

On Local γ -factors

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1 Introduction

Let G be a reductive algebraic group defined over a p -adic local field F . We assume that F is a finite extension of \mathbb{Q}_p for simplicity. Let \mathcal{W}_F be the local Weil group of F and ${}^L G$ be the Langlands dual group of G , which is a semi-product of the complex dual group G^\vee and the absolute Galois group $\Gamma_F =$

$\text{Gal}(\overline{F}/F)$. Consider continuous homomorphisms ϕ from the Weil-Deligne group $\mathcal{W}_F \times \text{SL}_2(\mathbb{C})$ to the Langlands dual group ${}^L G$, which is admissible in the sense of [B79]. The G^\vee -conjugacy class of such a homomorphism ϕ is called a local Langlands parameter. The set of local Langlands parameters is denoted by $\Phi(G/F)$. Let $\Pi(G/F)$ be the set of equivalence classes of irreducible admissible complex representations of $G(F)$.

The local Langlands conjecture for G over F asserts that for each local Langlands parameter $\phi \in \Phi(G/F)$, there should be a finite subset $\Pi(\phi)$, which is called the local L -packet attached to ϕ such that the set $\{\Pi(\phi) \mid \phi \in \Phi(G/F)\}$ is a partition of $\Pi(G/F)$, among other required properties ([B79]). The map $\phi \mapsto \Pi(\phi)$ is called the local Langlands correspondence or the local Langlands reciprocity law for G over F .

The main problem is of course how to construct the local Langlands reciprocity map $\phi \mapsto \Pi(\phi)$. From the classification-theoretic point of view, the local Langlands conjecture provides a classification for irreducible admissible representations up to L -packet. It is interesting to characterize the local L -packets in general. The most well known approach to characterize local L -packets is in terms of stability of distribution characters following from the idea of Arthur trace formula approach to the discrete spectrum of automorphic forms. We refer to [MW03], [KV05], [DR05], [R05] and [V93] for further discussions.

In this notes, we discuss the roles of local factors attached to irreducible admissible representations of $G(F)$. They yield information about the classification theory and the functorial structures of irreducible admissible representations of $G(F)$.

First, we recall the local Langlands conjecture for GL_n over F , which is proved by Harris-Taylor [HT01] and by Henniart [H00].

Theorem 1.1 ([HT01], [H00], [H93]). *There is a unique collection of bijections*

$$\text{rec}_F : \Pi(\text{GL}_n/F) \rightarrow \Phi(\text{GL}_n/F)$$

for every $n \geq 1$ such that

1. for $\pi \in \Pi(\text{GL}_1/F)$, $\text{rec}_F(\pi) = \pi \circ \text{Art}_F^{-1}$, where Art_F is the local Artin reciprocity map from F^\times to $\mathcal{W}_F^{\text{ab}}$;
2. for $\pi_1 \in \Pi(\text{GL}_{n_1}/F)$ and $\pi_2 \in \Pi(\text{GL}_{n_2}/F)$,

$$L(s, \pi_1 \times \pi_2) = L(s, \text{rec}_F(\pi_1) \otimes \text{rec}_F(\pi_2))$$

and

$$\epsilon(s, \pi_1 \times \pi_2, \psi) = \epsilon(s, \text{rec}_F(\pi_1) \otimes \text{rec}_F(\pi_2), \psi)$$

where ψ is a given nontrivial character of F .

3. for $\pi \in \Pi(GL_n/F)$ and $\chi \in \Pi(GL_1/F)$,

$$\text{rec}_F(\pi \otimes (\chi \circ \det)) = \text{rec}_F(\pi) \otimes \text{rec}_F(\chi);$$

4. for $\pi \in \Pi(GL_n/F)$ with central character $\omega_\pi = \chi$,

$$\det \circ \text{rec}_F(\pi) = \text{rec}_F(\chi);$$

5. for $\pi \in \Pi(GL_n/F)$, $\text{rec}_F(\pi^\vee) = \text{rec}_F(\pi)^\vee$, where \vee denotes the contragredient.

We note that the existence of the local Langlands correspondence (the reciprocity map satisfies conditions 1.–5.) is proved in [HT01] and [H00]. The uniqueness of the such maps is proved in [H93]. We refer to [HT01] for historical remarks on the proof of the local Langlands conjecture for $GL_n(F)$. The local factors on the $GL_n(F)$ side is given [JPSS83] and the local factors on the $\mathcal{W}_F \times SL_2(\mathbb{C})$ side is given in [T79]. One can define as in [JPSS83] the local γ -factors by

$$(1.1) \quad \gamma(s, \pi_1 \times \pi_2, \psi) = \epsilon(s, \pi_1 \times \pi_2, \psi) \cdot \frac{L(1-s, \pi_1^\vee \times \pi_2^\vee)}{L(s, \pi_1 \times \pi_2)}.$$

On the $\mathcal{W}_F \times SL_2(\mathbb{C})$ side, one defines the γ -factor in the same way [T79]. Note that for $GL_n(F)$, the local L -packets always contains one member. This fact follows from [H93] and the Bernstein-Zelevinsky classification theory ([BZ77] and [Z80]).

