(h)$, we obtain a rapidly decreasing function $\Lambda^T I_n(h, s)$ on a fundamental domain for $\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})$. Hence we have

$$\mathrm{Res}_{s=s_0} \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \Lambda^T I_{n,s}(h) dh = \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \Lambda^T I_{n,s_0}(h) dh. \quad (4.45)$$

As in Page 353, [2], we write

$$\Phi_s(h) := \Lambda^T I_{n,s}(h) - I_{n,s}(h). \quad (4.46)$$

Then we have

$$\begin{aligned} \mathcal{P}_{\mathrm{Sp}_{2n}}(I_{n,s}) &= \mathrm{Res}_{s=s_0} \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \Lambda^T I_{n,s}(h) dh \\ &\quad - \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \Phi_{s_0}(h) dh. \end{aligned} \quad (4.47)$$

As in (6,14) of [2], we write

$$\begin{aligned} \Lambda^T I_{n,s}(h) &= I_{n,s}(h) \\ &\quad - \sum_{\gamma \in P_0(k) \backslash \mathrm{Sp}_{2n}(k)} \mathcal{F}(\gamma h; \phi, s) \hat{\tau}_{P_0}(\log_{M_{P_0}}(m_{P_0}(\gamma h)) - T_{M_{P_0}}) \\ &\quad + \sum_{\gamma \in P_0(k) \backslash \mathrm{Sp}_{2n}(k)} \mathcal{F}(\gamma h; \phi, s) \hat{\tau}_{P_0}(\log_{M_{P_0}}(m_{P_0}(\gamma h)) - T_{M_{P_0}}) \\ &\quad + \Phi_s(h), \end{aligned} \quad (4.48)$$

It follows that the period

$$\begin{aligned} &\int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \Lambda^T I_{n,s}(h) dh \\ &= \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\gamma \in P_0(k) \backslash \mathrm{Sp}_{2n}(k)} \\ &\quad \times \mathcal{F}(\gamma h; \phi, s) (1 - \hat{\tau}_{P_0}(\log_{M_{P_0}}(m_{P_0}(\gamma h)) - T_{M_{P_0}})) dh \\ &\quad + \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\gamma \in P_0(k) \backslash \mathrm{Sp}_{2n}(k)} \\ &\quad \times \mathcal{F}(\gamma h; \phi, s) \hat{\tau}_{P_0}(\log_{M_{P_0}}(m_{P_0}(\gamma h)) - T_{M_{P_0}}) dh \\ &\quad + \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \Phi_s(h) dh. \end{aligned} \quad (4.49)$$

Following the same argument as in [13], [17], [1], and [2], we expect that the residue at $s = s_0$ of the following as a (meromorphic function in s)

$$\int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\gamma \in P_0 \backslash \mathrm{Sp}_{2n}} \times \mathcal{F}(\gamma h; \phi, s) \hat{\tau}_{P_0}(\log_{M_{P_0}}(m_{P_0}(\gamma h)) - T_{M_{P_0}}) + \Phi_s(h) dh$$

is equal to

$$\int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \Phi_{s_0}(h) dh.$$

The justification of this expectation involves some detailed computations of relevant intertwining operators and will be omitted here. The main idea of such explicit calculation is the same as the one we are going to deal with, which is the second integral in (4.49). It follows that

$$\begin{aligned} & \mathcal{P}_{\mathrm{Sp}_{2n}}(I_{n, s_0}) \\ &= \mathrm{Res}_{s=s_0} \int_{\mathrm{Sp}_{2n}(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\gamma \in P_0(k) \backslash \mathrm{Sp}_{2n}(k)} \mathcal{F}(\gamma h; \phi, s) (1 - \tau_c(H(\gamma h))) dh, \end{aligned} \tag{4.50}$$

where $\tau_c(H(g)) = \hat{\tau}_{P_0}(\log_{M_{P_0}}(m_{P_0}(\gamma h)) - c_{M_{P_0}})$ as defined in §4.1, [17]. Since the function $\mathcal{F}(\gamma h; \phi, s)(1 - \tau_c(H(\gamma h)))$ is truncated, the summation over $\gamma \in P_0 \backslash \mathrm{Sp}_{2n}$ is finite, which is absolutely convergent in particular. Hence the integral in (4.50) is equal to

$$\int_{P_0(k) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \mathcal{F}(h; \phi, s) (1 - \tau_c(H(h))) dh \tag{4.51}$$

