L-FUNCTIONS FOR SYMPLECTIC GROUPS USING FOURIER-JACOBI MODELS

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This paper is dedicated to Stephen Kudla on the occasion of his 60th birthday.

Abstract. In this note, we introduce a global integral of Rankin-Selberg and Shimura type, which represents the tensor L-function of a pair of irreducible, cuspidal, automorphic representations of $\mathrm{Sp}_{2k}(\mathbb{A})$ and $\mathrm{GL}_m(\mathbb{A})$, respectively. This construction is completely general and applies to any cuspidal representation of the symplectic group. We use this construction to characterize some CAP representations of symplectic groups and give some information on poles of L-functions.

1. Introduction

In this announcement, we introduce and study a family of new global integrals of Rankin-Selberg and Shimura type, which represent the tensor L-function of a pair of irreducible, cuspidal, automorphic representations π and τ of the symplectic group $\mathrm{Sp}_{2k}(\mathbb{A})$ and $\mathrm{GL}_m(\mathbb{A})$, respectively. Here $\mathbb{A}$ is the ring of Adeles of a number field F.

Following Langlands, one may define (for $\mathrm{Re}(s)$ large) the partial tensor L-function $L^S(\pi \times \tau, s)$, with S being a finite set of places of F, containing those at infinity, and outside of which, all local components of π and τ are unramified. In order to study further properties of the partial tensor L-function $L^S(\pi \times \tau, s)$, we introduce a global integral, which represents $L^S(\pi \times \tau, s)$ in the sense that (for decomposable data, unramified outside S) the global integral is an Eulerian product over all local places of F (Theorems 3.2 and 3.3) and the partial Eulerian product over the local places outside S yields $L^S(\pi\times\tau, s)$ (Theorem 4.3). In case π is generic (i.e. π has a nontrivial Whittaker-Fourier coefficient), such a global construction was introduced in [GRS98].

The Langlands conjecture on the functorial transfer from Sp_{2k} to GL_{2k+1} asserts that for any irreducible, cuspidal, automorphic representation π of $\mathrm{Sp}_{2k}(\mathbb{A})$, there exists an irreducible, automorphic representation Π of $\mathrm{GL}_{2k+1}(\mathbb{A})$, such that the correspondence between the local components π_ν and Π_ν is compatible with the local Langlands functorial transfer at every local place ν of F. This has been proved when π is generic ([CKPSS04]). When π may not be generic, an approach has been discussed in [S06], based on the theory of tensor L-functions discussed in this paper and the converse theorem of Cogdell and Piatetski-Shapiro ([CPS]).

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On the other hand, it is expected that the Arthur trace formula method will establish the Langlands conjecture on the functorial transfer from Sp_{2k} to GL_{2k+1} in general. It is also important to note that the Langlands functorial transfer from π to Π obtained from the trace formula method is compatible with the local Langlands functorial transfer only at unramified local places ν of F (i.e. where both π_ν and Π_ν are unramified). Once we have a weak version of the global Langlands functorial transfer, the analytic properties of the partial tensor L-functions $L^S(\pi \times \tau, s)$ will be deduced from the similar properties of Rankin-Selberg convolution L-functions for general linear groups.

It is clear that in order to establish the local Langlands functorial transfer at ramified local places and hence the complete version of global Langlands functorial transfer from Sp_{2k} to GL_{2k+1}, one has to define and study the local L-factors, ϵ-factors and γ-factors at ramified local places. It is well known that the Rankin-Selberg method is one of the major methods to accomplish such a goal.

In order to illustrate the potential of such an approach to the global theory of automorphic representations and to refined analytic properties of automorphic L-functions, we provide two applications (Theorems 5.1, 5.2 below). The first application (Theorem 5.1) is a characterization of certain cuspidal automorphic representations of $\mathrm{Sp}_{2(2m+k)}(\mathbb{A})$, whose unramified local components are equivalent to the unramified local components of a global parabolically induced representation of $\mathrm{Sp}_{2(2m+k)}(\mathbb{A})$, with generic cuspidal data of the form

$$(\mathrm{GL}_{2m} \times \mathrm{Sp}_{2k}, \tau \cdot |\det|^{\frac{1}{2}} \otimes \pi)$$

where $\mathrm{GL}_{2m} \times \mathrm{Sp}_{2k}$ is the Levi part, which determines the standard maximal parabolic subgroup of $\mathrm{Sp}_{2(2m+k)}$, and $\tau \otimes \pi$ is an irreducible generic cuspidal automorphic representation of $\mathrm{GL}_{2m}(\mathbb{A}) \times \mathrm{Sp}_{2k}(\mathbb{A})$. Such representations are called CAP representations of $\mathrm{Sp}_{2(2m+k)}(\mathbb{A})$, with respect to the data above. The second application (Theorem 5.2) is about location of poles of $L^S_\psi(\widetilde{\pi} \times \tau, s)$, where now $\widetilde{\pi}$ is on a metaplectic cover $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$ of $\mathrm{Sp}_{2k}(\mathbb{A})$.

In Section 2, we recall the definition of Fourier-Jacobi coefficients of automorphic forms on $\mathrm{Sp}_{2k}(\mathbb{A})$ and on $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$. In Section 3, we introduce two families of global integrals of Fourier-Jacobi type and prove Theorems 3.2 and 3.3, which show that these global integrals are factorized into Eulerian product of local integrals, based the local uniqueness of Fourier-Jacobi models (Theorem A in [Su] and Theorem 4.1 in §4). In Section 4, we discuss the local theory of these global integrals briefly in general. In particular, we sketch the proof of Theorem 4.1 which reduces the local uniqueness of Fourier-Jacobi models in general to the spherical case of the Fourier-Jacobi model, which is Theorem A of B.-Y. Sun ([Su]). We also outline the proof of Theorem 4.3 on the unramified calculation of the local integrals. In the last section (Section 5), we provide two applications of the construction of the global integrals of Fourier-Jacobi type, which are Theorems 5.1 and 5.2.

Finally, we would like to thank Binyong Sun for sending us his recent beautiful work ([Su]) during the preparation of this paper. We would also like to thank the referee for valuable remarks.

2. Fourier-Jacobi Coefficients

In this note, the symplectic group Sp_{2k} is written with respect to the antisymmetric matrix

$$\begin{pmatrix} & w_k \\ -w_k & \end{pmatrix},$$

where w_k is the $k \times k$ matrix, which has 1 on the anti-diagonal, and zero elsewhere. Our global integrals involve Fourier-Jacobi coefficients of automorphic forms on symplectic groups or metaplectic groups. We recall this notion now.

For $1 \le r \le k$, let P_r^k denote the standard maximal parabolic subgroup of Sp_{2k} whose Levi part is $\mathrm{GL}_1^r \times \mathrm{Sp}_{2(k-r)}$. Let V_r^k denote its unipotent radical. Since P_r^k is standard, V_r^k consists of upper unipotent matrices. Let ψ denote a nontrivial additive character of $F\backslash\mathbb{A}$. We define a character $\psi_{V_r^k}$ of $V_r^k(\mathbb{A})$ as follows. For $v = (v_{i,j}) \in V_r^k(\mathbb{A})$, when $r < k$, we define

$$\psi_{V_r^k}(v) := \psi(v_{1,2} + v_{2,3} + \cdots + v_{r-1,r}); \tag{1}$$

and when $r = k$, we define

$$\psi_{V_k^k,a}(v) := \psi(v_{1,2} + v_{2,3} + \cdots + v_{k-1,k} + av_{k,k+1}) \tag{2}$$

with $a \in F^*$. It is clear that both $\psi_{V_r^k}(v)$ and $\psi_{V_k^k,a}(v)$ are trivial on $V_r^k(F)$ and $V_k^k(F)$, respectively.

Let $\widetilde{\mathrm{Sp}}_{2l}(\mathbb{A})$ denote the double cover of $\mathrm{Sp}_{2l}(\mathbb{A})$. For a given nontrivial character ψ on $F\backslash\mathbb{A}$, there exists the Weil representation $\omega_{\psi,l}$ of the semi-direct product $\widetilde{\mathrm{Sp}}_{2l}(\mathbb{A}) \cdot \mathcal{H}_{2l+1}(\mathbb{A})$, where $\mathcal{H}_{2l+1}$ is the Heisenberg group in $2l+1$ variables. The Weil representation $\omega_{\psi,l}$ acts on the space $\mathcal{S}(\mathbb{A}^l)$ of Schwartz-Bruhat functions. For $\phi \in \mathcal{S}(\mathbb{A}^l)$, one defines the corresponding theta series

$$\widetilde{\theta}_{\psi,l}^{\phi}(xg) = \sum_{\xi \in F^l} \omega_{\psi,l}(xg)\phi(\xi),$$

where $x \in \mathcal{H}_{2l+1}(\mathbb{A}),\ g \in \widetilde{\mathrm{Sp}}_{2l}(\mathbb{A})$. Following [GRS98] and [GRS03], for instance, we write the elements of the Heisenberg group $\mathcal{H}_{2l+1}$ as $(v;t)$, where v is a row vector of $2l$ coordinates. When $v = 0$, $(0;t)$ is a general element of the center of $\mathcal{H}_{2l+1}$. We write the group law in $\mathcal{H}_{2l+1}$ as

$$(v_1;t_1)(v_2;t_2) = (v_1 + v_2; t_1 + t_2 + (v_1, v_2)),$$

where $(\ ,\ )$ denotes the symplectic form. (Note that we do not use $\frac{1}{2}(\ ,\ )$ in the group law. It is more convenient, especially when we need to embed the Heisenberg group in $\mathrm{Sp}_{2(l+2)}$.) For a given $a \in F^*$, let ψ^a denote the character of $F\backslash\mathbb{A}$ given by $t \mapsto \psi(at)$.

When $r < k$, we define a projection map

$$l_{k-r}\ :\ V_r^k \mapsto \mathcal{H}_{2(k-r)+1}$$

by $l_{k-r}(v) = (v_{r,r+1}, v_{r,r+2}, \cdots, v_{r,2k-r}; v_{r,2k-r+1})$. The r-th Fourier-Jacobi coefficient attached to an automorphic representation (V_π, π) of $\mathrm{Sp}_{2k}(\mathbb{A})$, with respect to (ψ, a) is defined by the following integral

$$\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^k},a}(\varphi_\pi)(h) := \int_{V_r^k(F)\backslash V_r^k(\mathbb{A})} \varphi_\pi(vh)\widetilde{\theta}^{\phi}_{\psi^a,k-r}(l_{k-r}(v)h)\psi_{V_r^k}(v)dv, \tag{3}$$

where $\varphi_\pi \in V_\pi$ and $\phi \in \mathcal{S}(\mathbb{A}^{k-r})$. It is clear that the r-th Fourier-Jacobi coefficient $\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^k},a}(\varphi_\pi)(h)$ is an automorphic form on $\widetilde{\mathrm{Sp}}_{2(k-r)}(\mathbb{A})$. When $r=0$, the group V_0^k is just the identity group. In this case, we define the Fourier-Jacobi coefficient $\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_0^k}}(\varphi_\pi)$ to be the product $\varphi_\pi \cdot \widetilde{\theta}^{\phi}_{\psi^a,k}$, as a function on $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$. Note here and in (3) that the automorphic form φ_π on $\mathrm{Sp}_{2k}(\mathbb{A})$ is extended as an automorphic form on $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$ trivially.

When $r=k$, we define the Fourier-Jacobi coefficient attached to π to be the Whittaker coefficient, which is given by

$$\mathcal{W}_{\psi_{V_k^k,a}}(\varphi_\pi)(h) := \int_{V_k^k(F)\backslash V_k^k(\mathbb{A})} \varphi_\pi(vh)\psi_{V_k^k,a}(v)dv.$$

It follows from [GRS03], Lemma 2.2, that every (nontrivial) automorphic representation π of $\mathrm{Sp}_{2k}(\mathbb{A})$ has a nonzero r-th Fourier-Jacobi coefficient, for some r and a. We fix ψ and let r_0 be the largest number such that π has a nonzero r_0-th Fourier-Jacobi coefficient, with respect to ψ and some a. We refer to this coefficient as a top Fourier-Jacobi coefficient attached to π. We also have a similar notion when both ψ and a are fixed.

