Construction of Endoscopy Transfers for Classical Groups

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This is a report of our work in progress on construction of endoscopy transfers for quasi-split classical groups.

This construction uses certain Fourier coefficients of residues of certain Eisenstein series as kernels of integral transforms. The formulation works for all quasi-split classical groups. However, we discuss in this notes a particular case, which explains the idea and the method behind the construction.

Let F be a number field, and $\mathbb{A}$ be the ring of adeles of F. Let τ be an irreducible unitary cuspidal automorphic representation of $GL_{2a}(\mathbb{A})$, with a positive integer a. Assume that τ is self-dual and with trivial central character. It is well-known that τ is either of symplectic type (i.e. the partial exterior square L-function $L^S(s, \tau, \wedge^2)$ has a pole at $s = 1$) or of orthogonal type (i.e. the partial symmetric square L-function $L^S(s, \tau, \vee^2)$ has a pole at $s = 1$).

For a positive integer m, consider the F-split even special orthogonal group SO_{2am}. Assume that if τ is of symplectic type, take m to be even; and if τ is of orthogonal type, take m to be odd. When m is odd and τ is of orthogonal type, we denote by ϵ the automorphic descent of τ from GL_{2a} to SO_{2a}, following the work of Ginzburg, Rallis, and Soudry [7].

When $m = 2n$, take the standard parabolic subgroup $P_{(2a)^n}$ with the Levi part being $GL_{2a}^{\times(n)}$; and when $m = 2n + 1$, take the standard parabolic subgroup $P_{(2a)^n}$ with the Levi part being $GL_{2a}^{\times(n)} \times SO_{2a}$. Consider the cuspidal data $(P_{(2a)^n}, \tau^{\otimes(n)})$ when $m = 2n$, and $(P_{(2a)^n}, \tau^{\otimes(n)} \otimes \epsilon)$ when $m = 2n+1$, respectively. Let $\underline{s} := (s_1, s_2, \cdots, s_n) \in \mathbb{C}^n$. Following Langlands, there are Eisenstein series $E(g, \Phi_{\tau^{\otimes(n)}}, \underline{s})$ when $m = 2n$, and $E(g, \Phi_{\tau^{\otimes(n)} \otimes \epsilon}, \underline{s})$ when $m = 2n + 1$, attached to the above cuspidal data. It is not hard to check that $E(g, \Phi_{\tau^{\otimes(n)}}$ has a pole at $\underline{s} = (n - \frac{1}{2}, n - \frac{3}{2}, \cdots, \frac{1}{2})$ when $m = 2n$, and $E(g, \Phi_{\tau^{\otimes(n)} \otimes \epsilon}, \underline{s})$ has a pole at $\underline{s} = (n, n - 1, \cdots, 1)$ when $m = 2n + 1$ ([8]).

We denote by $\mathcal{E}_{(\tau,m)}(g)$ the iterated residue of this Eisenstein series at the given point, which is a square-integrable automorphic form on $SO_{2am}(\mathbb{A})$.

In the following, we construct kernel functions for the integral transform, which gives certain types of endoscopy transfers ([1]). In order to explain the idea and the method of the construction, we consider the following type of endoscopy transfer

$$SO_{2ab} \times SO_{2k} \to SO_{2k+2ab}, \tag{1}$$

where b is a positive integer.

The kernel functions for such a construction are given by certain Fourier coefficients of the residues $\mathcal{E}_{(\tau,2k+b)}(g)$ on $SO_{2a(2k+b)}(\mathbb{A})$. Consider the standard parabolic subgroup $P_{(2k)^{a-1}}$ of $SO_{2a(2k+b)}$ with the Levi part being $GL_{2k}^{\times(a-1)} \times SO_{2ab+4k}$. The unipotent radical is denoted by $V_{a,k,l}$. It is clear that

$$V_{a,k,l}/[V_{a,k,l}, V_{a,k,l}] \cong M_{(2k)\times(2k)}^{\oplus(a-2)} \oplus M_{(2k)\times(ab+k)} \oplus M_{(2k)\times(2k)} \oplus M_{(2k)\times(ab+k)} \tag{2}$$