By the Iwasawa decomposition $\mathrm{Sp}_{2n}(\mathbb{A}) = P_0(\mathbb{A})K_{\mathrm{Sp}_{2n}}$, where $K_{\mathrm{Sp}_{2n}}$ is the standard maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{A})$, the Haar measure on $\mathrm{Sp}_n(\mathbb{A})$ at $g = nm\kappa$ can be chosen as

$$|\det m|^{-(n+1)} dn dm d\kappa.$$

By the Langlands decomposition

$$\mathrm{GL}_n(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A})^1 \cdot A^+,$$

we deduce that the integral in (4.51) is equal to

$$\int_{K_{\mathrm{Sp}_{2n}} \times \mathrm{GL}_n(k) \backslash \mathrm{GL}_n(\mathbb{A})} \mathcal{F}(m\kappa; \phi, s) (1 - \tau_c(H(m))) |\det m|^{-(n+1)} dm d\kappa. \tag{4.52}$$

It is easy to calculate the following

$$\begin{aligned} |\det m|^{-(n+1)} &= |a|_{\mathbb{R}}^{-nd(n+1)} \\ \mathcal{F}(m\kappa; \phi, s) &= |a|_{\mathbb{R}}^{nd(s+n)} \mathcal{F}(m^1\kappa; \phi, s) \\ (1 - \tau_c(H(mw_2v_1))) &= (1 - \tau_c(H(t_a))) \end{aligned}$$

where d is the number of the real archimedean places of the number field k . The second equation comes from (4.41) by the assumption that ϕ is A^+ -invariant. We deduce that the integral in (4.52) is equal to

$$\lambda(s) \cdot \int_{K_{\mathrm{Sp}_{2n}} \times \mathcal{Z}_n(\mathbb{A}) \mathrm{GL}_n(k) \backslash \mathrm{GL}_n(\mathbb{A})} \mathcal{F}(m\kappa; \phi, s) dm d\kappa. \quad (4.53)$$

where \mathcal{Z}_n denotes the center of GL_n and the function $\lambda(s)$ is given by

$$\lambda(s) := \mathrm{vol}(\mathbb{A}^1/k) \cdot \int_{\mathbb{R}^+} |a|^{nd(s-1)} (1 - \tau_c(H(t_a))) da^\times.$$

It is easy to check that

$$\int_{\mathbb{R}^+} |a|^{nd(s-1)} (1 - \tau_c(H(t_a))) da^\times = \frac{e^{nd(s-1)}}{nd(s-1)}.$$

Since the integral

$$\int_{K_{\mathrm{Sp}_{2n}} \times \mathcal{Z}_n(\mathbb{A}) \mathrm{GL}_n(k) \backslash \mathrm{GL}_n(\mathbb{A})} \mathcal{F}(m\kappa; \phi, s) dm d\kappa$$

represents a holomorphic in s , the possible of (4.53) can only come from the function $\lambda(s)$, which has the only simple pole at $s = 1$. Hence from (4.50) and (4.53) we obtain that s_0 must be 1 and

$$\mathcal{P}_{\mathrm{Sp}_{2n}}(I_{n,1}) = \frac{\mathrm{vol}(\mathbb{A}^1/k)}{nd} \int_{K_{\mathrm{Sp}_{2n}} \times \mathcal{Z}_n(\mathbb{A}) \mathrm{GL}_n(k) \backslash \mathrm{GL}_n(\mathbb{A})} \mathcal{F}(m\kappa; \phi, s) dm d\kappa. \quad (4.54)$$