Assume that the automorphic representation π of $\mathrm{Sp}_{2k}(\mathbb{A})$ is cuspidal and has a nonzero r-th Fourier-Jacobi coefficient, for some $0 \le r < k$ and $a \in F^*$. Then $\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^k},a}(\varphi_\pi)(h)$ defined in (3), when viewed as a function in the variable h, defines a genuine, automorphic function on $\widetilde{\mathrm{Sp}}_{2(k-r)}(\mathbb{A})$. If, in addition, this coefficient is a top Fourier-Jacobi coefficient, when (ψ, a) are fixed, then the space generated by the automorphic functions $\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^k},a}(\varphi_\pi)(h)$, is a genuine cuspidal automorphic representation of $\widetilde{\mathrm{Sp}}_{2(k-r)}(\mathbb{A})$. See [GRS03], Lemma 2.3 and [GRS99a], Theorem 8. In this case, let $(V_{\widetilde{\sigma}}, \widetilde{\sigma})$ denote an irreducible summand of the complex conjugate of this cuspidal representation. Then the following pairing is not identically zero.

$$\mathcal{FJP}_{\psi_{V_r^k},a}(\varphi_\pi, \phi, \varphi_{\widetilde{\sigma}}) := \int_{\mathrm{Sp}_{2(k-r)}(F)\backslash \mathrm{Sp}_{2(k-r)}(\mathbb{A})} \widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^k},a}(\varphi_\pi)(h)\varphi_{\widetilde{\sigma}}(h)dh.$$

Here, $\varphi_{\widetilde{\sigma}}$ is a cusp form in the space $V_{\widetilde{\sigma}}$. It is easy to check that the Fourier-Jacobi coefficients and Fourier-Jacobi periods above, can be obtained from the similar integrals, with $a=1$, by outer twisting by the similitude element $\mathrm{diag}(I_{k-r}, aI_{k-r})$. From now on, we assume, for simplicity, that $a=1$, and we will denote

$$\mathcal{FJP}_{\psi_{V_r^k},1} = \mathcal{FJP}_{\psi_{V_r^k}},$$

and

$$\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^k},1} = \widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^k}}.$$

Thus, the following pairing

$$\mathcal{FJP}_{\psi_{V_r^k}}(\varphi_\pi, \phi, \varphi_{\widetilde{\sigma}}) := \int_{\mathrm{Sp}_{2(k-r)}(F)\backslash \mathrm{Sp}_{2(k-r)}(\mathbb{A})} \widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_r^k}}(\varphi_\pi)(h)\varphi_{\widetilde{\sigma}}(h)dh \qquad (4)$$

is not identically zero. We call $\mathcal{FJP}_{\psi_{V_r^k}}(\varphi_\pi, \phi, \varphi_{\widetilde{\sigma}})$ the r-th Fourier-Jacobi period for the pair $(\pi, \widetilde{\sigma})$ with $0 \le r < k$.

It follows from the above discussion that for a given irreducible, cuspidal, automorphic representation π of $\mathrm{Sp}_{2k}(\mathbb{A})$, it is either generic (a nonzero Whittaker-Fourier coefficient, i.e. $r = k$), or has a nonzero r-th (top) Fourier-Jacobi period $\mathcal{FJP}_{\psi_{V_r^k}}(\varphi_\pi, \phi, \varphi_{\widetilde{\sigma}})$ for some $r < k$ and some choice of data. Since the generic cases ($r = k$) were treated in [GRS98], we will concentrate on the case when $r < k$. We remark that, as in the case of orthogonal groups ([GPSR97]), the generic case and the nongeneric case can be treated uniformly. We leave the details to future work of ours.

It is clear that the Fourier-Jacobi period can also be defined, in exactly the same way, for a pair $(\widetilde{\pi}, \sigma)$, where $\widetilde{\pi}$ is an irreducible cuspidal automorphic representation of $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$ and σ is an irreducible cuspidal automorphic representation of $\mathrm{Sp}_{2(k-r)}(\mathbb{A})$. See [GJR04], for instance.

3. The global integrals of Fourier-Jacobi type

Let π denote an irreducible, cuspidal, automorphic representation of $\mathrm{Sp}_{2k}(\mathbb{A})$. We assume that that π has the top Fourier-Jacobi coefficient occurring at $1 \leq r < k$, with respect to $(\psi, 1)$. Let $\widetilde{\sigma}$ denote a genuine, irreducible, cuspidal, automorphic representation of $\widetilde{\mathrm{Sp}}_{2(k-r)}(\mathbb{A})$, such that the (top) Fourier-Jacobi period $\mathcal{FJP}_{\psi_{V_r^k}}(\varphi_\pi, \phi, \varphi_{\widetilde{\sigma}})$, as defined in (4), is nonzero for some choice of data. Let τ denote an irreducible, cuspidal, automorphic representation of $\mathrm{GL}_m(\mathbb{A})$. Denote by $\widetilde{E}_{\tau,\widetilde{\sigma}}(h, s)$ an Eisenstein series on $\widetilde{\mathrm{Sp}}_{2(m+k-r)}(\mathbb{A})$, associated to a holomorphic section $\widetilde{f}_{\tau,\widetilde{\sigma},s}$ in the induced representation

$$\mathrm{Ind}_{\widetilde{P}(\mathbb{A})}^{\widetilde{\mathrm{Sp}}_{2(m+k-r)}(\mathbb{A})}(\gamma_\psi \tau |\det|^{s-\frac{1}{2}} \otimes \widetilde{\sigma}).$$

Here, P is the standard parabolic subgroup of $\mathrm{Sp}_{2(m+k-r)}$, whose Levi part is isomorphic to $\mathrm{GL}_m \times \mathrm{Sp}_{2(k-r)}$. Denote its unipotent radical by $U_{m,k-r}$. The group $\widetilde{P}(\mathbb{A})$ is the inverse image of $P(\mathbb{A})$ inside $\widetilde{\mathrm{Sp}}_{2(m+k-r)}(\mathbb{A})$. The induction data depend also on ψ, through the Weil factor γ_ψ. For s, with large real part, the Eisenstein series is an absolutely convergent series

$$\widetilde{E}_{\tau,\widetilde{\sigma}}(h, s) = \sum_{\gamma \in P(F) \backslash \mathrm{Sp}_{2(m+k-r)}(F)} \widetilde{f}_{\tau,\widetilde{\sigma},s}(\gamma h).$$

Assume, first, that $r \geq m$. In this case, we have that $k \geq m + k - r$. We define the global integral by

$$\begin{aligned} \mathrm{I}(\varphi_\pi, \phi, \widetilde{f}_{\tau,\widetilde{\sigma},s}) &:= \mathcal{FJP}_{\psi_{V_{r-m}^k}}(\varphi_\pi, \phi, \widetilde{E}_{\tau,\widetilde{\sigma}}) \\ &= \int_{\mathrm{Sp}_{2(m+k-r)}(F) \backslash \mathrm{Sp}_{2(m+k-r)}(\mathbb{A})} \widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_{r-m}^k}}(\varphi_\pi)(h) \widetilde{E}_{\tau,\widetilde{\sigma}}(h, s) dh. \end{aligned} \tag{5}$$

Next, when $r \leq m$, we have that $k \leq m + k - r$. In this case, we define the global integral by

$$\begin{aligned} \mathrm{I}(\varphi_\pi, \phi, \widetilde{f}_{\tau,\widetilde{\sigma},s}) &:= \mathcal{FJP}_{\psi_{V_{m-r}^{m+k-r}}}(\widetilde{E}_{\tau,\widetilde{\sigma}}, \phi, \varphi_\pi) \\ &= \int_{\mathrm{Sp}_{2k}(F) \backslash \mathrm{Sp}_{2k}(\mathbb{A})} \widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_{m-r}^{m+k-r}}}(\widetilde{E}_{\tau,\widetilde{\sigma}})(h) \varphi_\pi(h) dh. \end{aligned} \tag{6}$$

The Eisenstein series $\widetilde{E}_{\tau,\widetilde{\sigma}}(h,s)$ converges absolutely, for $\mathrm{Re}(s) > k-r+1+\frac{m}{2}$, and has meromorphic continuation to the whole complex plane $\mathbb{C}$ ([MW95]).

Lemma 3.1. *Both integrals* (5), (6) *converge absolutely and uniformly in vertical strips in $\mathbb{C}$, away from poles of the Eisenstein series $\widetilde{E}_{\tau,\widetilde{\sigma}}(h,s)$.*

The proof of convergence of the integrals (5) is entirely similar to the proof of Lemma 2.1 in [B-AS09]. Consider the global integral (6). Let S be a Siegel domain for $\mathrm{Sp}_{2k}(F)\backslash\mathrm{Sp}_{2k}(\mathbb{A})$. Then since φ_π is rapidly decreasing on S, it is enough to note that $\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V^{m+k-r}_{m-r}}}(\widetilde{E}_{\tau,\widetilde{\sigma}})$ is of moderate growth on $Sp_{2k}(\mathbb{A})$. Indeed, this follows from the facts that the Eisenstein series $g \mapsto \widetilde{E}_{\tau,\widetilde{\sigma}}(g,s)$ is of moderate growth on $\widetilde{\mathrm{Sp}}_{2(m+k-r)}(\mathbb{A})$, the theta series $l_k(v)h \mapsto \widetilde{\theta}^{\phi}_{\psi,k}(l_k(v)h)$ is of moderate growth on $\mathcal{H}_{2k+1}(\mathbb{A})\,\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$, and $V^{m+k-r}_{m-r}(F)\backslash V^{m+k-r}_{m-r}(\mathbb{A})$ is compact.

Next, we discuss the unfolding process of these global integrals. We start with the global integral (5)

Theorem 3.2. *In the notation above, assume that $r \geq m$. For $\mathrm{Re}(s)$ large enough,*

(7)
$$\mathrm{I}(\varphi_\pi,\phi,\widetilde{f}_{\tau,\widetilde{\sigma},s}) = \int \int_{R(\mathbb{A})} \mathcal{FJP}_{\psi_{V^k_r}}(\pi(wxh)\varphi_\pi, [\omega_\psi(l_{m+k-r}(x)h)\phi]_{k-r}, \rho(h)\widetilde{f}_{W_\tau,\widetilde{\sigma},s})dxdh,$$

where the dh-integration is over $\mathrm{Sp}_{2(k-r)}(\mathbb{A})V^{m+k-r}_m(\mathbb{A})\backslash\mathrm{Sp}_{2(m+k-r)}(\mathbb{A})$. Here, w is a certain Weyl element and R is a certain F-unipotent group, given explicitly in (13) *and* (19)*, respectively. For a Schwartz-Bruhat function $\phi' \in S(\mathbb{A}^{m+k-r})$, ϕ'_{k-r} denotes its restriction to $\mathbb{A}^{k-r}$, embedded in $\mathbb{A}^{m+k-r}$ as $0\times\mathbb{A}^{k-r}$. Finally, $\rho(h)\widetilde{f}_{W_\tau,\widetilde{\sigma},s}$ denotes the cusp form in $\widetilde{\sigma}$ obtained by the application of the ψ-Whittaker coefficient to the following cusp form in the space of τ, $a \mapsto \widetilde{f}_{\tau,\widetilde{\sigma},s}(agh)$, for fixed h, viewed as a function of $g \in \widetilde{Sp}_{2(k-r)}(\mathbb{A})$; $a \in \mathrm{GL}_m(\mathbb{A})$, naturally embedded in the Adelic points of the Levi part of P.*