For general reductive groups local factors have been defined by for many cases. When irreducible admissible representations π of $G(F)$ are generic, i.e. have nonzero Whittaker models, the Shahidi's theory of local coefficients defines the local L -, ϵ -, and γ -factors. It is expected that the local factors defined by Shahidi should be essentially the same as the ones defined by the Rankin-Selberg method if they are available, although it has to be verified case by case. It should be mentioned that for nongeneric representations, there are cases where the local factors can be defined by the Rankin-Selberg

method ([GPSR97] and [LR05], and also that the work [FG99] has the potential to define the local factors for nongeneric representations, which can be viewed as the natural extension of Shahidi's work. Of course, one may define the local factors by means of the conjectured local Langlands conjecture for G over F , and this definition should be consistent with all other definitions.

We recall the local Langlands functoriality principle. Let G and H be reductive algebraic groups defined over F . For an admissible homomorphism ${}^L\rho$ ([B79]) from the Langlands dual group LH to the Langlands dual group LG , there should be a functorial transfer ρ from $\Pi(H/F)$ to $\Pi(G/F)$, which takes L -packets of $H(F)$ to L -packets of $G(F)$, and satisfies the following conditions.

1. For any local Langlands parameter $\phi_H \in \Phi(H/F)$, ${}^L\rho \circ \phi_H$ is a local Langlands parameter in $\Phi(G/F)$, such that the functorial transfer ρ takes the local L -packet $\Pi(\phi_H)$ to the local L -packet $\Pi({}^L\rho \circ \phi_H)$.
2. For any finite-dimensional complex representation r of LG and $\sigma \in \Pi(\phi_H)$, one has

$$L(s, \rho(\sigma), r) = L(s, \sigma, r \circ {}^L\rho),$$

and

$$\epsilon(s, \rho(\sigma), r, \psi) = \epsilon(s, \sigma, r \circ {}^L\rho, \psi).$$

It follows that $\gamma(s, \rho(\sigma), r, \psi) = \gamma(s, \sigma, r \circ {}^L\rho, \psi)$. From the formulation of the local Langlands conjecture for $\mathrm{GL}_n(F)$ (Theorem 1.1), the functorial transfer should be characterized by the conditions similar to conditions 1.–5. in Theorem 1.1, in particular, by

$$L(s, \sigma \times \tau) = L(s, \rho(\sigma) \times \tau)$$

and

$$\epsilon(s, \sigma \times \tau, \psi) = \epsilon(s, \rho(\sigma) \times \tau, \psi),$$

for all irreducible supercuspidal representations τ of $\mathrm{GL}_n(F)$ for all integers $n \geq 1$. For the local γ -factors, one expects

$$(1.2) \quad \gamma(s, \sigma \times \tau, \psi) = \gamma(s, \rho(\sigma) \times \tau, \psi).$$

Of course, if one assumes the validity of the local Langlands functoriality from reductive groups to the general linear group, then one may use (1.2) to

define the twisted local γ -factors in general. We refer [K95], [K00], [BK00] and [GK99] for some very interesting discussions in this aspect.

For a given $\pi \in \Pi(G/F)$, we have two collections of local γ -factors: one is

$$(1.3) \quad \{\gamma(s, \pi \times \tau, \psi) \mid \text{for all } \tau \in \Pi(\mathrm{GL}_n/F), \text{ and for all } n = 1, 2, \dots\},$$

the twisted local γ -factors of π , and the other is

$$(1.4) \quad \{\gamma(s, \pi, r, \psi) \mid \text{for all } r\},$$

the local γ -factors attached to all finite-dimensional complex representations r of ${}^L G$. Although the exact definition of these collections of local γ -factors is still conjectural in general, it is clear that they are invariants attached to irreducible admissible representations π of $G(F)$ up to equivalence. The basic questions are the following.

1. How do the collections of local γ -factors classify the irreducible admissible representations? (The Local Converse Theorem)
2. How do the explicit analytic properties of the local γ -factors determine the functorial structures of the irreducible admissible representations? (a local version of Langlands problem)

We first recall some basic properties of local γ -factors, and then discuss these two basic problems in details, including some typical known examples in the following sections.

2 Basic Properties of Local γ -factors

We recall briefly some basic properties of the local γ -factors. Among them are mainly the multiplicativity and stability of the local γ -factors.

2.1 Multiplicativity

For the twisted local γ -factors, one expects the multiplicativity holds. More precisely, it can be stated as follows. For an irreducible admissible representation π of $G(F)$, there is a supercuspidal datum $(M(F), \sigma)$ where $P = MN$ is a parabolic subgroup of G defined over F and σ is an irreducible supercuspidal representation of $M(F)$ such that π is isomorphic to an irreducible

subrepresentation of the induced representation $\text{Ind}_{P(F)}^{G(F)}(\sigma)$. Then one expects

$$(2.1) \quad \gamma(\pi \times \tau, s, \psi) = \gamma(\sigma \times \tau, s, \psi)$$

for all irreducible supercuspidal representations τ of $\text{GL}_l(F)$ with $l = 1, 2, \dots$. Further, if $M = \text{GL}_r \times H$, then one may write $\sigma = \tau_r \otimes \sigma'$, and expect

$$(2.2) \quad \gamma(\sigma \times \tau, s, \psi) = \gamma(\tau_r \times \tau, s, \psi) \cdot \gamma(\sigma' \times \tau, s, \psi).$$

Properties (2.1) and (2.2) are called the *Multiplicativity* of the local γ -factors. F. Shahidi proves in [Sh90b] the multiplicativity for irreducible generic representations π of all F -quasisplit reductive algebraic groups $G(F)$ by using his theory on the local coefficients. For $G = \text{GL}(n)$, it is proved by Soudry by the Rankin-Selberg method ([S00]). One may expect that the work ([GPSR97] and [FG99]) has implication in this aspect for irreducible admissible representations, which may not be generic.