By (4.38), it is easy to see that the integration in variable m in (4.54) yields the following integral

$$\begin{aligned} &\int_{V_{\gamma_n}(\mathbb{A})} \int_{\mathcal{Z}_n(\mathbb{A}) \mathrm{GL}_n(k) \backslash \mathrm{GL}_n(\mathbb{A})} \int_{V_{n,n}(k) \backslash V_{n,n}(\mathbb{A})} \\ &\quad \times F(vmw_nv_{\gamma_n}\kappa; \phi, s) \psi_{V_{n,n}}(v) dv dm dv_{\gamma_n}. \end{aligned} \quad (4.55)$$

It follows that the integration over $\mathcal{S}_n = V_{n,n} \rtimes \mathrm{GL}_n$ is the Shalika period for the cuspidal datum (GL_{2n}, π) . We set

$$\mathcal{P}_{\mathcal{S}_n, \psi}(F(\cdot; \phi))(g) := \int_{\mathcal{S}_n(k) \backslash \mathcal{S}_n(\mathbb{A})} F(vmg; \phi, s) \psi_{n,n}(v) dv dm. \quad (4.56)$$

Hence we obtain the main identity relating the ‘inner period’, the Shalika period for the cuspidal datum on GL_{2n} to the ‘outer’ period, the generalized Shalika period for the residue of Eisenstein series on SO_{4n} associated to the given cuspidal datum.

Theorem 4.4. *The period $\mathcal{P}_{H,\psi}(E_{s_0}(\cdot, \phi))$ is zero unless $s_0 = 1$. If $s_0 = 1$, we have*

$$\begin{aligned} & \mathcal{P}_{H,\psi}(E_1(\cdot, \phi)) \\ &= \frac{\mathrm{vol}(\mathbb{A}^1/k)}{nd} \int_{K_{\mathrm{Sp}_{2n}} \times V_{\gamma_n}(\mathbb{A})} \mathcal{P}_{\mathcal{S}_n,\psi}(F(\cdot; \phi))(w_n v_{\gamma_n} \kappa) dv_{\gamma_n} d\kappa. \end{aligned}$$

By Theorem 4.4, if the generalized Shalika period $\mathcal{P}_{H,\psi}(E_1(\cdot, \phi))$ is nonzero, then the Shalika period $\mathcal{P}_{\mathcal{S}_n,\psi}(F(\cdot; \phi))$ is nonzero. Conversely, if the Shalika period $\mathcal{P}_{\mathcal{S}_n,\psi}(F(\cdot; \phi))$ is nonzero, by the argument in Page 179-180, [13], in Theorems 5.2, 5.5, [17], and in Theorem 3.2, [1], one can prove that the generalized Shalika period $\mathcal{P}_{H,\psi}(E_1(\cdot, \phi))$ is nonzero. We state this as a proposition below without giving any further details here.

Proposition 4.5. *With the notation as above, the non-vanishing of the Shalika period $\mathcal{P}_{\mathcal{S}_n,\psi}(F(\cdot; \phi))$ is equivalent to the nonvanishing of the generalized Shalika period $\mathcal{P}_{H,\psi}(E_1(\cdot, \phi))$.*

By Propositions 4.1 and 4.5, we obtain a sufficient condition (in terms of period) for the existence of the pole at $s = 1$ of the (complete) exterior square L-function $L(s, \pi, \Lambda^2)$, which is a theorem of Jacquet and Shalika in [16] proved by a different method, i.e. the Rankin-Selberg integral representation method. We record it here for completeness.

Corollary 4.6. *Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$. If the Shalika period*

$$\mathcal{P}_{\mathcal{S}_n,\psi}(\varphi_\pi) = \int_{S_n(k) \backslash S_n(\mathbb{A})} \varphi_\pi(vh) \psi_{n,n}(v) dv dh$$

does not vanish for some $\varphi_\pi \in \pi$, then

1. *the Eisenstein series $E(g; \phi_\pi, s)$ has a pole at $s = 1$, and*
2. *the exterior square L-function $L(s, \pi, \Lambda^2)$ has a pole at $s = 1$.*

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