Proof. Consider the global integral (5), $\mathrm{I}(\varphi_\pi,\phi,\widetilde{f}_{\tau,\widetilde{\sigma},s})$. For $\mathrm{Re}(s)$ large, we unfold the Eisenstein series and obtain

$$\begin{aligned}
\mathrm{I}(\varphi_\pi,\phi,\widetilde{f}_{\tau,\widetilde{\sigma},s}) &= \int_{P(F)\backslash\mathrm{Sp}_{2(m+k-r)}(\mathbb{A})} \widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V^k_{r-m}}}(\varphi_\pi)(h)\widetilde{f}_{\tau,\widetilde{\sigma},s}(h)dh \\
&= \int_{P(F)\backslash\mathrm{Sp}_{2(m+k-r)}(\mathbb{A})} \widetilde{f}_{\tau,\widetilde{\sigma},s}(h) \\
&\quad \cdot \int_{[V^k_{r-m}]} \varphi_\pi(vh)\widetilde{\theta}^{\phi}_{\psi,m+k-r}(l_{m+k-r}(v)h)\psi_{V^k_{r-m}}(v)dvdh, \qquad (8)
\end{aligned}$$

where we set $[V^k_{r-m}] := V^k_{r-m}(F)\backslash V^k_{r-m}(\mathbb{A})$. We will use this notation for other F-algebraic groups, as well. We may repeat the proof of the last lemma, in the domain of convergence of the series defining $\widetilde{E}_{\tau,\widetilde{\sigma}}(h,s)$, replacing it by its series, $\widetilde{f}_{\tau,\widetilde{\sigma},s}$ by its absolute value, and similarly, for the cusp form and the theta series. Thus, this step is justified. Write the elements of $\mathcal{H}_{2(m+k-r)+1}$ in the form $(v_1,v_2,v_3;z)$, where v_1, v_3 are row vectors of dimension m, v_2 is a row vector of dimension $2(k-r)$ and

z is the center of the Heisenberg group $\mathcal{H}_{2(m+k-r)+1}$. Decompose

$$\mathcal{H}_{2(m+k-r)+1} = H_2 \rtimes H_1,$$

where $H_1 = \{(v_1, 0, 0; 0) \in \mathcal{H}_{2(m+k-r)+1}\}$, $H_2 = \{(0, v_2, v_3; z) \in \mathcal{H}_{2(m+k-r)+1}\}$. Then, unfolding the theta series, we get, for $v \in V_{r-m}^k(\mathbb{A})$, (for simplicity, we set $\omega_{m+k-r,\psi} = \omega_\psi$)

$$\begin{aligned}
\widetilde{\theta}^{\phi}_{\psi,m+k-r}(l_{m+k-r}(v)g) &= \sum_{\xi_1\in F^m, \xi_2\in F^{k-r}} \omega_\psi(l_{m+k-r}(v)g)\phi(\xi_1,\xi_2)\\
&= \sum_{\xi_1\in F^m, \xi_2\in F^{k-r}} \omega_\psi((\xi_1,0,0;0)l_{m+k-r}(v)g)\phi(0,\xi_2)\\
&= \sum_{\gamma\in H_1(F)} \sum_{\xi_2\in F^{k-r}} \omega_\psi(\gamma l_{m+k-r}(v)g)\phi(0,\xi_2)\\
&= \sum_{\gamma\in H_2(F)\backslash\mathcal{H}(F)} \sum_{\xi_2\in F^{k-r}} \omega_\psi(\gamma l_{m+k-r}(v)g)\phi(0,\xi_2), \qquad (9)
\end{aligned}$$

where we set $\mathcal{H} := \mathcal{H}_{2(m+k-r)+1}$ for this formula. Let ι denote the standard F-embedding of $\mathcal{H}_{2(m+k-r)+1}$ inside V_{r-m}^k. Thus, $V_{r-m}^k = V_{r-m-1}^k \rtimes \iota(\mathcal{H}_{2(m+k-r)+1})$. As in [GJR04], we substitute (9) into the above integral. We write the dv-integration as an integration along $[V_{r-m-1}^k]$, followed by integration along $[\iota(\mathcal{H}_{2(m+k-r)+1})]$. The proof of Lemma 3.1 remains valid also when we replace, in the theta series, written as a convergent sum, $\omega_\psi(l_{m+k-r}(v)g)(\phi)$ by $|\omega_\psi(l_{m+k-r}(v)g)(\phi)|$. Thus, we may switch the order of integration along $[V_{r-m-1}^k]$ with the outer summation in (9). We obtain, (writing $\mathcal{H} = \mathcal{H}_{2(m+k-r)+1}$, for short again)

$$\int \widetilde{f}_{\tau,\widetilde{\sigma},s}(h) \int_{H_2(F)\backslash\mathcal{H}(\mathbb{A})} \int_{[V_{r-m-1}^k]} \varphi_\pi(v\iota(b)h) \sum_{\xi_2\in F^{k-r}} \omega_\psi(bh)\phi(0,\xi_2)\psi_{V_{r-m}^k}(v)dvdbdh$$

where the dh-integration is along $P(F)\backslash \mathrm{Sp}_{2(m+k-r)}(\mathbb{A})$. This integral is equal to

$$\begin{aligned}
\int \int_{H_1(\mathbb{A})} \widetilde{f}_{\tau,\widetilde{\sigma},s}(h) \int_{[H_2]} \int_{[V_{r-m-1}^k]} &\varphi_\pi(v\iota(b_2)\iota(b_1)h)\\
&\cdot \sum_{\xi_2\in F^{k-r}} \omega_\psi(b_2b_1h)\phi(0,\xi_2)\psi_{V_{r-m}^k}(v)dvdb_2db_1dh,
\end{aligned}$$

where the dh-integration is as before. Next, we write the integration domain $P(F)\backslash \mathrm{Sp}_{2(m+k-r)}(\mathbb{A})$ as

$$P(F)U(\mathbb{A})\backslash \mathrm{Sp}_{2(m+k-r)}(\mathbb{A}) \times U(F)\backslash U(\mathbb{A})$$

where we denote, for short, $U = U_{m,k-r}$, which is the unipotent radical of P. Let us denote the Levi part of P by M, which is isomorphic to $\mathrm{GL}_m \times \mathrm{Sp}_{2(k-r)}$. For an element of $U(\mathbb{A})$

$$u = \begin{pmatrix} I_m & a & c\\ & I_{2(k-r)} & a'\\ & & I_m \end{pmatrix},$$

an element of $H_1(\mathbb{A})$, $b_1 = (y_1, 0, 0; 0)$, and an element of $H_2(\mathbb{A})$, $b_2 = (0, y_2, y_3; z)$, we have

$$\iota(b_2b_1)u = u\iota(0, y_2+y_1a, y_3+y_2a'+y_1c; z+(y_2, y_1a)+2(y_3+y_2a', y_1)+(y_1, y_1c))\iota(b_1).$$

Recall that (,) denotes the symplectic form. Now, we can perform a series of changes of variables in the db_2-integration by taking

$$\begin{aligned} y_2 &\mapsto y_2 - y_1 a, \\ y_3 &\mapsto y_3 - y_2 a' - y_1 c, \\ z &\mapsto z - (y_2, y_1 a) - 2(y_3 + y_2 a', y_1) - (y_1, y_1 c). \end{aligned}$$

We therefore obtain the following integral

$$\int_{M(F)U(\mathbb{A})\backslash \mathrm{Sp}_{2(m+k-r)}(\mathbb{A})} \int_{H_1(\mathbb{A})} \widetilde{f}_{\tau,\widetilde{\sigma},s}(h) \int_{[U]} \int_{[H_2]} \int_{[V^k_{r-m-1}]} \varphi_\pi(vu\iota(b_2)\iota(b_1)h)$$

$$\cdot \sum_{\xi_2 \in F^{k-r}} \omega_\psi(ub_2b_1h)\phi(0,\xi_2)\psi_{V^k_{r-m}}(v)dvdb_2dudb_1dh, \tag{10}$$

It is easy to see, in the notation above, that, for $b_2 = (0, y_2, y_3; z)$, we have

$$\omega_\psi(u(0, y_2, y_3; z)b_1h)\phi(0, \xi_2) = \omega_\psi((0, y_2, 0; z)b_1h)\phi(0, \xi_2).$$

Note that the elements $(0, y_2, 0; z)$ form the Heisenberg group $\mathcal{H}_{2(k-r)+1}$, which we denote, for short, by $\mathcal{H}'$. We embed $\mathbb{A}^{k-r}$ into $\mathbb{A}^{m+k-r}$ as $0 \times \mathbb{A}^{k-r}$, the last $(k-r)$-coordinates and still denote the image as $\mathbb{A}^{k-r}$. For a Schwartz-Bruhat function $\phi' \in \mathcal{S}(\mathbb{A}^{m+k-r})$, we denote by ϕ'_{k-r} its restriction to $\mathbb{A}^{k-r}$. Then, for $b' \in \mathcal{H}'(\mathbb{A})$, we have

$$\sum_{\xi_2 \in F^{k-r}} \omega_\psi(b'b_1h)\phi(0, \xi_2)) = \widetilde{\theta}^{[\omega_\psi(b_1h)\phi]_{k-r}}_{k-r,\psi}(b').$$

Then the integral (10) becomes

$$\int_{M(F)U(\mathbb{A})\backslash \mathrm{Sp}_{2(m+k-r)}(\mathbb{A})} \int_{H_1(\mathbb{A})} \widetilde{f}_{\tau,\widetilde{\sigma},s}(h) \int_{[\mathcal{H}']} \varphi^\psi_\pi(\iota(b')\iota(b_1)h)\widetilde{\theta}^{[\omega_\psi(b_1h)\phi]_{k-r}}_{k-r,\psi}(b')db'db_1dh, \tag{11}$$

where

$$\varphi^\psi_\pi(g) = \int_{[U]} \int_{F^m\backslash \mathbb{A}^m} \int_{[V^k_{r-m-1}]} \varphi_\pi(vu\iota(0,0,y_3;0)g)\psi_{V^k_{r-m}}(v)dvdy_3du. \tag{12}$$

Consider the Weyl element

$$w = \begin{pmatrix} & I_m & & & \\ I_{r-m} & & & & \\ & & I_{2(k-r)} & & \\ & & & & I_{r-m} \\ & & & I_m & \end{pmatrix} \in \mathrm{Sp}_{2k}. \tag{13}$$

Denote $w(V^k_{r-m-1}U\iota(\mathcal{H}'))w^{-1} = E$. Then we can write

$$E = \left\{ e = \begin{pmatrix} I_m & 0 & 0 & * & * & * & * \\ * & \zeta & x & * & * & * & * \\ & & 1 & 0 & 0 & * & * \\ & & & I_{2(k-r)} & 0 & * & * \\ & & & & 1 & x' & 0 \\ & & & & & \zeta^* & 0 \\ & & & & & * & I_m \end{pmatrix} \in \mathrm{Sp}_{2k} \middle| \zeta \quad \textit{is upper unipotent} \right\}. \tag{14}$$

The character of $E(F)\backslash E(\mathbb{A})$:

$$e = wvui(b')w^{-1} \mapsto \psi_{V^k_{r-m}}(v) = \psi_E(e),$$

where $v \in [V^k_{r-m-1}]$, $u \in [U]$, and $h' \in [\mathcal{H}']$ becomes, in the notation of (14), the character

$$\psi\begin{pmatrix} \zeta & x \\ & 1 \end{pmatrix} = \psi(\zeta_{1,2} + \cdots + z_{r-m-2,r-m-1} + x_{r-m-1}).$$

Thus, (12) becomes

$$\varphi^{\psi}_{\pi}(g) = \int\limits_{E(F)\backslash E(\mathbb{A})} \varphi_\pi(ewg)\psi_E(e)de. \tag{15}$$