2.2 Stability

Another significant property of twisted local γ -factors is the *Stability*, which can be stated as follows. For irreducible admissible representations π_1 and π_2 of $G(F)$, there exists a highly ramified character χ of F^\times such that

$$\gamma(\pi_1 \times \chi, s, \psi) = \gamma(\pi_2 \times \chi, s, \psi).$$

It was proved by Jacquet-Shalika ([JS85]) for the group $\text{GL}_m \times \text{GL}_n$. For irreducible generic representations of classical groups, it was proved in [CPS98] and [CKPSS04]. For irreducible generic representations of general F -quasisplit groups, the approach is taken in [CPSS05]. For F -split classical groups of either symplectic or orthogonal type, the stability of local γ -factors has been proved for general irreducible admissible representations via the doubling method ([RS05]).

2.3 Remarks

It is important to mention that E. Lapid and S. Rallis ([LR05]) determines the sign of the local ϵ -factors via the doubling method. They introduce the ten properties, called the *Ten Commandments* of the local γ -factors, which determines the local γ -factors uniquely.

We would also like to mention the explicit calculations of the local γ -factors for irreducible supercuspidal representations via the local Rankin-Selberg method (see [jK00] for example).

3 Local Converse Theorems

The local converse theorem is to find the smallest subcollection of twisted local γ -factors $\gamma(s, \pi \times \tau, \psi)$ which classifies the irreducible admissible representation π up to equivalence. However, this is usually not the case in general. From the local Langlands conjecture, one may expect a certain subcollection of local γ -factors classifies the irreducible representation π up to L -packet. On the other hand, if the irreducible admissible representations under consideration have additional structures, then one may still expect that a certain subcollection of local γ -factors classifies the irreducible representation π up to equivalence.

3.1 The case of $\mathrm{GL}_n(F)$

Let π be an irreducible admissible representation of $\mathrm{GL}_n(F)$. Then there is a partition $n = \sum_{j=1}^r n_j$ ($n_j > 0$) and an irreducible supercuspidal representation

$$\tau_1 \otimes \cdots \otimes \tau_r$$

of $\mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$ such that the representation π can be realized as a subrepresentation of the (normalized) induced representation

$$(3.1) \quad I(\tau_1, \dots, \tau_r) = \mathrm{Ind}_{P_{[n_1 \cdots n_r]}(F)}^{\mathrm{GL}_n(F)}(\tau_1 \otimes \cdots \otimes \tau_r).$$

By the multiplicativity of the local γ -factors ([Sh90b] and [S00]), we have

$$(3.2) \quad \gamma(s, \pi \times \tau, \psi) = \prod_{j=1}^r \gamma(s, \tau_j \times \tau, \psi)$$

for all irreducible admissible representations τ of $\mathrm{GL}_l(F)$ for all $l \geq 1$. It reduces the problem for the case of general irreducible admissible representations to the case when the irreducible admissible representations are supercuspidal. It should be remarked that even if the irreducible supercuspidal

representations can be determined by the twisted local γ -factors up to equivalence, it is the best one can expect that in general the twisted local γ -factors determines the irreducible admissible representations up to the equivalence of supercuspidal data.

We first consider the case of irreducible supercuspidal representations of $\mathrm{GL}_n(F)$. The first local converse theorem (**LCT**) for $\mathrm{GL}_n(F)$ is proved by G. Henniart in [H93], which can be stated as follows.

Theorem 3.1 (LCT(n,n-1) [H93]). *Let π_1, π_2 be irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ with the same central character. If the twisted local γ -factors are the same, i.e.*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi)$$

for all irreducible supercuspidal representations τ of $\mathrm{GL}_l(F)$ with $l = 1, 2, \dots, n - 1$, then π_1 and π_2 are equivalent.

It follows that an irreducible supercuspidal representation π can be determined up to equivalence by the subcollection of twisted local γ -factors

$$\{\gamma(s, \pi \times \tau, \psi) \mid \text{for all } \tau \text{ as in the theorem}\}.$$

The remaining problem is to reduce the ‘size’ of the subcollection of twisted local γ -factors, that is, to prove **LCT(n,r)** for $r < n - 1$. In this direction, we have

Theorem 3.2 (LCT(n,n-2) [C96], [CPS99]). *Let π_1, π_2 be irreducible supercuspidal representations of $\mathrm{GL}_n(F)$. If the twisted local γ -factors are the same, i.e.*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi)$$

for all irreducible supercuspidal representations τ of $\mathrm{GL}_l(F)$ with $l = 1, 2, \dots, n - 2$, then π_1 and π_2 are equivalent.

This theorem is proved in [C96] by a purely local argument, and prove in [CPS99] as a consequence of the global converse theorem for automorphic forms. It is well known to expect

Conjecture 3.3 (Jacquet). *Let π_1, π_2 be irreducible supercuspidal representations of $\mathrm{GL}_n(F)$. If the twisted local γ -factors are the same, i.e.*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi)$$

for all irreducible supercuspidal representations τ of $\mathrm{GL}_l(F)$ with $l = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$, then π_1 and π_2 are equivalent.

