

On Local Descent from $GL(n)$ to Classical Groups

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1 Introduction

The descent method of Ginzburg, Rallis and Soudry enables one to construct, for an irreducible, self-dual, automorphic, cuspidal representation τ of $GL_m(\mathbb{A})$ (\mathbb{A} – the Adele ring of a number field k) an irreducible, automorphic, cuspidal and globally generic representation σ on the Adele points of the appropriate orthogonal (split or quasi-split), or symplectic group G , or metaplectic group (which we also denote by $G(\mathbb{A})$) such that σ lifts to τ at almost all places. See [GRS1]–[GRS5], [S2]. This method works also for a representation (cuspidal, generic) σ of a quasi-split unitary group $G(\mathbb{A}) = U_m(\mathbb{A})$, associated to a quadratic extension E of k , and its lift τ on $GL_m(\mathbb{A}_E)$. In case $L(\tau, \Lambda^2, s)$ has a pole at $s = 1$, and hence $m = 2n$ is even, we showed that σ lifts to τ at all places [JS1], [JS2]. This was done by local descent, which is the local counterpart of the global descent method, with almost complete analogy. It allows us to construct, when G is orthogonal, symplectic, or metaplectic, for an irreducible, self-dual, supercuspidal representation τ of $GL_m(F)$, where F is a p -adic field, an irreducible, supercuspidal, generic representation σ of $G(F)$, such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$, or equivalently, $L(\sigma \times \tau, s)$ has a pole at $s = 0$. This works also for quasi-split unitary groups. See [GRS3], where the representations (irreducible, supercuspidal) τ of $GL_{2n}(F)$, whose local exterior square L -function has a pole at $s = 0$, are treated. Here, the representation σ , as above, is on the metaplectic group $\tilde{Sp}_{2n}(F)$. The case of even quasi-split unitary groups is treated in [T]. In this note, we will present a similar result for an irreducible, supercuspidal representation τ of $GL_m(F)$, whose symmetric square L -function has a pole at $s = 0$. In this case, $G = Sp_{2n}$, when $m = 2n + 1$, and $G = SO_{2n}$, when $m = 2n$ (split, or quasi-split over F , according to the central character of τ being trivial or (nontrivial) quadratic.)

2 Local gamma factors

Let F be a p -adic field, and $G = G(F)$ be a local orthogonal, symplectic, or metaplectic group, as in the introduction. Let σ, τ be irreducible, generic representations of $G, GL_m(F)$, respectively. The local gamma factor $\gamma(\sigma \times \tau, s, \psi)$ (ψ is a nontrivial character of F) is obtained via a local functional equation, which arises from the theory of global integrals of Rankin-Selberg type, or Shimura type, and represent the standard L -functions for $G \times GL_m$. See [G], [S1], [GRS3], [GRS6], for G odd orthogonal, symplectic, or metaplectic. The case where G is even orthogonal

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is treated in [G], [K]. We restrict ourselves to $\text{rank}(G) < m$. The local functional equation has the form

$$\frac{\gamma(\sigma \times \tau, s, \psi)}{c(\tau, s, \psi)} \mathcal{L}(W_\sigma, D^\psi(f_{\tau, s})) = \mathcal{L}(W_\sigma, D^\psi(M(f_{\tau, s}))).$$

Here, W_σ is in the Whittaker model of σ (with respect to a given character), $f_{\tau, s}$ is a holomorphic section in $\rho_{\tau, s} = \text{Ind}_P^H \tau | \det \cdot |^{s - \frac{1}{2}}$, where H is an appropriate split classical group, or a metaplectic group, and $P \subset H$ is a Siegel type parabolic subgroup, with Levi part isomorphic to GL_m , according to the following table, where we also specify $c(\tau, s, \psi)$.

| G | H | $c(\tau, s, \psi)$ | ρ |
|--------------------------------|-----------------------------|---|-----------------------|
| 1. $\text{SO}_{2n+1}(F)$ | $\text{SO}_{2m}(F)$ | $\gamma(\tau, \Lambda^2, 2s - 1, \psi)$ | Λ^2 |
| 2. $\tilde{\text{Sp}}_{2n}(F)$ | $\text{Sp}_{2m}(F)$ | $\gamma(\tau, \Lambda^2, 2s - 1, \psi) \gamma(\tau, s - \frac{1}{2}, \psi)$ | $\Lambda^2 \oplus st$ |
| 3. $\text{SO}_{2n}(F)$ | $\text{SO}_{2m+1}(F)$ | $\gamma(\tau, \text{sym}^2, 2s - 1, \psi)$ | sym^2 |
| 4. $\text{Sp}_{2n}(F)$ | $\tilde{\text{Sp}}_{2m}(F)$ | $\gamma(\tau, \text{sym}^2, 2s - 1, \psi)$ | sym^2 |