We preform Fourier expansions along the column in E, which lies above x, in the notation of (14), and then along the columns above ζ, one after one, from right to left, and we stop at the first column above ζ. This "practical" description translates into the following identity:

$$\varphi^{\psi}_{\pi}(g) = \int\limits_{R'(\mathbb{A})} \int\limits_{[E']} \varphi_\pi(e'xwg)\psi_{E'}(e')de'dx, \tag{16}$$

where

$$E' = \left\{ e' = \begin{pmatrix} I_m & y & * & * & * & * & * \\ & \zeta & x & * & * & * & * \\ & & 1 & 0 & 0 & * & * \\ & & & I_{2(k-r)} & 0 & * & * \\ & & & & 1 & x' & * \\ & & & & & \zeta* & y' \\ & & & & & & I_m \end{pmatrix} \in \mathrm{Sp}_{2k} \right\}$$

where in the matrix e', ζ is upper unipotent and $y_{1,1} = y_{2,1} = \cdots = y_{m,1} = 0$. For $e' \in E'(\mathbb{A})$, as above, we have the character

$$\psi_{E'}(e') = \psi\begin{pmatrix} \zeta & x \\ & 1 \end{pmatrix}.$$

Finally,

$$R' = \left\{ x = \begin{pmatrix} I_m & & & & \\ x & I_{r-m-1} & & & \\ & & I_{2(k-r+1)} & & \\ & & & I_{r-m-1} & \\ & & & x' & I_m \end{pmatrix} \in \mathrm{Sp}_{2k} \right\}.$$

The proof of (16) is exactly as the proof of (4.17) in [S05]. We continue our Fourier expansions, and exactly as one does, when writing a cusp form on $GL_{m+1}(\mathbb{A})$ as a sum of Whittaker functions ("Shalika expansion"), we get

$$\int\limits_{[E']} \varphi_\pi(e'g)\psi_{E'}(e')de' = \sum_{\gamma \in Z_m(F)\backslash \mathrm{GL}_m(F)} \int\limits_{[V^k_{r-1}]} \varphi_\pi(v\gamma^\wedge g)\psi_{V^k_r}(v)dv, \tag{17}$$

where, for $\gamma \in \mathrm{GL}_m(F)$, $\gamma^\wedge = \mathrm{diag}(\gamma, I_{2(k-m)}, \gamma^*)$, and Z_m is the standard maximal unipotent subgroup of GL_m. See, for example, the proof of Prop. 2.3 in [B-AS09], for a similar derivation. Substituting (15)-(17) in (11), we get

$$\int\int_{R(\mathbb{A})} \widetilde{f}_{\tau,\widetilde{\sigma},s}(h) \int_{[V_r^k]} \varphi_\pi(vwxh)\widetilde{\theta}_{k-r,\psi}^{[\omega_\psi(l_{m+k-r}(x)h)\phi]_{k-r}}(l_{k-r}(v))\psi_{V_r^k}(v)dvddxdh, \tag{18}$$

where the dh-integration is over $Z_m(F)\mathrm{Sp}_{2(k-r)}(F)U(\mathbb{A})\backslash\mathrm{Sp}_{2(m+k-r)}(\mathbb{A})$ and

$$R = w^{-1}R'w\iota(H_1) = \left\{ x = \begin{pmatrix} I_{r-m} & y & & & \\ & I_m & & & \\ & & I_{2(k-r)} & & \\ & & & I_m & y' \\ & & & & I_{r-m} \end{pmatrix} \in \mathrm{Sp}_{2k} \right\}. \tag{19}$$

Note that $w\iota(\mathcal{H}')w^{-1}$ is exactly $\iota(\mathcal{H}_{2(k-r)+1})$. The arguments used to obtain (18) can all be justified. The justifications are quite technical, and not suitable for this happy occasion.

When we factor the dh integration, through $[Z_m]$, we get the ψ-Whittaker coefficient along Z_m, applied to the following cusp form in the space of τ,

$$a \mapsto \widetilde{f}_{\tau,\widetilde{\sigma},s}(ah),$$

for fixed h. Here $a \in \mathrm{GL}_m(\mathbb{A})$ is naturally embedded in the Adelic points of the Levi part of P. We denote the resulting function by $\widetilde{f}_{W_\tau,\widetilde{\sigma},s}(h)$. The integral (18) becomes

$$\int\int_{R(\mathbb{A})} \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(h) \int_{[V_r^k]} \varphi_\pi(vwxh)\widetilde{\theta}_{k-r,\psi}^{[\omega_\psi(l_{m+k-r}(x)h)\phi]_{k-r}}(l_{k-r}(v))\psi_{V_r^k}(v)dvdxdh, \tag{20}$$

where the dh-integration is over $\mathrm{Sp}_{2(k-r)}(F)V_m^{m+k-r}(\mathbb{A})\backslash\mathrm{Sp}_{2(m+k-r)}(\mathbb{A})$. Finally, factoring the dh-integration through $\mathrm{Sp}_{2(k-r)}(F)\backslash\mathrm{Sp}_{2(k-r)}(\mathbb{A})$, we obtain as an inner integral the r-th Fourier-Jacobi period of the pair $(\varphi_\pi, \widetilde{\sigma})$ as defined in (4).

Denote, for fixed h, by $\rho(h)\widetilde{f}_{W_\tau,\widetilde{\sigma},s}$, the cusp form in the space of $\widetilde{\sigma}$ given by $g \mapsto \widetilde{f}_{W_\tau,\widetilde{\sigma},s}(gh)$. Then we get from (20) that the global integral $\mathrm{I}(\varphi_\pi, \phi, \widetilde{f}_{\tau,\widetilde{\sigma},s})$ is equal to

$$\int\int_{R(\mathbb{A})} \mathcal{FJP}_{\psi_{V_r^k}}(\pi(wxh)\varphi_\pi, [\omega_\psi(l_{m+k-r}(x)h)\phi]_{k-r}, \rho(h)\widetilde{f}_{W_\tau,\widetilde{\sigma},s})dxdh, \tag{21}$$

where the dh-integration is over $\mathrm{Sp}_{2(k-r)}(\mathbb{A})V_m^{m+k-r}(\mathbb{A})\backslash\mathrm{Sp}_{2(m+k-r)}(\mathbb{A})$. We finish the proof. □

We will now consider the global integral (6). Here, although we give a fair amount of details, the proof is still a little sketchy. See also Prop. 6.6 in [GJR04], where the unfolding process of these integrals is described.

Theorem 3.3. *In the notation above, assume that $r \le m$. For $\mathrm{Re}(s)$ large enough,*

$$\begin{aligned} &\mathrm{I}(\varphi_\pi, \phi, \widetilde{f}_{\tau,\widetilde{\sigma},s}) \\ = &\int_{V_r^k(\mathbb{A})\mathrm{Sp}_{2(k-r)}(\mathbb{A})\backslash\mathrm{Sp}_{2k}(\mathbb{A})} \int_{V^w(\mathbb{A})\backslash V_{m-r}^{m+k-r}(\mathbb{A})} \\ &\mathcal{FJP}_{\psi_{V_r^k}}(\pi(h)\varphi_\pi, (\omega_{\psi,k}(l_k(v)h)\phi)^{k-r}, \rho(wvh)\widetilde{f}_{W_\tau,\widetilde{\sigma},s})\psi_{V_{m-r}^{m+r-k}}(v)dvdh. \end{aligned} \tag{22}$$

Here, w is a certain Weyl element (see (30)), and $V^w = V_{m-r}^{m+k-r} \cap w^{-1}Pw$. For a Schwartz function $\phi' \in \mathcal{S}(\mathbb{A}^k)$, $(\phi')^{k-r}$ denotes the Schwartz-Bruhat function in $\mathcal{S}(\mathbb{A}^{k-r})$, defined by $x \mapsto \phi'(e_r, x)$, where $e_r = (0, ..., 0, 1)$. Finally, $\rho(h)\widetilde{f}_{W_\tau,\widetilde{\sigma},s}$ is defined as in Theorem 3.2 (except for a slight twist in the Whittaker coefficient of τ).

Proof. We first focus attention on the Fourier-Jacobi coefficient of our Eisenstein series

$$\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_{m-r}^{m+k-r}}}(\widetilde{E}_{\tau,\widetilde{\sigma}})(h,s) = \int\limits_{[V_{m-r}^{m+k-r}]} \widetilde{E}_{\tau,\widetilde{\sigma}}(vh,s)\widetilde{\theta}^{\phi}_{\psi,k}(l_k(v)h)\psi_{V_{m-r}^{m+k-r}}(v)dv \tag{23}$$

Let $\mathcal{H} = \mathcal{H}_{2k+1}$ and denote by i its embedding inside $\mathrm{Sp}_{2(m+k-r)}$. Now, we will write its elements as $(x;t)$, where x is a row vector in $2k$ coordinates. Denote the center of $\mathcal{H}$ by C. Let $N_{m-r}^{m+k-r} = V_{m-r-1}^{m+k-r}\iota(C)$, and denote by $\psi_{N_{m-r}^{m+k-r}}$ the character of $N_{m-r}^{m+k-r}(\mathbb{A})$, obtained by extending $\psi_{V_{m-r}^{m+k-r}}$ from $V_{m-r-1}^{m+k-r}(\mathbb{A})$ to $\iota(C)(\mathbb{A})$ by $\psi(\iota(0,0;t)) = \psi(t)$. Consider the Fourier coefficient

$$\widetilde{E}_{\tau,\widetilde{\sigma}}^{\psi_{N_{m-r}^{m+k-r}}}(g,s) = \int\limits_{[N_{m-r}^{m+k-r}]} \widetilde{E}_{\tau,\widetilde{\sigma}}(vg,s)\psi_{N_{m-r}^{m+k-r}}(v)dv.$$

Then

$$\widetilde{\mathcal{FJ}}^{\phi}_{\psi_{V_{m-r}^{m+k-r}}}(\widetilde{E}_{\tau,\widetilde{\sigma}})(h,s) = \int\limits_{\mathcal{H}(F)C(\mathbb{A})\backslash\mathcal{H}(\mathbb{A})} \widetilde{E}_{\tau,\widetilde{\sigma}}^{\psi_{N_{m-r}^{m+k-r}}}(\iota(b)h,s)\widetilde{\theta}^{\phi}_{\psi,k}(bh)db. \tag{24}$$

We remark here, that the Adele points of unipotent groups U of $\mathrm{Sp}_{2(m+k-r)}$ split in the metaplectic cover of $\mathrm{Sp}_{2(m+k-r)}(\mathbb{A})$, and so we treat $U(\mathbb{A})$ as a subgroup of the metaplectic group. Similarly $\mathrm{Sp}_{2(m+k-r)}(F)$ splits in $\widetilde{\mathrm{Sp}}_{2(m+k-r)}(\mathbb{A})$, and we view it as a subgroup.

Next, we unfold $\widetilde{E}_{\tau,\widetilde{\sigma}}^{\psi_{N_{m-r}^{m+k-r}}}(g,s)$. For $\mathrm{Re}(s)$ large enough,

$$\begin{aligned}\widetilde{E}_{\tau,\widetilde{\sigma}}^{\psi_{N_{m-r}^{m+k-r}}}(g,s) &= \int\limits_{[N_{m-r}^{m+k-r}]} \sum_{\gamma\in P(F)\backslash \mathrm{Sp}_{2(m+k-r)}(F)} \widetilde{f}_{\tau,\widetilde{\sigma},s}(\gamma vg)\psi_{N_{m-r}^{m+k-r}}(v)dv \\ &= \sum_{\gamma} \int\limits_{N^\gamma(F)\backslash N_{m-r}^{m+k-r}(\mathbb{A})} \widetilde{f}_{\tau,\widetilde{\sigma},s}(\gamma vg)\psi_{N_{m-r}^{m+k-r}}(v)dv, \qquad (25)\end{aligned}$$

where the summation is over $\gamma \in P(F)\backslash \mathrm{Sp}_{2(m+k-r)}(F)/N_{m-r}^{m+k-r}(F)$. Here, $N^\gamma = N_{m-r}^{m+k-r} \cap \gamma^{-1}P\gamma$.