There are not strong evidence to support this conjecture, which is known for $n = 2, 3, 4$ for example. On the other hand, one may expect an even stronger version of this conjecture from the conjectural global converse theorem in [CPS94]. In order to prove a better local converse theorem, it is expected to use the explicit construction of irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ and reduce to the case over finite fields. On the other hand, it is also important to consider the local converse theorem for general reductive groups.

3.2 A conjectural LCT

For a general reductive algebraic group G defined over F , the collection of twisted local γ -factors $\gamma(s, \pi \times \tau, \psi)$ is expected to determine the irreducible supercuspidal representation π up to the local L -packet. Note that all irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ are generic, i.e. have nonzero Whittaker models. It is natural to consider the local converse theorem for irreducible generic supercuspidal representations of $G(F)$ in general.

We recall the notion of Whittaker models for F -quasisplit reductive algebraic group $G(F)$. Fix an F -Borel subgroup $B = TU$. Let $\Phi(G, T)$ be the root system with the positive roots Φ^+ determined by U and Δ be the set of the simple roots. Choose an F -split $\{X_\alpha\}$, where X_α is a basis vector in the one-dimensional F -root space of α . Then we have

$$(3.3) \quad U/[U, U] \cong \bigoplus_{\alpha \in \Delta} F \cdot X_\alpha.$$

Let ψ be a character of $U(F)$. Then ψ factorizes through $U(F)/[U(F), U(F)]$, which is isomorphic as abelian groups to $\bigoplus_{\alpha \in \Delta} F \cdot X_\alpha$. A character ψ of $U(F)$ is called *generic* if ψ is nontrivial at each of the simple root α , via the isomorphism above. By the Pontriagin duality, such characters of $U(F)$ is parameterized by r -tuples

$$\underline{a} = (a_1, \dots, a_r) \in (F^\times)^r,$$

where r is the F -rank of G , i.e. the number of simple roots in Δ . An irreducible admissible representation (π, V_π) of $G(F)$ is called generic or ψ -generic if the following space

$$\mathrm{Hom}_{U(F)}(V_\pi, \psi) \cong \mathrm{Hom}_{G(F)}(V_\pi, \mathrm{Ind}_{U(F)}^{G(F)}(\psi))$$

is nonzero. For any nonzero functional $\ell_\psi \in \text{Hom}_{U(F)}(V_\pi, \psi)$, under the above isomorphism, there is $G(F)$ -equivariant homomorphism

$$v \in V_\pi \mapsto W_v^\psi(g) = \ell_\psi(\pi(g)(v)).$$

The subspace $\{W_v^\psi(g) \mid v \in V_\pi\}$ is called the ψ -Whittaker model associated to π . By the uniqueness of local Whittaker models ([Shl74]), the functional ℓ_ψ is unique up to scalar multiple.

For $t \in T(F)$, we define $t \circ \psi(u) = \psi(t^{-1}ut)$. If ψ is generic, then $t \circ \psi$ is generic for all $t \in T(F)$. Also it is clear that

$$\text{Ind}_{U(F)}^{G(F)}(\psi) \cong \text{Ind}_{U(F)}^{G(F)}(t \circ \psi)$$

for all $t \in T(F)$. It follows that if an irreducible admissible representation π of $G(F)$ is ψ -generic, then π is also $t \circ \psi$ -generic for all $t \in T(F)$. In other words, the genericity of irreducible admissible representations of $G(F)$ depends on the $T(F)$ -orbit of the generic characters. It is an easy exercise to show ([K02]) that

Proposition 3.4. *The set of $T(F)$ -orbits the generic characters of $U(F)$ is in one-to-one correspondence with $H^1(\Gamma_F, Z_G)$, where Γ_F is the absolute Galois group of F and Z_G is the center of G .*

For an irreducible admissible representation π of $G(F)$, we define

$$(3.4) \quad \mathcal{F}(\pi) = \{\psi \mid \pi \text{ is } \psi\text{-generic}\},$$

and call it the set of generic characters attached to π . It is clear that the set $\mathcal{F}(\pi)$ is $T(F)$ -stable, and the $T(F)$ -orbits $\mathcal{F}(\pi)/T(F)$ determines the genericity of π . The following conjecture of Shahidi ([Sh90a]) is fundamental to the harmonic analysis of p -adic groups and representations.

Conjecture 3.5 (Shahidi [Sh90a]). *Every tempered local L -packet contains a generic member.*

In general some nontempered local L -packets may also contains generic members ([JS04]). We call a local L -packet with generic members a generic local L -packet. Shahidi's conjecture is known to be true for the case of $\text{GL}_n(F)$ ([BZ77], [Z80]), for the case of $\text{SL}_n(F)$ ([LS86]), for the case of $\text{SO}_{2n+1}(F)$ ([JS04], [M98]), and for the case of $\text{U}_{2,1}(F)$ ([GRS97]). More

recently, it is proved to be true for cuspidal local L -packets of F -quasisplit groups ([DR05]). For reductive algebraic groups at archimedean local fields, it follows from [L89]. Some relevant discussions can be found in [MT02], and global applications can be found in [Ar89] and [KS99].

The author proposes the following refinement of the Shahidi conjecture.