In case (4), we have to consider $\rho_{\tau, s} = \text{Ind}_P^H \gamma_\psi \tau | \det \cdot |^{s - \frac{1}{2}}$ instead (γ_ψ is the Weil factor); M is the intertwining operator corresponding to the long Weyl element;

$$\mathcal{L}(W_\sigma, D^\psi(f_{\tau, s})) = \int_{N \backslash G} W_\sigma(g) D^\psi(f_{\tau, s})(g) dg,$$

where N is the (standard) maximal unipotent subgroup of G ; $D^\psi(f_{\tau, s})$ is given as an integral along the unipotent radical of the standard parabolic subgroup, which preserves a maximal flag in a totally isotropic subspace of dimension ℓ , and factors through the Jacquet module of $\rho_{\tau, s}$, which furnishes a Gelfand-Graev (resp. Fourier-Jacobi) model of $\rho_{\tau, s}$, stabilized by G , when H is orthogonal (resp. symplectic or metaplectic). In fact D^ψ defines an isomorphism with this Jacquet module when τ is supercuspidal. Denote this Jacquet module by $\sigma_{\psi, \ell}(\rho_{\tau, s})$. The number ℓ is determined easily by n and m ; it is $m - n - 1$, in cases (1), (3), (4), and it is $m - n$, in case (2). The gamma factor thus defined is the same as the Shahidi gamma factor, at least up to a multiple by an exponential function, and hence it has the same set of poles and zeroes.

3 Descent

The local functional equation above, defining the local gamma factor, implies the following theorem.

Theorem 3.1. *Let σ, τ be irreducible, supercuspidal representations of $G, \text{GL}_m(F)$ respectively. Then $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$, if and only if $L(\tau, \rho, s)$ has a pole at $s = 0$ and σ pairs with the Jacquet module above, where we replace $\rho_{\tau, 1}$ with its image π_τ by the intertwining operator M at $s = 1$.*

Since $L(\tau, \rho, s)$ has a pole at $s = 0$, π_τ is the Langlands quotient of $\rho_{\tau, 1}$. We call $\sigma_{\psi, \ell}(\tau) =$

$\sigma_{\psi,\ell}(\pi_\tau)$ the descent of τ to G . Consider the cases of functoriality.

| | $\mathrm{GL}_m(F)$ | <i>pole at $s = 0$</i> | H | G | <i>descent</i> |
|----|-------------------------|-----------------------------------|-------------------------------------|--|---|
| 1. | $\mathrm{GL}_{2n}(F)$ | $L(\tau, \Lambda^2, s)$ | $\mathrm{SO}_{4n}(F)$ | $\mathrm{SO}_{2n+1}(F) : \ell = n - 1$ | $\sigma_{\psi,n-1}(\tau)$ |
| 2. | $\mathrm{GL}_{2n}(F)$ | $L(\tau, \Lambda^2, s)$ | $\mathrm{Sp}_{4n}(F)$ | $\widetilde{\mathrm{Sp}}_{2n}(F) : \ell = n - 1$ | $\sigma_{\psi,n-1}(\tau)$ |
| 3. | $\mathrm{GL}_{2n}(F)$ | $L(\tau, \mathrm{sym}^2, s)$ | $\mathrm{SO}_{4n+1}(F)$ | $\mathrm{SO}_{2n,\alpha}(F) : \ell = n$ | $\sigma_{\psi,n,\alpha}(\tau), \omega_\tau = \chi_\alpha$ |
| 4. | $\mathrm{GL}_{2n+1}(F)$ | $L(\tau, \mathrm{sym}^2, s)$ | $\widetilde{\mathrm{Sp}}_{4n+2}(F)$ | $\mathrm{Sp}_{2n}(F) : \ell = n$ | $\sigma_{\psi,n}(\tau), \omega_\tau = 1$ |

Here, ω_τ is the central quadratic character of τ . If it corresponds to $\alpha \in F^*$, we denote it also by χ_α , and then we denote by $\mathrm{SO}_{2n,\alpha}(F)$ the corresponding quasi-split (or split when α is a square) orthogonal group in $2n$ variables. We can prove the following theorem.

Theorem 3.2. *In all these cases, the descent of τ is a nontrivial, supercuspidal, multiplicity free representation of G , all of whose irreducible summands σ are ψ -generic and are such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$. Moreover, any irreducible, supercuspidal and ψ -generic representation σ of G , such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$ is isomorphic to a summand of the descent of τ .*

The proof in cases (1), (2) appears in [GRS3], [JS1]. The proof in cases (3), (4) will appear in detail in a forthcoming paper by the authors. It uses the tower property of local descents (see [GRS3], Sec. 2), local gamma factors, the theory of global descent, summarized in [S2]; it will appear in great detail in [GRS7]. The proof uses also the existence of the weak functorial lift from cuspidal generic representations of $G(\mathbb{A})$ to $\mathrm{GL}_m(\mathbb{A})$ (appropriate m), proved in [CKPSS]. Assume that F is completion at a place v of the number field k . Consider a self-dual, supercuspidal τ , as above, and an irreducible summand σ of its descent to G . By globalizing σ to an irreducible, automorphic, cuspidal, generic representation of $G(\mathbb{A})$, and lifting it to $\mathrm{GL}_m(\mathbb{A})$, we get