We first describe a set of representatives of the double coset decomposition

$$P(F)\backslash \mathrm{Sp}_{2(m+k-r)}(F)/Q(F),$$

where Q is the standard parabolic subgroup, whose Levi part is isomorphic to $\mathrm{GL}_{m-r-1} \times \mathrm{Sp}_{2(k+1)}$. These are determined by the pairs of integers (s,t), such that $s \leq m$ and $r+1 \leq s-t \leq k+1$. The corresponding representative can be chosen to be $w_{s,t}$, which is equal to

(26)
$$\mathrm{diag}(I_t, \begin{pmatrix} 0 & 0 & I_{s-t} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-s} & 0 \\ I_{s-t-r-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{2(k+1-s+t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{s-t-r-1} \\ 0 & -I_{m-s} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{s-t} & 0 & 0 \end{pmatrix}, I_t).$$

The elements of the group $Q^{s,t} = Q \cap w_{s,t}^{-1} P w_{s,t}$ are the elements of $\mathrm{Sp}_{2(m+k-r)}$, of the form

$$g = \begin{pmatrix} a & x_1 & x_2 & y_1 & y_2 & y_3 & z_1 & z_2 & z_3 \\ 0 & b & x_3 & 0 & y_4 & y_5 & 0 & z_4 & z'_2 \\ 0 & 0 & c & 0 & 0 & y_6 & 0 & 0 & z'_1 \\ & & & d & u_1 & u_2 & y'_6 & y'_5 & y'_3 \\ & & & 0 & e & u'_1 & 0 & y'_4 & y'_2 \\ & & & 0 & 0 & d^* & 0 & 0 & y'_1 \\ & & & & & & c^* & x'_3 & x'_2 \\ & & & & & & 0 & b^* & x'_1 \\ & & & & & & 0 & 0 & a^* \end{pmatrix}, \tag{27}$$

where $a \in \mathrm{GL}_t$, $b \in \mathrm{GL}_{s-t-r-1}$, $c \in \mathrm{GL}_{m-s}$, $d \in \mathrm{GL}_{s-t}$, $e \in \mathrm{Sp}_{2(k-s+t+1)}$. For g as in (27), we have

$$w_{s,t} g w_{s,t}^{-1} = \begin{pmatrix} a & y_1 & z_1 & x_1 & y_2 & z_2 & -x_2 & y_3 & z_3 \\ 0 & d & y'_6 & 0 & u_1 & y'_5 & 0 & u_2 & y'_3 \\ 0 & 0 & c^* & 0 & 0 & x'_3 & 0 & 0 & x'_2 \\ & & & b & y_4 & z_4 & -x_3 & y_5 & z'_2 \\ & & & 0 & e & y'_4 & 0 & u'_1 & y'_2 \\ & & & 0 & 0 & b^* & 0 & 0 & x'_1 \\ & & & & & & c & -y_6 & -z'_1 \\ & & & & & & 0 & d^* & y'_1 \\ & & & & & & 0 & 0 & a^* \end{pmatrix}. \tag{28}$$

Now it is clear that we can take the representatives above to be of the form $\gamma_{s,t,\epsilon,\eta} = w_{s,t}\mathrm{diag}(\epsilon, \eta, \epsilon^*)$, where ϵ is a Weyl element in $\mathrm{GL}_{m-r-1}(F)$, modulo the Weyl group of $\mathrm{GL}_t(F) \times \mathrm{GL}_{s-t-r-1}(F) \times \mathrm{GL}_{m-s}(F)$, from the left, and η can be taken in a set of representatives of $R^{s,t}(F)\backslash \mathrm{Sp}_{2(k+1)}(F)/N'(F)$, where N' is the image of C inside $\mathrm{Sp}_{2(k+1)}$, and $R^{s,t}$ is the standard parabolic subgroup of $\mathrm{Sp}_{2(k+1)}$, whose Levi part is isomorphic to $\mathrm{GL}_{s-t} \times \mathrm{Sp}_{2(k-s+t+1)}$. We claim that, if $t > 0$, then for any representative $\gamma = \gamma_{s,t,\epsilon,\eta}$, we have

$$\int\limits_{N^\gamma(F)\backslash N_{m-r}^{m+k-r}(\mathbb{A})} \widetilde{f}_{\tau,\widetilde{\sigma},s}(\gamma v g)\psi_{N_{m-r}^{m+k-r}}(v)dv = 0. \tag{29}$$

Indeed, from (27), (28), we see that if ϵ conjugates a simple root subgroup in Z_{m-r-1} into a root subgroup which corresponds to one in the block coordinates x_1, x_2, x_3 in (27), then $\gamma_{s,t,\epsilon,\eta}$ conjugates this simple root subgroup to the unipotent radical of P, and we get in the integral in (29) an inner integration of the character

$\psi_{N_{m-r}^{m+k-r}}$ along this simple root subgroup, which results in zero. Ruling out all such ϵ, it is easy to see that we may take

$$\epsilon = \begin{pmatrix} & & I_t \\ & I_{s-t-r-1} & \\ I_{m-s} & & \end{pmatrix}.$$

Consider the subgroup $N_{\epsilon,\eta}$ of N_{m-r}^{m+k-r}, consisting of elements of the form

$$\begin{pmatrix} I_{m-t-r-1} & & & & & & \\ & I_t & (0 & y_2 & y_3)\eta & * & \\ & & I_{s-t} & 0 & 0 & * & \\ & & & I_{2(k-s+t+1)} & 0 & * & \\ & & & & I_{s-t} & * & \\ & & & & & I_t & \\ & & & & & & I_{m-t-r-1} \end{pmatrix}.$$

In this matrix, the second row is the product of the $t \times (2k+2)$ matrix $(0, y_2, y_3)$ by the matrix η in $\mathrm{Sp}_{2(k+1)}(F)$; $y_2 \in M_{t\times 2(k-s+t+1)}$, $y_3 \in M_{t\times(s-t)}$; η appears in the formation data of the representative $\gamma_{s,t,\epsilon,\eta}$. It is easy to see that if the first column of η is not zero from line $m-r$ downwards, then $\psi_{N_{m-r}^{m+k-r}}$ is nontrivial on $N_{\epsilon,\eta}(\mathbb{A})$, while the conjugation of $N_{\epsilon,\eta}$ by $\gamma_{s,t,\epsilon,\eta}$ sends this subgroup to the unipotent radical of P. Otherwise, we may assume that the first column of η has coordinates $1, 0, ..., 0$, and then $\gamma_{s,t,\epsilon,\eta}$ conjugates $\iota(C)$ into the unipotent radical of P. In both these cases, this results with an inner integration in (29) of a nontrivial character $\psi_{N_{m-r}^{m+k-r}}$ along a subgroup $N_{\epsilon,\eta}$, or $\iota(C)$ of N_{m-r}^{m+k-r}, and hence we get zero.

Denote $\gamma_{s,0,\epsilon,\eta} = \gamma_{s,\eta}$, with

$$\epsilon = \begin{pmatrix} & I_{s-r-1} \\ I_{m-s} & \end{pmatrix}.$$

Now $r+1 \le s \le k+1$. Let us write the representatives η following double coset decomposition

$$R^{s,0}(F)\backslash \mathrm{Sp}_{2(k+1)}(F)/\mathrm{Sp}_{2k}(F)\mathcal{H}(F).$$

This set contains exactly three elements, when $s < k+1$. These are $I_{2(k+1)}$,

$$\eta_0 = \begin{pmatrix} 0 & I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & I_{2(k-r)} & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_r & 0 \end{pmatrix},$$

and

$$\eta_1 = \begin{pmatrix} & I_s & & & \\ 1 & & & & \\ & & I_{2(k-s)} & & \\ & & & & 1 \\ & & & I_s & \end{pmatrix}.$$

When $s = k+1$, there are only two representatives, and these are the first two of the three elements above. Similar arguments show, with more work, the following facts, which we now list briefly.

The contribution of $\eta = I_{2(k+1)}$ to (25) is zero, for all $r+1 \le s \le k+1$, and, similarly, the contribution of η_0 is zero, for all $r+1 < s \le k+1$. The reason for these two facts is similar to the case $t > 0$ above.

When $r+1 \le s < k+1$ and $s \ne m$, the contribution of η_1 to (25) is such that the dv-integration factors through a subgroup, X, such that $\psi_{N^{m+k-r}_{m-r}}$ is trivial on $X(\mathbb{A})$, and is conjugated by γ to the subgroup of GL_m, embedded in the Levi part of P as

$$Z'_{m,s} = \left\{ \begin{pmatrix} I_{m-s} & y \\ & I_{m-s} \end{pmatrix} \right\}.$$

We get, as an inner integration, the constant term of τ, along the unipotent radical $Z'_{m,s}$. Since τ is cuspidal, this is zero.

When $s = m$, η_1 does contribute to (25), but its "end contribution" to the global integral (6) is zero, due to the cuspidality of π. Again, we get, as inner integration, a constant term of π, along the unipotent radical of the standard parabolic subgroup, whose Levi part is isomorphic to

$$\mathrm{GL}_m \times \mathrm{Sp}_{2(k-m)}.$$

Thus, the only nonzero contribution to (25) comes from $s = r+1$ and $\eta = \eta_0 \iota(\beta)\alpha$, where $\alpha \in \mathrm{Sp}_{2k}(F)$ and $\beta \in \mathcal{H}(F)/C(F)$. Note that in this case $\epsilon = I_{m-r-1}$. Denote

$$w = \gamma_{r+1,0,I_{m-r-1},\eta_0}.$$

Then

$$w = \begin{pmatrix} 0 & I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{m-r} \\ 0 & 0 & I_{2(k-r)} & 0 & 0 \\ -I_{m-r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_r & 0 \end{pmatrix}. \tag{30}$$

It follows that the only representatives, which contribute to (6) are $w\delta\beta\alpha$, where $\delta \in N(F)^w \backslash N(F)$, $\beta \in \mathcal{H}(F)^w \backslash \mathcal{H}(F)$, and $\alpha \in Q^{w,k}(F) \backslash \mathrm{Sp}_{2k}(F)$. Here, $N^w = N \cap w^{-1} P w$, etc.

A computation shows that $Q^{w,k}$ is the standard parabolic subgroup of Sp_{2k}, whose Levi part is isomorphic to $\mathrm{GL}_r \times \mathrm{Sp}_{2(k-r)}$. The subgroup N^w consists of the elements

$$\begin{pmatrix} z & v(y_1, y_2) & 0 \\ & I_{2(k+1)} & v'(y'_2, y'_1) \\ & & z^* \end{pmatrix},$$

where $z \in Z_{m-r-1}$ (i.e. upper unipotent),

$$v(y_1, y_2) = (y_1, 0_{(m-r-1)\times(2k-r)}, y_2, 0_{(m-r-1)\times r})$$

with y_1 being a column vector and y_2 being an $(m-r-1)\times r$ matrix. This element is conjugated by w into

$$\begin{pmatrix} I_r & 0 & y'_2 \\ & 1 & y'_1 \\ & & z^* \end{pmatrix}^{\wedge} = \mathrm{diag}(\begin{pmatrix} I_r & 0 & y'_2 \\ & 1 & y'_1 \\ & & z^* \end{pmatrix}, I_{2(k-r)}, \begin{pmatrix} z & y_1 & -y_2 \\ & 1 & 0 \\ & & I_r \end{pmatrix}). \tag{31}$$

The elements of $\mathcal{H}(F)^w$ have the form $(0_{2k-r}, x; 0)$ (x has r coordinates). Now, the global integral (6) becomes, for $\mathrm{Re}(s)$ large,

$$\int\limits_{Q^{w,k}(F)\backslash \mathrm{Sp}_{2k}(\mathbb{A})} \varphi_\pi(h) \int\limits_{\mathcal{H}(F)^w C(\mathbb{A})\backslash \mathcal{H}(\mathbb{A})}$$

$$\int\limits_{N(F)^w\backslash N(\mathbb{A})} \widetilde{f}_{\tau,\tilde{\sigma},s}(wv\iota(b)h)\psi_{N_{m-r}^{m+k-r}}(v)\widetilde{\theta}^{\phi}_{\psi,k}(bh)dvdbdh. \tag{32}$$

Factor the dv-integration through $N^w(F)\backslash N^w(\mathbb{A})$. Then the conjugation (31) shows that we get, as an inner integral, the Fourier coefficient along the subgroup of elements of the form (31)

$$\int \widetilde{f}_{\tau,\tilde{\sigma},s}(\begin{pmatrix} I_{r+1} & y \\ & z \end{pmatrix}^{\wedge} wv\iota(b)h)\psi^{-1}(y_{r+1,1} + z_{1,2} + z_{2,3} \cdots + z_{m-r-2,m-r-1})dydz.$$