Conjecture 3.6 (Refinement of Shahidi's conjecture). *In a generic local L -packet $\Pi(\phi)$, for any generic members $\pi_1, \pi_2 \in \Pi(\phi)$, the sets $\mathcal{F}(\pi_1)$ and $\mathcal{F}(\pi_2)$ are disjoint and the union of $T(F)$ -orbits of generic characters over the subset $\Pi^g(\phi)$ of $\Pi(\phi)$ consisting of all generic members in $\Pi(\phi)$, i.e.*

$$\cup_{\pi \in \Pi^g(\phi)} \mathcal{F}(\pi) / T(F)$$

is in one-to-one correspondence with $H^1(\Gamma_F, Z_G)$.

It is not difficult to check that Conjecture 3.6 holds if one knows the complete structure of local L -packets. This should be the case for the cases where Shahidi's conjecture holds. However, Conjecture 3.6 may be verified before one knows completely the structure of local l -packets. We will explain in the next section that Conjecture 3.6 holds for $G = \mathrm{SO}_{2n+1}$ by means of the local converse theorem.

Based on Conjecture 3.6, we formulate the following general version of the local converse theorem.

Conjecture 3.7 (LCT). *For any irreducible admissible generic representations π_1 and π_2 of $G(F)$, if the following two conditions hold*

1. *the intersection of $\mathcal{F}(\pi_1)$ and $\mathcal{F}(\pi_2)$ is not empty, and*
2. *the twisted local γ -factors are equal, i.e.*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi)$$

holds for all irreducible supercuspidal representations τ of $\mathrm{GL}_l(F)$ with $l = 1, 2, \dots, \lfloor \frac{r}{2} \rfloor$, where r is F -rank of G ,

then $\pi_1 \cong \pi_2$.

We remark that any theorem of this nature should be called a local converse theorem. The number of twists up to the half of the F -rank of G is an imitation of Jacquet's conjecture for GL_n . We have no strong evidence about this claim. In the next section we discuss the author's joint work with David Soudry for SO_{2n+1} .

3.3 The case of $\mathrm{SO}_{2n+1}(F)$

We review briefly here the joint work with Soudry ([JS03]) on **LCT** for SO_{2n+1} .

Theorem 3.8 (LCT for SO_{2n+1} [JS03]). *Let π and π' be irreducible admissible generic representations of $\mathrm{SO}_{2n+1}(k)$. If the twisted local gamma factors $\gamma(\pi \times \tau, s, \psi)$ and $\gamma(\pi' \times \tau, s, \psi)$ are the same, i.e.*

$$\gamma(\pi \times \tau, s, \psi) = \gamma(\pi' \times \tau, s, \psi)$$

for all irreducible supercuspidal representations τ of $\mathrm{GL}_l(k)$ with $l = 1, 2, \dots, 2n - 1$, then the representations π and π' are equivalent.

We remark that this theorem was proved in [JS03] by using Theorem 2.1, from the work [H93]. It is clear that we can improve this theorem to **LCT(n,n-2)** by using the work of [C96] or [CPS99]. We did not do this in [JS03] because it was good enough for applications in that paper. We remark that the **LCT** for generic representations of $\mathrm{U}(2, 1)$ and for $\mathrm{GSp}(4)$ was established by E. M. Baruch in [B95] and [B97].

In order to point out the essence of the local converse theorem for general reductive groups, we would like to recall some important applications of the local converse theorem for SO_{2n+1} to the theory of automorphic forms.

Let k be a number field and $\mathbb{A} = \mathbb{A}_k$ be the ring of adèles of k , First we obtain the injectivity of the weak Langlands functorial lifting established in [CKPSS01] (which is proved for example in [JS03] and [JS04] to strong Langlands functoriality).

Theorem 3.9 (Theorem 5.2 [JS03]). *Let $\Pi^{gca}(\mathrm{SO}_{2n+1}/\mathbb{A})$ be the set of all equivalence classes of irreducible generic cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ and $\Pi^a(\mathrm{GL}_{2n}/\mathbb{A})$ be the set of all equivalence classes of irreducible automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A})$. Then the Langlands functorial lifting from $\Pi^{gca}(\mathrm{SO}_{2n+1}/\mathbb{A})$ to $\Pi^a(\mathrm{GL}_{2n}/\mathbb{A})$ is an injective map.*

The second global application is to determine the generic cuspidal data for an irreducible cuspidal automorphic representations. It is a well-known theorem that every irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ is generic (i.e. having non-zero Whittaker-Fourier coefficients). This follows from the Whittaker-Fourier expansion of cuspidal automorphic forms of $\mathrm{GL}_n(\mathbb{A})$. In general, we consider a cuspidal datum (P, σ) , where P is a

parabolic subgroup of GL_n and σ is an irreducible cuspidal automorphic representation of $M(\mathbb{A})$ with $P = MN$ being the Levi decomposition. Jacquet and Shalika proved the following theorem.

Theorem 3.10 (Theorem 4.4 [JS81]). *Let $(P; \sigma)$ and $(Q; \tau)$ be two pairs of cuspidal data of $\mathrm{GL}(n)$. If the two induced representations $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})}(\sigma)$ and $\mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})}(\tau)$ share the same irreducible unramified local constituents at almost all places, then the two pairs of cuspidal data are associate.*

By the Langlands functorial lifting from SO_{2n+1} to GL_{2n} and the local converse theorem for SO_{2n+1} , we prove in [JS05] an analogue of Jacquet-Shalika's Theorem for SO_{2n+1} with generic cuspidal data. For the trivial parabolic subgroup $P = \mathrm{SO}_{2n+1}$, this was proved in [JS03] (and also in [GRS01]).