Theorem 3.3. *Let τ be an irreducible, self-dual, supercuspidal representation of $\mathrm{GL}_m(F)$. Assume that $L(\tau, \rho, s)$ has a pole at $s = 0$, where $\rho = \Lambda^2, \mathrm{sym}^2$. Then we can globalize τ to an irreducible, self-dual, automorphic, cuspidal representation T of $\mathrm{GL}_m(\mathbb{A})$, such that $L(T, \rho, s)$ has a pole at $s = 1$.*

The same argument shows

Theorem 3.4. *Let v_1, \dots, v_r be r finite places of the number field k . Let τ_1, \dots, τ_r be r irreducible, self-dual, supercuspidal representations of $\mathrm{GL}_m(k_{v_1}), \dots, \mathrm{GL}_m(k_{v_r})$, respectively. Assume that $L(\tau_i, \rho, s)$ has a pole at $s = 0$, for all $i \leq r$, where $\rho = \Lambda^2, \mathrm{sym}^2$. Then we can globalize $\tau_1 \otimes \dots \otimes \tau_r$ to an irreducible, self-dual, automorphic, cuspidal representation T of $\mathrm{GL}_m(\mathbb{A})$, such that $L(T, \rho, s)$ has a pole at $s = 1$.*

We simply pick irreducible summands $\sigma_1, \dots, \sigma_r$ in the local descents of τ_1, \dots, τ_r , respectively, globalize $\sigma_1 \otimes \dots \otimes \sigma_r$ to an irreducible, automorphic, cuspidal, generic representation, and lift it to $\mathrm{GL}_m(\mathbb{A})$, as before. In case (2) of the last table, the local descent is irreducible. This is proved in [GRS3]. Using this and the local theta correspondence, we proved that the descent is irreducible in case (1), as well. See [JS1]. This means that σ in the last theorem is unique. For a long time we tried to address the irreducibility question of the descent in cases (3), (4),

without success. Here is our new idea. Let us add two more cases to the last table

| | $\mathrm{GL}_m(F)$ | <i>pole at $s = 0$</i> | H | G | <i>descent</i> |
|----|-------------------------|-----------------------------------|-------------------------|--------------------------------------|--------------------------------------|
| 5. | $\mathrm{GL}_{2n}(F)$ | $L(\tau, \mathrm{sym}^2, s)$ | $\tilde{S}p_{4n}(F)$ | $\mathrm{Sp}_{2n}(F) : \ell = n - 1$ | $\sigma'_{\psi, n-1}(\tau)$ |
| 6. | $\mathrm{GL}_{2n+1}(F)$ | $L(\tau, \mathrm{sym}^2, s)$ | $\mathrm{SO}_{4n+3}(F)$ | $\mathrm{SO}_{2n+2}(F) : \ell = n$ | $\sigma'_{\psi, n}, \omega_\tau = 1$ |

Here, we denote the descent by $\sigma'_{\psi, \ell}$, in order to distinguish it from the one in cases (4), (5). We can prove

Theorem 3.5. *Let τ be an irreducible, supercuspidal representation of $\mathrm{GL}_m(F)$, such that $L(\tau, \mathrm{sym}^2, s)$ has a pole at $s = 0$. In case m is odd, assume that $\omega_\tau = 1$. Then the descent in cases (5), (6) above is a nonzero irreducible, supercuspidal, ψ -generic representation σ of G , such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$; these properties determine σ uniquely.*

Consider case (5) and denote $\sigma = \sigma'_{\psi, n-1}(\tau)$. The local lift of σ to $\mathrm{GL}_{2n+1}(F)$ must be $\tau \times \omega_\tau$. We conclude that $L(\sigma \times \omega_\tau, s)$ has a pole at $s = 0$. We then prove that σ is the local ψ -theta lift from an irreducible, supercuspidal, ψ -generic representation π of $\mathrm{O}_{2n, \alpha}(F)$, where $\omega_\tau = \chi_\alpha$. We know that the restriction of π to $\mathrm{SO}_{2n, \alpha}(F)$ is either irreducible or a direct sum of two irreducible representations of the form $\pi_1 \oplus \pi_1^\varepsilon$, where $\varepsilon \in \mathrm{O}_{2n, \alpha}(F)$, with $\det(\varepsilon) = -1$; π_1^ε denotes the outer conjugation of π_1 by ε . Thus,

Theorem 3.6. *Let τ be an irreducible, supercuspidal representation of $\mathrm{GL}_{2n}(F)$, such that $L(\tau, \mathrm{sym}^2, s)$ has a pole at $s = 0$. Let $\omega_\tau = \chi_\alpha$. Then there is an irreducible, supercuspidal, ψ -generic representation σ of $\mathrm{SO}_{2n, \alpha}(F)$, such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$, and it is unique up to outer conjugation by ε .*

Similarly, in case (6), we get

Theorem 3.7. *Let τ be an irreducible, supercuspidal, self-dual representation of $\mathrm{GL}_{2n+1}(F)$, with $\omega_\tau = 1$. Then there is a unique irreducible, supercuspidal, ψ -generic representation σ of $\mathrm{Sp}_{2n}(F)$, such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$.*

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