Now, we apply, as in the previous case, the Shalika expansion for cusp forms in the space of τ, and obtain that the last Fourier coefficient is equal to

$$\sum_{\gamma \in Z_r(F)\backslash GL_r(F)} \widetilde{f}_{W_\tau,\tilde{\sigma},s}(\gamma' wv\iota(b)h),$$

where $\gamma' = \mathrm{diag}(\gamma, I_{2(m-k)}, \gamma^*)$. Note that $w^{-1}\gamma' w = \hat{\gamma}$ lies in the GL_r subgroup of the Levi part of $Q^{w,k}$. Write ϕ as an element is the Schwartz-Bruhat space of $\mathbb{A}^r \times \mathbb{A}^{k-r}$, and write the elements of $\mathcal{H}$ as $(a, b, c; t)$, where a, c have r coordinates and b has $2(k-r)$ coordinates. Then $\mathcal{H}^w$ is the subgroup of all elements of the form $(0, 0, c; 0)$. We may choose the Whittaker coefficient on τ, such that

$$\widetilde{f}_{W_\tau,\tilde{\sigma},s}(w\iota(0,0,c;0)g) = \psi^{-1}(c_1)\widetilde{f}_{W_\tau,\tilde{\sigma},s}(wg).$$

We will keep the same notation, for convenience. Note that

$$\omega_{\psi,k}((0,0,c)g)\phi(x,\xi) = \psi(c_1x_r + \cdots c_rx_1)\omega_{\psi,k}(g)\phi(\xi,x).$$

These arguments show that the integral (32) is equal to

$$\int \varphi_\pi(h) \int\limits_{V(\mathbb{A})^w\backslash V_{m-r}^{m+k-r}(\mathbb{A})} \widetilde{f}_{W_\tau,\tilde{\sigma},s}(wvh)\psi_{V_{m-r}^{m+k-r}}(v) \sum_{\xi\in F^{k-r}} \omega_{\psi,k}(l_k(v)h)\phi(e_r,\xi)dvdh. \tag{33}$$

Here the dh-integration is over $V_r^k(F)\mathrm{Sp}_{2(k-r)}(F)\backslash \mathrm{Sp}_{2k}(\mathbb{A})$,

$$V^w = w^{-1}V_{m-r}^{m+k-r}w \cap P,$$

and e_r is the r-th vector in the standard basis of F^r. Denote by ϕ^{k-r} the restriction of ϕ to $e_r \times \mathbb{A}^{k-r}$, viewed as a function on $\mathbb{A}^{k-r}$. Then it is easy to check that

$$\omega_{\psi,k}(ug)\phi(e_r,\xi) = \omega_{\psi,k-r}(l_{k-r}(u))(\omega_{\psi,k}(g)\phi)^{k-r}(\xi),$$

for all $u \in V_r^k(\mathbb{A})$.

Now, we factor the dh-integration through $V_r^k(F)\backslash V_r^k(\mathbb{A})$ and obtain, from (27), (28) and the last observation, that the integral (33) becomes

$$\int\limits_{V_r^k(\mathbb{A})\mathrm{Sp}_{2(k-r)}(F)\backslash \mathrm{Sp}_{2k}(\mathbb{A})} \int\limits_{V_r^k(F)\backslash V_r^k(\mathbb{A})} \varphi_\pi(uh)\psi_{V_r^k}(u) \int\limits_{V(\mathbb{A})^w\backslash V_{m-r}^{m+k-r}(\mathbb{A})}$$

$$\widetilde{f}_{W_\tau,\tilde{\sigma},s}(wvh)\psi_{V_{m-r}^{m+k-r}}(v)\widetilde{\theta}_{\psi,k-r}^{(\omega_{\psi,k}(l_k(v)h)\phi)^{k-r}}(l_{k-r}(u))dvdudh. \tag{34}$$

Note that the inner du-integration provides a Fourier-Jacobi coefficient for $\pi(h)\varphi_\pi$. More precisely, it occurs as follows.

Recall that P_r^k denotes the standard parabolic subgroup of Sp_{2k}, whose Levi part is isomorphic to $\mathrm{GL}_1^r \times \mathrm{Sp}_{2(k-r)}$. Let $b \in \mathrm{Sp}_{2(k-r)}(\mathbb{A})$, embedded in the Levi subgroup of $P_r^k(\mathbb{A})$. Of course, we regard Sp_{2k} as a subgroup of the Levi part of P inside $\mathrm{Sp}_{2(m+k-r)}$. Note that b commutes with w, it normalizes both $V^w(\mathbb{A})$ and $V_{m-r}^{m+k-r}(\mathbb{A})$, and preserves $\psi_{V_{m-r}^{m+k-r}}$. Denote, as before, for fixed $g \in \widetilde{\mathrm{Sp}}_{2(m+k-r)}(\mathbb{A})$, by $\rho(g)\widetilde{f}_{W_\tau,\tilde{\sigma},s}$ the cusp form in the space of $\tilde{\sigma}$ given by $\tilde{b} \mapsto \widetilde{f}_{W_\tau,\tilde{\sigma},s}(\tilde{b}g)$. Now, when we switch the order of the dv and du-integrations in (34), let us consider the inner du-integration in (34), as a function of h. Then, for fixed h, its value in bh is equal to

$$(35) \qquad \int_{V_r^k(F)\backslash V_r^k(\mathbb{A})} \varphi_\pi(ubh)\psi_{V_r^k}(u)\widetilde{\theta}_{\psi,k-r}^{(\omega_{\psi,k}(l_k(v)bh)\phi)^{k-r}}(l_{k-r}(u))du$$

which is denoted by $\widetilde{\mathcal{FJ}}_{\psi_{V_r^k}}^{(\omega_{\psi,k}(b^{-1}l_k(v)bh)\phi)^{k-r}}(\pi(h)\varphi_\pi)(bh)$, following the definition of the Fourier-Jacobi coefficient as in (3).

Finally, by factoring the dh-integration in (34) through $\mathrm{Sp}_{2(k-r)}(F)\backslash\mathrm{Sp}_{2(k-r)}(\mathbb{A})$, we get the following "Eulerian" expression for the integral $\mathrm{I}(\varphi_\pi, \phi, \widetilde{f}_{\tau,\widetilde{\sigma},s})$, as in (6)

$$\begin{aligned} &\mathrm{I}(\varphi_\pi, \phi, \widetilde{f}_{\tau,\widetilde{\sigma},s}) \\ = &\int_{V_r^k(\mathbb{A})\mathrm{Sp}_{2(k-r)}(\mathbb{A})\backslash\mathrm{Sp}_{2k}(\mathbb{A})} \int_{V^w(\mathbb{A})\backslash V_{m-r}^{m+k-r}(\mathbb{A})} \end{aligned}$$

$$(36) \qquad \mathcal{FJP}_{\psi_{V_r^k}}(\pi(h)\varphi_\pi, (\omega_{\psi,k}(l_k(v)h)\phi)^{k-r}, \rho(wvh)\widetilde{f}_{W_\tau,\tilde{\sigma},s})\psi_{V_{m-r}^{m+r-k}}(v)dvdh.$$

This is (22). □

We remark that the unfolding process shows, in both cases, the compatibility of Fourier-Jacobi models with parabolic induction. The decomposition of the integrals in (7) and (22) into an Eulerian product depends on the local uniqueness of the local version of the Fourier-Jacobi period, which we call the local Fourier-Jacobi model. This will be discussed in the next section.

A similar construction is also available when we consider (genuine) irreducible, cuspidal automorphic, representations $\widetilde{\pi}$ of $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$. As above, we define a family of integrals $\widetilde{\mathrm{I}}(\varphi_{\widetilde{\pi}}, \phi, f_{\tau,\sigma,s})$. Here, σ is an irreducible, cuspidal, automorphic representation of $\mathrm{Sp}_{2(k-r)}(\mathbb{A})$, such that the corresponding Fourier-Jacobi periods are not zero. In (5), (6), we simply switch the roles of π with $\widetilde{\pi}$ and $\widetilde{\sigma}$ with σ.

4. Some local theory

In this section, we discuss the local integrals derived from the global integrals in the unfolding process described in the previous section. The first issue is the Eulerian factorizability of the global integrals $\mathrm{I}(\varphi_\pi, \phi, \widetilde{f}_{\tau,\widetilde{\sigma,s}})$, as expressed in (7), (22), when $\mathrm{Re}(s)$ is large. The key issue is the uniqueness property of the local Fourier-Jacobi functionals or models, which correspond to the global Fourier-Jacobi period $\mathcal{FJP}_{\psi_{V_r^k}}(\varphi_\pi, \phi, \varphi_{\widetilde{\sigma}})$.

Assume that $\pi \cong \otimes_\nu \pi_\nu$, and similarly for τ and $\widetilde{\sigma}$. Here, ν runs over all places of the number field F. For each place ν, the global Fourier-Jacobi period, as defined in (4), induces a local trilinear form $\mathcal{T}$ which satisfies the property

$$\mathcal{T}(\pi_\nu(vg)\xi_{\pi_\nu}, \omega_\psi(l_{k-r}(v)g)\phi_\nu, \widetilde{\sigma}(g)w_{\widetilde{\sigma}}) = \psi_{V_r^k}(v)\mathcal{T}(\xi_{\pi_\nu}, \phi_\nu, w_{\widetilde{\sigma}}). \tag{37}$$

Here, ξ_{π_ν} is a vector in the space of π_ν and $w_{\widetilde{\sigma}}$ is a vector in the space of $\widetilde{\sigma}$. Also, ϕ_ν is a vector in the space of the Weil representation. Then we have

Theorem 4.1. *Let ν denote a local finite place. Denote by $\mathcal{FJF}(\pi_\nu, \widetilde{\sigma}_\nu, \psi_\nu)$ the space of all trilinear forms which satisfy the equivariance property* (37). *Then*

$$\dim_{\mathbb{C}} \mathcal{FJF}(\pi_\nu, \widetilde{\sigma}_\nu, \psi_\nu) \leq 1.$$

The key point here is Theorem A in [Su], which proves this theorem for the case when $r = 0$. See also [AGRS], [SZ] and [JSZ] for local uniqueness of Bessel models.

In general, consider the local integrals obtained from the global integrals in the previous section, where we replace the global Fourier-Jacobi period $\mathcal{FJP}$ by a local analogue. For example, over F_ν, the local integral derived from (21) is denoted by $\mathrm{I}_\nu(\xi_{\pi_\nu}, \phi_\nu, \widetilde{f}_{\tau_\nu, \widetilde{\sigma}_\nu, s})$ and has the form

$$\int_g \int_{R(F_\nu)} b_\nu(\pi_\nu(wxg)(\xi_{\pi_\nu}), [\omega_{\psi_\nu}(l_{m+k-r}(x)g)\phi_\nu]_{k-r}, \rho(g)\widetilde{f}_{\tau_\nu, \widetilde{\sigma}_\nu, s}) dx dg. \tag{38}$$

Here, the integration $\int_g$ is over $\mathrm{Sp}_{2(k-r)}(F_\nu)V_m^{m+k-r}(F_\nu)\backslash \mathrm{Sp}_{2(m+k-r)}(F_\nu)$, b_ν is taken from $\mathcal{FJF}(\pi_\nu, \widetilde{\sigma}_\nu, \psi_\nu)$, and ξ_{π_ν} is a vector in the space of π_ν, etc. Then the local integral $\mathrm{I}_\nu(\xi_{\pi_\nu}, \phi_\nu, \widetilde{f}_{\tau_\nu, \widetilde{\sigma}_\nu, s})$ converges absolutely for $\mathrm{Re}(s)$ large enough and is a rational function of q_ν^{-s}. We have

Lemma 4.2. *Assume that b_ν is nonzero. Then given a complex number s_0, there is a choice of data such that the local integral (defined through b_ν) $\mathrm{I}_\nu(\xi_{\pi_\nu}, \phi_\nu, \widetilde{f}_{\tau_\nu, \widetilde{\sigma}_\nu, s})$ is holomorphic and nonzero at s_0.*

This is a standard result, which was carried out in many cases. See [GRS98], for example.