Theorem 3.11 (Theorem 3.2 [JS05]). *Let $(P; \sigma)$ and $(Q; \tau)$ be two pairs of generic cuspidal data of $\mathrm{SO}_{2n+1}(\mathbb{A})$. If the two induced representations $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{SO}_{2n+1}(\mathbb{A})}(\sigma)$ and $\mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{SO}_{2n+1}(\mathbb{A})}(\tau)$ share the same irreducible unramified local constituent at almost all places, then $(P; \sigma)$ and $(Q; \tau)$ are associate.*

It has the following consequences which are important to the understanding of structure of the discrete spectrum of $\mathrm{SO}_{2n+1}(\mathbb{A})$.

Theorem 3.12 (Corollary 3.3 [JS05]). *With notations as above, we have*

- (1) *Irreducible generic cuspidal automorphic representations π of $\mathrm{SO}_{2n+1}(\mathbb{A})$ can not be a CAP with respect to a generic, proper, cuspidal datum (P, σ) , i.e π can not be nearly equivalent to any irreducible constituent of $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{SO}_{2n+1}(\mathbb{A})}(\sigma)$.*
- (2) *If two pairs of generic cuspidal data $(P; \sigma)$ and $(Q; \tau)$, are nearly associate, i.e. their local components are associate at almost all local places, then they are globally associate.*
- (3) *The generic cuspidal datum $(P; \sigma)$ is an invariant for irreducible automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ up to near equivalence.*

We refer to [JS05] for more detailed discussions on their relation with the Arthur conjecture.

4 Poles of Local γ -factors

In his recent paper ([L04]), R. Langlands gave detailed discussion on a conjecture relating the order of the pole at $s = 1$ of automorphic L-functions to his functoriality principle. This conjecture can be stated as follows.

Let k be a number field and \mathbb{A} be the ring of adeles of k . For any reductive algebraic group G defined over k , ${}^L G$ denotes its Langlands dual group ([B79]). Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Langlands defined automorphic L-functions $L(s, \pi, \rho)$ for all finite-dimensional complex representation ρ of ${}^L G$ ([B79]). It is a theorem of Langlands that the automorphic L-function $L(s, \pi, \rho)$ converges absolutely for the real part of s large. It is a basic conjecture of Langlands that every automorphic L-function $L(s, \pi, \rho)$ should have meromorphic continuation to the whole complex plane \mathbb{C} and satisfy a functional equation relating s to $1 - s$. This basic conjecture has in fact been verified in many cases through the spectral theory of automorphic forms. See [Bmp04] and [GS88] for some detailed account on this aspect.

Problem 4.1 (Langlands [L04]). *For a given irreducible cuspidal automorphic representation π of $G(\mathbb{A})$, there exists an algebraic subgroup \mathcal{H}_π of ${}^L G$ such that for every finite-dimensional complex representation ρ of ${}^L G$, the order of the pole at $s = 1$, denoted by $m_\pi(\rho)$, of the automorphic L-function $L(s, \pi, \rho)$ is equal to the multiplicity, denoted by $m_{\mathcal{H}_\pi}(\rho)$, of the trivial representation of \mathcal{H}_π occurring in the representation ρ of ${}^L G$ when restricted to the subgroup \mathcal{H}_π . That is, the following identity*

$$(4.1) \quad m_\pi(\rho) = m_{\mathcal{H}_\pi}(\rho)$$

holds for all finite-dimensional complex representations ρ of ${}^L G$.

In [L04], Langlands discussed in length some relations of this conjecture to many basic problems in arithmetic and number theory and suggested a trace formula approach to certain important cases. Some detailed discussions for a special case, along the main idea of [L04] has been carried out by A. Venkatesh in his thesis [V04]. For $G = \mathrm{SO}_{2n+1}$, the relation between the order of the pole at $s = 1$ of the automorphic L-functions attached to irreducible unitary generic cuspidal automorphic representation π and the fundamental representations of the complex dual group $\mathrm{Sp}_{2n}(\mathbb{C})$ and the endoscopy structure of π has been discussed in detail in [J05], along the line of the Langlands Problem.

The main point here is to develop the local theory for the Langlands problem and the local analogy of [J05]. For simplicity of discussions below, it is assumed that all reductive algebraic groups considered below are F -split, and the Langlands dual group ${}^L G$ is taken to be the complex dual group $G^\vee(\mathbb{C})$, without action of the absolute Galois group.

Let π be an irreducible supercuspidal representation of $G(F)$ and ρ be a finite-dimensional complex representation of the complex dual group $G^\vee(\mathbb{C})$. One may define the local γ -factor attached to (π, ρ, ψ) (for a fixed nontrivial additive character of F) to be

$$(4.2) \quad \gamma(s, \pi, \rho, \psi) := \epsilon(s, \pi, \rho, \psi) \times \frac{L(1-s, \pi^\vee, \rho^\vee)}{L(s, \pi, \rho)},$$

where $\epsilon(s, \pi, \rho, \psi)$ is the local ϵ -factor attached to (π, ρ, ψ) . This definition is based on the assumption of the local Langlands conjecture for $G(F)$.