where $M_{m\times n}$ denotes the matrix of size $m \times n$. The elements on the right hand side is denoted by $(X_1, \cdots, X_{a-2}; Y_1, Y_2, Y_3)$. Take a non-trivial additive character ψ of $F\backslash\mathbb{A}$, and define a character of $V_{a,k,l}(\mathbb{A})$ by

$$\psi_{a,k,l}(v) := \psi(trace(X_1 + \cdots X_{a-2} + Y_2)). \tag{3}$$

It is left $V_{a,k,l}(F)$-invariant. It is easy to check that the stabilizer of the character $\psi_{a,k,l}$ in the Levi subgroup $GL_{2k}^{\times(a-1)} \times SO_{2ab+4k}$ is $SO_{2k}^{\Delta(a)} \times SO_{2ab+2k}$. The elements

$$(g, \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}) \in SO_{2k} \times SO_{2ab+2k}$$

correspond to the elements

$$(g^{\Delta(a-1)}, \begin{pmatrix} h_1 & & h_2 \\ & g & \\ h_3 & & h_4 \end{pmatrix}) \in GL_{2k}^{\times(a-1)} \times SO_{2ab+4k}.$$

The Fourier coefficient defined by

$$\Theta^{\tau,\psi}_{a,k,l}(g,h) := \int_{V_{a,k,l}(F)\backslash V_{a,k,l}(\mathbb{A})} \mathcal{E}_{(\tau,2k+b)}(v(g,h))\overline{\psi}_{a,k,l}(v)dv \tag{4}$$

is automorphic over $SO_{2k}(\mathbb{A}) \times SO_{2ab+2k}(\mathbb{A})$. Let σ and π be irreducible cuspidal automorphic representations of $SO_{2k}(\mathbb{A})$ and $SO_{2ab+2k}(\mathbb{A})$, respectively. Then the main integral is defined as follows:

$$I^{\psi}_{a,k,l}(\tau,\sigma;\pi) := \int_{[SO_{2k}]} \int_{[SO_{2ab+2k}]} \Theta^{\tau,\psi}_{a,k,l}(g,h)\varphi_{\sigma}(g)\varphi_{\pi^\vee}(h)dhdg, \tag{5}$$

where $[SO_{2k}] := SO_{2k}(F)\backslash SO_{2k}(\mathbb{A})$ and $[SO_{2ab+2k}] := SO_{2ab+2k}(F)\backslash SO_{2ab+2k}(\mathbb{A})$. We make the following conjecture.

Conjecture: *Let σ and π be irreducible cuspidal automorphic representations of $SO_{2k}(\mathbb{A})$ and $SO_{2ab+2k}(\mathbb{A})$, respectively. Assume that if τ is of symplectic type, take b to be even; and if τ is of orthogonal type, take b to be odd, and assume that the integral $I^{\psi}_{a,k,l}(\tau,\sigma;\pi)$ is nonzero for some choice of $\varphi_\sigma \in V_\sigma$ and $\varphi_{\pi^\vee} \in V_{\pi^\vee}$. Then σ has a global Arthur parameter $\psi_{SO_{2k}}$ if and only if π has a global Arthur parameter $\psi_{SO_{2k}} \oplus (\tau, b)$.*

Note that the global Arthur parameters are referred to [1]. The condition that if τ is of symplectic type, take b to be even; and if τ is of orthogonal type, take b to be odd is to make (τ, b) a global Arthur parameter for SO_{2ab}.

The following special cases of our conjecture are proved in [4] and [5].

Theorem: *The above conjecture holds for tempered representation σ and for either $b = 1$ if τ is orthogonal or $b = 2$ if τ is symplectic.*

In [6], we explore the problem on cuspidality of the above construction for a fixed σ with b varying. This leads to the problem on the first occurrence in such endoscopy tower, generalizing the classical theory on first occurrence in theta correspondences.

References

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