It is clear that the local integral in (38) defines an element of

$$\mathcal{FJF}(\pi_\nu, \mathrm{Ind}_{\widetilde{P}(F_\nu)}^{\widetilde{\mathrm{Sp}}_{2(m+k-r)}(F_\nu)}(\gamma_{\psi_\nu} \cdot \tau_\nu |\det|^{s-\frac{1}{2}} \otimes \widetilde{\sigma}_\nu), \psi_\nu),$$

and hence, we obtain an embedding

$$\mathcal{FJF}(\pi_\nu, \widetilde{\sigma}_\nu, \psi_\nu) \hookrightarrow \mathcal{FJF}(\pi_\nu, \mathrm{Ind}_{\widetilde{P}(F_\nu)}^{\widetilde{\mathrm{Sp}}_{2(m+k-r)}(F_\nu)}(\gamma_{\psi_\nu} \cdot \tau_\nu |\det|^{s-\frac{1}{2}} \otimes \widetilde{\sigma}_\nu), \psi_\nu). \tag{39}$$

Take, for example, $m = r$ and τ_ν any irreducible generic representation of $\mathrm{GL}_m(F_\nu)$. Then, for q_ν^{-s} in general position, so that

$$\mathrm{Ind}_{\widetilde{P}(F_\nu)}^{\widetilde{\mathrm{Sp}}_{2k}(F_\nu)}(\gamma_{\psi_\nu} \cdot \tau_\nu |\det|^{s-\frac{1}{2}} \otimes \widetilde{\sigma}_\nu)$$

is irreducible, Theorem A of [Su] implies that the right hand side of (39) is at most one-dimensional. We conclude that it is one-dimensional, since it is nonzero by the non-triviality of the local integral in (38). Hence the left hand side of (39) is at most one-dimensional. This proves the local uniqueness of Fourier-Jacobi models for p-adic local fields, which is Theorem 4.1. The argument reduces to the case proved in Theorem A of [Su].

For archimedean local fields, the local uniqueness of Fourier-Jacobi models is expected as well, but it seems that the proof is more technically involved and no proof is known in general, except some lower rank cases ([BPSR] and [BR07]).

The arguments of the global unfolding, described before, have a straightforward analog to the local setting. See, for example, [S93], Sec. 8, or [GRS99b], Sec. 1.1, where the proof of local uniqueness runs in parallel to the unfolding process of the global integrals in [GRS98]. Thus, locally, the analogue of unfolding (5) is that, except for a finite set of values of q_ν^{-s}, we have embeddings

$$\mathcal{FJF}(\pi_\nu, \mathrm{Ind}_{\widetilde{P}(F_\nu)}^{\widetilde{\mathrm{Sp}}_{2(m+k-r)}(F_\nu)}(\gamma_{\psi_\nu}\cdot\tau_\nu|\det|^{s-\frac{1}{2}}\otimes\widetilde{\sigma}_\nu), \psi_\nu) \hookrightarrow \mathcal{FJF}(\pi_\nu, \widetilde{\sigma}_\nu, \psi_\nu), \tag{40}$$

and the analogue for (6) is that except for a finite set of values of q_ν^{-s},

$$\mathcal{FJF}(\mathrm{Ind}_{\widetilde{P}(F_\nu)}^{\widetilde{\mathrm{Sp}}_{2(m+k-r)}(F_\nu)}(\gamma_{\psi_\nu}\cdot\tau_\nu|\det|^{s-\frac{1}{2}}\otimes\widetilde{\sigma}_\nu), \pi_\nu, \psi_\nu) \hookrightarrow \mathcal{FJF}(\pi_\nu, \widetilde{\sigma}_\nu, \psi_\nu). \tag{41}$$

Similar variants hold for the other cases mentioned in the end of the last section. From (39), we conclude that these embeddings are isomorphisms. The proofs of (40), (41) follow from a computation of the corresponding Jacquet module, with respect to the Fourier-Jacobi character, of the parabolic induction from

$$\gamma_{\psi_\nu}\tau_\nu|\det|^{s-\frac{1}{2}}\otimes\widetilde{\sigma}_\nu.$$

It runs in parallel with computing the analogous coefficient of the Eisenstein series above. A detailed study of these Jacquet modules with respect to Fourier-Jacobi characters (and of Gelfand-Graev characters) of classical groups will appear in a book in preparation by Ginzburg, Rallis and Soudry. See also [GGP].

We remark that when $r = 1$, and when all data are unramified, the uniqueness is proved in [MS91]. Also, in [BR00], the general uniqueness is proved for the group Sp_4. Of course, (40), (41) give rise, as usual, to local functional equations, defining corresponding local gamma factors. See [GPSR97], [FG99], [S06], for similar cases in classical groups.

It follows from Theorem 4.1 that the global integrals introduced in Sec. 2 are factorizable, at least up to the archimedean places, which can be "lumped" together. In other words, if all vectors in the corresponding spaces of our representations are factorizable, then we have, for $\mathrm{Re}(s)$ large enough,

$$\mathrm{I}(\varphi_\pi, \phi, \widetilde{f}_{\tau,\widetilde{\sigma},s}) = \mathrm{I}_\infty((\varphi_\pi)_\infty, \phi_\infty, (\widetilde{f}_{\tau,\widetilde{\sigma},s})_\infty)\cdot\prod_{\nu<\infty}\mathrm{I}_\nu((\varphi_\pi)_\nu, \phi_\nu, (\widetilde{f}_{\tau,\widetilde{\sigma},s})_\nu)$$

where $\mathrm{I}_\infty(\cdots)$ takes care of all archimedean parts and for $\nu < \infty$, $\mathrm{I}_\nu(\cdots)$ denotes, as before, the corresponding local integral at ν. The local integral at every finite local place ν,

$$\mathrm{I}_\nu((\varphi_\pi)_\nu, \phi_\nu, (\widetilde{f}_{\tau,\widetilde{\sigma},s})_\nu)$$

and the archimedean integral

$$\mathrm{I}_\infty((\varphi_\pi)_\infty, \phi_\infty, (\widetilde{f}_{\tau,\widetilde{\sigma},s})_\infty)$$

converge absolutely for $\mathrm{Re}(s)$ large, and continue to a meromorphic function in the whole complex plane. In particular, at finite places ν, $\mathrm{I}_\nu((\varphi_\pi)_\nu, \phi_\nu, (\widetilde{f}_{\tau,\widetilde{\sigma},s})_\nu)$ is rational in q_ν^{-s}.

Next, we consider the unramified computations. We have

Theorem 4.3. *Assume that ν is a finite place. Assume that all data $(\varphi_\pi)_\nu, \phi_\nu, \ldots$ are unramified and suitably normalized. Then*

$$\mathrm{I}_\nu((\varphi_\pi)_\nu, \phi_\nu, (\widetilde{f}_{\tau,\widetilde{\sigma},s})_\nu) = \frac{L(\pi_\nu \times \tau_\nu, s)}{L_{\psi_\nu}(\widetilde{\sigma}_\nu \times \tau_\nu, s+\frac{1}{2})L(\tau_\nu, sym^2, 2s)}. \tag{42}$$

We mention that in case $r=1$, these computations were done in [MS91]. Our idea here is to compute the left hand side of (42) in an open dense set of unramified parameters of π_ν, $\widetilde{\sigma}_\nu$, and, in this set, we prove (42), using our unramified calculations from [GRS98]. Then we argue by rationality and prove the theorem. The exact same idea works also for the proof of (3.4) in [S06]. Details will appear elsewhere.

We mention that similar results to those stated in this section also apply to the global integrals $\widetilde{\mathrm{I}}(\varphi_{\widetilde{\pi}}, \phi, f_{\tau,\sigma,s})$. For example, let ν be a finite place, let $\widetilde{\pi}_\nu$ and σ_ν be irreducible, unramified representations of $\widetilde{\mathrm{Sp}}_{2k}(F_\nu)$ and $\mathrm{Sp}_{2(k-r)}(F_\nu)$, respectively, such that $\widetilde{\pi}_\nu$ has a local Fourier-Jacobi model with respect to σ_ν. Let τ_ν be an irreducible, generic, unramified representation of $\mathrm{GL}_m(F_\nu)$. Then for unramified data, suitably normalized, we have

$$\widetilde{\mathrm{I}}_\nu(\xi_{\pi_\nu}, \phi_\nu, f_{\tau_\nu,\sigma_\nu,s}) = \frac{L_{\psi_\nu}(\widetilde{\pi}_\nu \times \tau_\nu, s)}{L(\sigma_\nu \times \tau_\nu, s+\frac{1}{2})L(\tau_\nu, \wedge^2, 2s)}. \tag{43}$$

5. Applications

In this section, we give two applications to our global construction. For the first application, let π and τ denote irreducible cuspidal automorphic representations of $\mathrm{Sp}_{2(2m+k)}(\mathbb{A})$ and $\mathrm{GL}_{2m}(\mathbb{A})$, respectively. Assume that $\mathcal{O}(\pi) = (2(m+k)(2m))$. Here, $\mathcal{O}(\pi)$ is the set of unipotent orbits of $\mathrm{Sp}_{2(2m+k)}$, such that $\mathcal{O} \in \mathcal{O}(\pi)$, if π has a nontrivial Fourier coefficient corresponding to $\mathcal{O}$, and for all $\widetilde{\mathcal{O}} > \mathcal{O}$, π has no nontrivial Fourier coefficients corresponding to $\widetilde{\mathcal{O}}$. For more details on the set $\mathcal{O}(\pi)$, see [GRS03] Sec. 1. Then the Fourier-Jacobi coefficient

$$\mathcal{FJ}^{\phi}_{\psi_{V^{2m+k}_{m+k}},a}(\varphi_\pi)(h) = \int\limits_{V^{2m+k}_{m+k}(F)\backslash V^{2m+k}_{m+k}(\mathbb{A})} \varphi_\pi(vh)\widetilde{\theta}^{\phi}_{\psi^a,m}(l_m(v)h)\psi_{V^{2m+k}_{m+k}}(v)dv$$

as in (3) is not identically zero, for some $a \in F^*$, and defines an automorphic function on $\widetilde{\mathrm{Sp}}_{2m}(\mathbb{A})$. By an appropriate conjugation of v in φ_π by a certain rational diagonal element, we may assume, after replacing ψ by ψ^a, in the definition of $\psi_{V^{2m+k}_{m+k}}$, that $a=1$. Let $\widetilde{\sigma}'$ denote the automorphic representation of $\widetilde{\mathrm{Sp}}_{2m}(\mathbb{A})$ generated by the above functions, as φ_π and ϕ vary. It follows from [GRS03] Lemma 2.3 and Lemma 2.6 that $\widetilde{\sigma}'$ is a (nontrivial) generic cuspidal automorphic representation. Let $\widetilde{\sigma}$ denote an irreducible generic summand of the complex conjugate of $\widetilde{\sigma}'$. From this we deduce that the Fourier-Jacobi period $\mathcal{FJP}_{\psi_{V^{2m+k}_{m+k}}}(\varphi_\pi, \phi, \varphi_{\widetilde{\sigma}})$ is not zero for some choice of data. Therefore, we can write, as in (5), the global integral $\mathrm{I}(\varphi_\pi, \phi, \widetilde{f}_{\tau,\widetilde{\sigma},s})$ for this case.