Problem 4.2 (Local Version of the Langlands Problem). *Let G be a reductive algebraic group defined over F . For an irreducible unitary supercuspidal representation π of $G(F)$, there exists an algebraic subgroup \mathcal{H}_π of $G^\vee(\mathbb{C})$ such that if ρ is a finite-dimensional complex representation of $G^\vee(\mathbb{C})$, then the multiplicity $m_{\mathcal{H}_\pi}(\rho)$ of the trivial representation of \mathcal{H}_π occurring in the restriction of ρ to \mathcal{H}_π is the order $m_\pi(\rho)$ of the pole of the local γ -factor $\gamma(s, \pi, \rho, \psi)$ at $s = 1$, i.e. the following identity*

$$m_\pi(\rho) = m_{\mathcal{H}_\pi}(\rho)$$

holds for all finite-dimensional complex representations of $G^\vee(\mathbb{C})$.

We remark that the key point here is to define the group \mathcal{H}_π for a given π and to study the relation between the structure of \mathcal{H}_π and the endoscopy structure of π . We refer to [J05] for discussion of \mathcal{H}_π in terms of the *observable groups* in the classical invariant theory. In the following we discuss the local analogy of [J05].

4.1 The case of $G = \mathrm{SO}_{2n+1}$

Let $G = \mathrm{SO}_{2n+1}$ be the F -split odd special orthogonal group. Then its complex dual group is $\mathrm{Sp}_{2n}(\mathbb{C})$. The fundamental representations of $\mathrm{Sp}_{2n}(\mathbb{C})$ are the finite-dimensional irreducible complex representations associated to

the fundamental weights. They can be constructed by the following split exact sequence

$$(4.3) \quad 0 \rightarrow V_{\rho_a}^{(2n)} \rightarrow \Lambda^a(\mathbb{C}^{2n}) \rightarrow \Lambda^{a-2}(\mathbb{C}^{2n}) \rightarrow 0,$$

where $\Lambda^a(\mathbb{C}^{2n})$ denotes the a -th exterior power of \mathbb{C}^{2n} , the contraction map from $\Lambda^a(\mathbb{C}^{2n})$ onto $\Lambda^{a-2}(\mathbb{C}^{2n})$ is as defined in Page 236, [GW98], and its kernel is denoted by $V_{\rho_a}^{(2n)}$. By Theorem 5.1.8 in [GW98], $V_{\rho_a}^{(2n)}$ is the space of the irreducible representation ρ_a of $\mathrm{Sp}_{2n}(\mathbb{C})$ with the a -th fundamental weight. Let ι be the natural embedding of $\mathrm{Sp}_{2n}(\mathbb{C})$ into $\mathrm{GL}_{2n}(\mathbb{C})$. Let Λ^2 be the exterior square representation of $\mathrm{GL}_{2n}(\mathbb{C})$ on the vector space $\Lambda^2(\mathbb{C}^{2n})$, which has dimension $2n^2 - n$. The composition $\Lambda^2 \circ \iota$ of Λ^2 with ι is a complex representation of $\mathrm{Sp}_{2n}(\mathbb{C})$. By (2.1) and by complete reducibility of representations of $\mathrm{Sp}_{2n}(\mathbb{C})$, we obtain

$$(4.4) \quad \Lambda^2 \circ \iota = \rho_2 \oplus \mathbf{1}_{\mathrm{Sp}_{2n}}$$

where ρ_2 is the second fundamental complex representation of $\mathrm{Sp}_{2n}(\mathbb{C})$, which is irreducible and has dimension $2n^2 - n - 1$, and $\mathbf{1}_{\mathrm{Sp}_{2n}}$ is the trivial representation of $\mathrm{Sp}_{2n}(\mathbb{C})$.

Let $\tau = \tau(\pi)$ be the image of π under the Langlands functorial transfer from $\mathrm{SO}_{2n+1}(F)$ to $\mathrm{GL}_{2n}(F)$ for irreducible admissible generic representations, which was established in [JS03] and [JS04]. One may expect the following identities hold,

$$(4.5) \quad L(s, \pi, \rho_a) = \frac{L(s, \tau(\pi), \Lambda^a)}{L(s, \tau(\pi), \Lambda^{a-2})}$$

and

$$(4.6) \quad \gamma(s, \pi, \rho_a, \psi) = \frac{\gamma(s, \tau(\pi), \Lambda^a, \psi)}{\gamma(s, \tau(\pi), \Lambda^{a-2}, \psi)}.$$

For the second exterior power representation ρ_2 , the above identities have been verified by G. Henniart ([H03]). The following theorem relates the endoscopy structure of σ to the order of pole at $s = 1$ of the second fundamental local γ -factors.