Let ϵ denote an irreducible cuspidal automorphic representation of $\mathrm{Sp}_{2k}(\mathbb{A})$. We will say that π is a CAP representation with respect to the pair (τ, ϵ), if for almost all places ν, the representation π_ν is isomorphic to the unramified constituent of $\mathrm{Ind}_{Q(F_\nu)}^{\mathrm{Sp}_{2(2m+k)}(F_\nu)}(\tau_\nu|\cdot|^{1/2} \times \epsilon_\nu)$. Here Q is the maximal parabolic subgroup of $\mathrm{Sp}_{2(2m+k)}$, whose Levi part is $\mathrm{GL}_{2m} \times \mathrm{Sp}_{2k}$. We prove

Theorem 5.1. *With the above notations and assumptions on π, the following are equivalent.*

(1) *The partial L-function $L^S(\pi\times\tau,s)$ has a right most simple pole at $s=3/2$. Here, S is a finite set of places, including the archimedean places, such that outside S all data are unramified.*
(2) *There exist data, such that the global integral $\mathrm{I}(\varphi_\pi,\phi,\widetilde{f}_{\tau,\widetilde{\sigma},s})$ has a simple pole at $s=\frac{3}{2}$.*
(3) *The partial exterior square L-function $L^S(\tau,\wedge^2,s)$ has a simple pole at $s=1$, and there exists an irreducible, generic, cuspidal, automorphic representation ϵ of $\mathrm{Sp}_{2k}(\mathbb{A})$, such that π is a CAP representation with respect to the pair (τ,ϵ).*

To prove that Part (1) implies Part (2), we use the global integrals $\mathrm{I}(\varphi_\pi,\phi,\widetilde{f}_{\tau,\widetilde{\sigma},s})$. Indeed, using the local results stated in Sec. 3, this implication follows.

That Part (3) implies Part (1) follows from the facts that

$$L^S(\pi\times\tau,s)=L^S(\tau\times\tau,s-1/2)L^S(\tau\times\tau,s+1/2)L^S(\tau\times\epsilon,s),$$

and that the partial exterior square L-function $L^S(\tau,\wedge^2,s)$ has a simple pole at $s=1$. Indeed, we know that $L^S(\tau\times\tau,s+1/2)$ is holomorphic and nonzero at $s=\frac{3}{2}$, since τ is unitary. Also, $L^S(\tau\times\epsilon,s)$ is holomorphic and nonzero at $s=3/2$, due to the fact that ϵ is generic. (For example, by [CKPSS04], and [S05], ϵ lifts to an irreducible automorphic representation of $\mathrm{GL}_{2k+1}(\mathbb{A})$, which is an isobaric sum of certain unitary cuspidal representations.) Now, since $L^S(\tau\times\tau,s-\frac{1}{2})$ has a pole at $s=\frac{3}{2}$, it follows that $L^S(\pi\times\tau,s)$ has a pole at $s=\frac{3}{2}$. The same reasoning also shows that this is the rightmost possible pole.

To prove that Part (2) implies Part (3), we first notice that since the global integral has a simple pole at $s=\frac{3}{2}$, this implies that the Eisenstein series $\widetilde{E}_{\tau,\widetilde{\sigma}}(g,s)$ has a simple pole at that point. Since we induce from a maximal parabolic subgroup, and since all data are generic, it follows from an analogue of Lemma 2.4 in [K99], that $L^S_\psi(\tau\times\widetilde{\sigma},s)$ has a simple pole at $s=1$. By [GRS99a], [GRS99b], and [S05], it follows that $L^S(\tau,\wedge^2,s)$ has a simple pole at $s=1$, and that τ is the (ψ-) functorial lift of $\widetilde{\sigma}$ (and also that $L^S(\tau,\frac{1}{2})\neq 0$). To prove that π is a CAP representation, we use the construction of such CAP representations, described in [G03]. From that construction, our result will follow, if we can prove that the integral

$$\int_{\mathrm{Sp}_{2(2m+k)}(F)\backslash\mathrm{Sp}_{2(2m+k)}(\mathbb{A})}\int_{U(F)\backslash U(\mathbf{A})}\varphi_\pi(g)\theta_\tau(u(1,g))\psi_U(u)dudg$$

is not zero for some choice of data. Here θ_τ is a vector in a certain residual representation depending on τ only. Also, the group U is a certain F-unipotent group. See [G03] for details. Using certain Fourier expansions, and a study of Fourier-Jacobi coefficients of residual representations, the non-vanishing of the above integral follows from the non-vanishing of $\mathrm{Res}_{s=\frac{3}{2}}\mathrm{I}(\varphi_\pi,\phi,\widetilde{f}_{\tau,\widetilde{\sigma},s})$.

We make some remarks on the theorem and its proof. We do not know a direct proof in general, in the framework of global integrals, which shows that Part (2) implies Part (1); for this it seems that one needs a complete theory of the corresponding local integrals at ramified local places. By using Part (3), we are able to do so for the current case. The result characterizes a family of CAP representations in terms of periods and poles of tensor product L-functions. Since the cuspidal data are generic, it confirms for the current cases the general CAP conjecture as stated in [JS]

The second application is the following theorem, which is proved by similar reasoning.

Theorem 5.2. *Let $\widetilde{\pi}$ and σ be irreducible cuspidal automorphic representations of $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$ and $\mathrm{Sp}_{2(k-r)}(\mathbb{A})$ $(r < k)$ respectively. Assume that σ is globally generic, and that the Fourier-Jacobi period for the pair $(\widetilde{\pi}, \sigma)$, $\mathcal{FJP}_{\psi_{V_r^k}}(\varphi_{\widetilde{\pi}}, \phi, \varphi_\sigma)$, is not identically zero. Let τ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$. Let $s_0 \geq \frac{1}{2}$ be a (real) pole of the partial L-function $L_\psi^S(\widetilde{\pi} \times \tau, s)$. Then either $s_0 = 1$, in which case $L^S(\tau, \wedge^2, s)$ has a pole at $s = 1$, or $s_0 = \frac{3}{2}$, in which case, $L^S(\tau, sym^2, s)$ has a pole at $s = 1$.*

The proof uses (43). Note that $L^S(\sigma \times \tau, s + \frac{1}{2}) L^S(\tau, \wedge^2, 2s)$ is nonzero at s_0. By the genericity of σ, we conclude that the global integral $\widetilde{\mathrm{I}}(\varphi_{\widetilde{\pi}}, \phi, f_{\tau,\sigma,s})$ has a pole at s_0, for some choice of data. Hence the corresponding Eisenstein series $E_{\tau,\sigma,s}$, on $\mathrm{Sp}_{2(m+k-r)}(\mathbb{A})$ has a pole at s_0. By taking constant terms, this implies that the corresponding intertwining operator has a pole at s_0. Note that $s_0 - \frac{1}{2} \geq 0$. Using Theorem 11.1 in [CKPSS04], we get that $L(\sigma \times \tau, s - \frac{1}{2}) L(\tau, \wedge^2, 2s - 1)$ has a pole at s_0. Thus, either $2s_0 - 1 = 1$, or $s_0 - \frac{1}{2} = 1$, and the assertion of the theorem follows. Note that in the second case $L(\sigma \times \tau, s)$ has a pole at $s = 1$, and we conclude that $L^S(\tau, sym^2, s)$ has a pole at $s = 1$. See [S05].

This theorem gives an information on the location of poles of the tensor product L-function $L_\psi^S(\widetilde{\pi} \times \tau, s)$. It implies that τ is of symplectic type or of orthogonal type. This is compatible with Arthur's conjecture on the discrete spectrum.

We remark that the same proof works for the similar theorem for orthogonal groups and cuspidal representations having global Gelfand-Graev models with respect to generic data. See [S06].

References

[AGRS] A. Aizenbud, D. Gourevitch, S. Rallis, G. Schiffmann, *Multiplicity One Theorems*, to appear in the Annals of Mathematics.

[BPSR] M.Baruch, I.Piatetski-Shapiro, S.Rallis, *On the uniqueness of Fourier Jacobi models for representations of* $\mathrm{U}(2,1)$. Lie groups and symmetric spaces, 47–56, Amer. Math. Soc. Transl. Ser. 2, 210, Amer. Math. Soc., Providence, RI, 2003.

[BR00] M. Baruch, S. Rallis, *A uniqueness theorem of Fourier Jacobi models for representations of* $\mathrm{Sp}(4)$. J. London Math. Soc. (2) 62 (2000), no. 1, 183–197.

[BR07] M. Baruch, S. Rallis, *On the uniqueness of Fourier Jacobi models for representations of* $\mathrm{U}(n,1)$. Represent. Theory 11 (2007), 1–15 (electronic)

[B-AS09] A. Ben-Artzi, D. Soudry, *L-functions for $U_m \times R_{E/F}GL_n$ $(n \leq [\frac{m}{2}]$*. Automorphic forms and L-functions I. Global Aspects. A workshop in honor of Steve Gelbart, May 2006, Rehovot and Tel-Aviv, Israel (Edited by D. Ginzburg, E. Lapid, D. Soudry). Israel Math. Proc., Contemporary Math. Vol. 488, (2009), 13–61.

[CKPSS04] J. Cogdell, H. Kim, I. Piatetski-Shapiro, F. Shahidi, *Functoriality for the classical groups.* Publ. Math. Inst. Hautes Etudes Sci. No. 99 (2004), 163–233.

[CPS] J. Cogdell, I. Piatetski-Shapiro, *Converse theorems for* GL_n. Inst. Hautes tudes Sci. Publ. Math. No. 79 (1994), 157–214.

[FG99] S. Friedberg, D. Goldberg, *On local coefficients for non-generic representations of some classical groups.* Compositio Math. 116 (1999), 133-166.

[GGP] W.T. Gan-B. Gross, D. Prasad, *Symplectic local root numbers, central critical L-values and restriction problems in the representation theory of classical groups.* Preprint (2008).

[G03] D. Ginzburg, *A construction of* CAP *representations in classical groups.* Int. Math. Res. Not. (2003), no. 20, 1123–1140.

[GJR04] D. Ginzburg, D. Jiang, S. Rallis, *On Nonvanishing of the Central Value of Rankin-Selberg L-functions.* Journal of the American Mathematical Society 17 (2004), 679-722.

[GPSR97] D. Ginzburg, I. Piatetski-Shapiro, S. Rallis, *L-functions for the orthogonal group.* Memoirs of the AMS 128, No. 611 (1997).

[GRS98] D. Ginzburg, S. Rallis, D. Soudry, *L-functions for symplectic groups.* Bull. Soc. Math. France 126 (1998), no. 2, 181-244.

[GRS99a] D. Ginzburg, S. Rallis, D. Soudry, *On explicit lifts of cusp forms from* $GL(m)$ *to classical groups.* Annals of Math. 150 (1999), 807-866.

[GRS99b] D. Ginzburg, S. Rallis, D. Soudry, *On a correspondence between cuspidal representations of* GL_{2n} *and* $\widetilde{Sp}_{2n}$. JAMS 12 (1999), 849-907.

[GRS03] D. Ginzburg, S. Rallis, D. Soudry, *On Fourier coefficients of automorphic forms of symplectic groups.* Manuscripta Math. 111 (2003), no. 1, 1-16.

[JS] D. Jiang, D. Soudry, *On the genericity of cuspidal automorphic forms of* SO_{2n+1}, J. Reine Angew. Math. 604 (2007), 187–209.

[JSZ] D. Jiang , B. Sun, C-B. Zhu, *Uniqueness of Bessel models: the archimedean case*, arXiv:0908.1728

[K99] H. Kim, *Langlands-Shahidi method and poles of automorphic L-functions: Applications to exterior square L-functions.* Canad. J. Math. 51 (1999), 835-849.

[MW95] C. Moeglin and J.-L. Waldspurger, *Spectral Decomposition and Eisenstein Series.* Cambridge University Press, 1995.

[MS91] A. Murase, T. Sugano, *Whittaker-Shintani functions on the symplectic group of Fourier-Jacobi type.* Compositio Math. 79 (1991), no. 3, 321-349.

[S93] D. Soudry, *Rankin-Selberg convolutions for* SO_{2l+1}*: Local theory.* Memoirs of the AMS, Vol. 105, No. 500 (1993).

[S05] D. Soudry, *On Langlands functoriality from classical groups to* GL_n. Formes Automorphes I: Actes du Semestre du CEB, Printemps 2000 (edited by H. Carayol, M. Harris, J. Tilouine, M.-F. Vigneras), Asterisque 298 (2005), 335-390.

[S06] D. Soudry, *Rankin-Selberg integrals, the descent method, and Langlands functoriality.* Proc. of the ICM, Madrid, Spain (2006),1311-1325.

[Su] B. Sun, *Multiplicity one theorems for symplectic groups*, arXiv:0903.1417.

[SZ] B. Sun, C-B. Zhu, *Multiplicity one theorems: the Archimedean case*, arXiv:0903.1413

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