Theorem 4.3. *Let π be an irreducible generic supercuspidal representation of $\mathrm{SO}_{2n+1}(F)$ and ρ_2 be the second fundamental complex representation of $\mathrm{Sp}_{2n}(\mathbb{C})$. Then the second fundamental local γ -factors $\gamma(s, \pi, \rho_2, \psi)$ are meromorphic functions over \mathbb{C} and have the following properties.*

1. The second fundamental local γ -factor $\gamma(s, \pi, \rho_2, \psi)$ has a pole of order $r - 1$ at $s = 1$ if and only if there exists a partition $n = \sum_{j=1}^r n_j$ with $n_j > 0$ such that π is a Langlands functorial lifting from an irreducible generic supercuspidal representation $\pi_1 \otimes \cdots \otimes \pi_r$ of

$$\mathrm{SO}_{2n_1+1}(F) \times \cdots \times \mathrm{SO}_{2n_r+1}(F).$$

2. The partition $[n_1 \cdots n_r]$ is uniquely determined by the irreducible generic supercuspidal representation π . More precisely, the set of positive integers

$$\{n_1, n_2, \cdots, n_r\}$$

consists of all positive integers m such that there exists an irreducible supercuspidal representation τ of $\mathrm{GL}_m(F)$ such that the tensor product local γ -factor $\gamma(s, \pi \times \tau, \psi)$ has a pole at $s = 1$.

3. The set

$$\{\pi_1, \pi_2, \cdots, \pi_r\}$$

of irreducible generic supercuspidal representations of $\mathrm{SO}_{2n_i+1}(F)$ is completely determined by the irreducible generic supercuspidal representation π , up to equivalence, namely, it is the set of irreducible generic supercuspidal representations π' (up to equivalence) of $\mathrm{SO}_{2l+1}(F)$ such that the tensor product local γ -factor

$$\gamma(s, \pi \times \tau(\pi'), \psi)$$

has a pole at $s = 1$, where $\tau(\pi')$ is the local Langlands transfer of π' to $\mathrm{GL}_{2l}(F)$ and is irreducible and supercuspidal.

4. We have $m_\pi(\rho_2) = m_{H_{[n_1 \cdots n_r]}^\vee}(\rho_2)$, where

$$H_{[n_1 \cdots n_r]}^\vee = \mathrm{Sp}_{2n_1}(\mathbb{C}) \times \cdots \times \mathrm{Sp}_{2n_r}(\mathbb{C}).$$

The proof of this theorem follows essentially from [JS03] and [JS04], and the arguments in [J05].

The relation between the theorem and the endoscopy can be briefly discussed as below. The theory of twisted endoscopy can be found in [KS99]. For simplicity, we first recall from [Ar04] and [Ar05] the basic structure of

all standard elliptic endoscopy groups of SO_{2n+1} . Let $n = n_1 + n_2$ with $n_1, n_2 > 0$. Take a semisimple element

$$s_{n_1, n_2} = \begin{pmatrix} -I_{n_1} & & 0 \\ & I_{2n_2} & \\ 0 & & -I_{n_1} \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{C}).$$

Then the centralizer of s_{n_1, n_2} in $\mathrm{Sp}_{2n}(\mathbb{C})$ is given by

$$H_{[n_1, n_2]}^\vee = \mathrm{Cent}_{\mathrm{Sp}_{2n}(\mathbb{C})}(s_{n_1, n_2}) = \mathrm{Sp}_{2n_1}(\mathbb{C}) \times \mathrm{Sp}_{2n_2}(\mathbb{C}).$$

The standard elliptic endoscopy group associated to the partition $n = n_1 + n_2$ is

$$H_{[n_1, n_2]} = \mathrm{SO}_{2n_1+1} \times \mathrm{SO}_{2n_2+1},$$

and the groups $H_{[n_1, n_2]}$ exhaust all standard elliptic endoscopy groups of SO_{2n+1} , in the sense of [KS99].

In general, an endoscopy transfer of admissible representations from an endoscopy group H of G to G takes a local Arthur packet of admissible representations of $H(F)$ to a local Arthur packet of admissible representations of $G(F)$, which is characterized by the stability of certain distributions from the geometric side of the Arthur trace formula ([Ar05]). Since the admissible representations of $H(\mathbb{A})$ and $G(\mathbb{A})$ considered in this paper are generic and tempered, following the Arthur conjecture on the structure of the local Arthur packets, the admissible representations we are considering in this paper should be the distinguished representatives of the corresponding local Arthur packets. This must also take the Shahidi conjecture on the genericity of tempered local L -packets into account, which is the case in Arthur's formulation of his conjecture. Then by the relation between the local Arthur packets (A -packets) and the local Langlands packets (L -packets), the endoscopy transfer should take the distinguished member of a local Arthur packet to its image of the Langlands functorial lifting from H to G . In other words, the endoscopy transfer from H to G for the distinguished members of local Arthur packets should be the same as the Langlands functorial lifting from H to G . In the above theorem, the Langlands functorial lifting can be viewed as the local endoscopy transfer from $H_{[n_1, \dots, n_r]}$ to SO_{2n+1} .

As in the global case considered in [J05], we can discuss the order of the pole at $s = 1$ of the higher fundamental local γ -factors explicitly and its precise relation with the structure of the endoscopy group $H_{[n_1, \dots, n_r]}$ to

SO_{2n+1} , i.e. complete determination of the set

$$\{n_1, n_2, \dots, n_r\}.$$

Since the argument for the local case is about the same as that for the global case, we omit the details here.

4.2 Other classical groups

We also remark that there exist analogy of the above discussions for other classical groups. The functoriality from the classical groups to the general linear groups for generic representations are known through more recent work of [AS], [CKPSS04], [KK05], and [S05]. On the other hand, the analogy of the local theory for SO_{2n+1} in [JS03] and [JS04] is still our work in progress.

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