ON SYMPLECTIC SUPERCUSPIDAL REPRESENTATIONS OF GL($2n$) OVER P-ADIC FIELDS

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Abstract. This is the second part of the authors' work started from [JNQ08]. In this paper, we consider the complete relations among the local theta correspondence, local Langlands transfer, and the local descent attached to a given irreducible symplectic supercuspidal representation of p-adic GL_{2n}. This is the natural extension of the work of Ginzburg-Rallis-Soudry ([GRS99]) and of Jiang-Soudry ([JS03]) on the local descents and the local Langlands transfers. The approach undertaken in this paper is purely local. A mixed approach with both local and global method, which works for more general classical groups, has been considered by Jiang-Soudry and was announced in [S08].

1. Introduction

Let $\mathcal{F}$ be a p-adic local field of characteristic zero. Let τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}_{2n}(\mathcal{F})$. By the local Langlands conjecture for $\mathrm{GL}_{2n}(\mathcal{F})$, which is now a theorem of Harris and Taylor ([HT01]) and of Henniart ([H00]), there exists an

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irreducible admissible $2n$-dimensional representation ϕ of the local Weil group $\mathcal{W}_{\mathcal{F}}$, i.e. the local Langlands parameter

$$\phi \ : \ \mathcal{W}_{\mathcal{F}} \to \mathrm{GL}_{2n}(\mathbb{C}),$$

corresponding to τ with a set of required conditions. We say that τ is of symplectic type if the image $\phi(\mathcal{W}_{\mathcal{F}})$ is contained in the symplectic subgroup $\mathrm{Sp}_{2n}(\mathbb{C})$ of the complex dual group $\mathrm{GL}_{2n}(\mathbb{C})$ of $\mathrm{GL}_{2n}(\mathcal{F})$.

Because of the deep connection with Galois representations, supercuspidal representations (or more importantly cuspidal automorphic representations) of symplectic type attract a lot of attentions in recent research (see [GJR04] and [CC09] for instance). It is desirable to understand the implications of the symplectic type of the supercuspidal representations to the other aspects of representations and harmonic analysis of p-adic groups. Various characterizations of the irreducible unitary supercuspidal representations of $\mathrm{GL}_{2n}(\mathcal{F})$ to be of symplectic type have been found from the accumulation of the earlier work of many people ([Sh90], [Sh92], [JR96], [GRS99], [JS03], [JS04], [JQ07], and [JNQ08], for instance), and were discussed in detail in [JNQ08], §5. We state them as follows and the notation and terminology used in the theorem will be defined or explained in §2.

Theorem 1.1. *Let τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}_{2n}(\mathcal{F})$. Then the following are equivalent.*

1. *τ is of symplectic type.*
2. *The local exterior square L-factor $L(s,\tau,\Lambda^2)$ has a pole at $s=0$.*
3. *The local exterior square γ-factor $\gamma(s,\tau,\Lambda^2,\psi)$ has a pole at $s=1$.*
4. *τ has a nonzero Shalika model.*
5. *The unitarily induced representation $\mathrm{I}^{\mathrm{SO}_{4n}}(s,\tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ is reducible at $s=1$. In this case, $\mathrm{I}^{\mathrm{SO}_{4n}}(1,\tau)$ has the unique Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1,\tau)$, which has a nonzero generalized Shalika model.*
6. *τ is a local Langlands functorial transfer from $\mathrm{SO}_{2n+1}(\mathcal{F})$.*
7. *τ has a nonzero linear model, i.e. $\mathrm{GL}_n(\mathcal{F})\times\mathrm{GL}_n(\mathcal{F})$-invariant functionals.*
8. *The unitarily induced representation $\mathrm{I}^{\mathrm{Sp}_{4n}}(s,\tau)$ of $\mathrm{Sp}_{4n}(\mathcal{F})$ is reducible at $s=\frac{1}{2}$. In this case, $\mathrm{I}^{\mathrm{Sp}_{4n}}(\frac{1}{2},\tau)$ has the unique Langlands quotient $\mathcal{L}^{\mathrm{Sp}_{4n}}(\frac{1}{2},\tau)$, which has a nonzero symplectic linear model, i.e. $\mathrm{Sp}_{2n}(\mathcal{F})\times\mathrm{Sp}_{2n}(\mathcal{F})$-invariant functionals.*
9. *τ is a local Langlands functorial ψ-transfer from $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$.*

If one of the above holds for τ, then τ is self-dual.

We remark that the local Langlands functorial ψ-transfer from an irreducible ψ-generic supercuspidal representation $\widetilde{\pi}$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$ to the irreducible supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ is given by the Corollary in §1.5 of [GRS99]. Also the local exterior square L-function and gamma factor are given by the Shahidi method.

The equivalence of the various characterizations in Theorem 1.1 can be explained by the following diagram. Recall that the complex dual groups of $\mathrm{SO}_{2n+1}(\mathcal{F})$ and the double metaplectic cover $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$ of $\mathrm{Sp}_{2n}(\mathcal{F})$ are the same, which is $\mathrm{Sp}_{2n}(\mathbb{C})$.

$$
\begin{array}{ccccc}
\mathrm{SO}_{4n} & & \overset{\mathrm{tc}}{\longleftrightarrow} & & \mathrm{Sp}_{4n} \\
 & & (A) & & \\
 & \nwarrow^{\mathrm{lq}} & & {}^{\mathrm{lq}}\nearrow & \\
\mathrm{gg}\downarrow & (B) & \mathrm{GL}_{2n} & (C) & \downarrow \mathrm{fj} \\
 & \nearrow_{\mathrm{lt}} & & {}_{\mathrm{lt}}\nwarrow & \\
 & & (D) & & \\
\mathrm{SO}_{2n+1} & & \overset{\mathrm{tc}}{\longleftrightarrow} & & \widetilde{\mathrm{Sp}}_{2n}
\end{array}
\tag{1.1}
$$

where the mappings are explained as follows. The mapping tc stands for the local theta correspondence for the reductive dual pairs $(\mathrm{SO}_{4n}, \mathrm{Sp}_{4n})$ and $(\mathrm{SO}_{2n+1}, \widetilde{\mathrm{Sp}}_{2n})$, respectively. The mapping gg stands for the local Gelfand-Graev coefficient which takes representations from SO_{4n} to SO_{2n+1} and the mapping fj stands for the local Fourier-Jacobi coefficient which takes representations from Sp_{4n} to $\widetilde{\mathrm{Sp}}_{2n}$. The mapping lq stands for the composition of the parabolic induction from the standard parabolic subgroups with Levi subgroup isomorphic to GL_{2n} in SO_{4n} and Sp_{4n}, respectively, and taking the unique Langlands quotient from the induced representations of SO_{4n} and Sp_{4n}, respectively. It is clear that by composing of the mapping lq with the mappings gg and fj, respectively, one obtains that $\mathrm{gg} \circ \mathrm{lq}$ and $\mathrm{fj} \circ \mathrm{lq}$ are the local descents from from GL_{2n} to SO_{2n+1} and $\widetilde{\mathrm{Sp}}_{2n}$, respectively, in the sense of Ginzburg-Rallis-Soudry. Finally the mapping lt stands for the local Langlands functorial transfer from SO_{2n+1} to GL_{2n} and from $\widetilde{\mathrm{Sp}}_{2n}$ to GL_{2n}, respectively.

For a given irreducible unitary symplectic supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, the mappings in Diagram (1.1) can be realized by

$$
\begin{array}{ccccccc}
\mathcal{L}^{\mathrm{SO}_{4n}}(1,\tau) & & & \overset{\mathrm{tc}}{\longleftrightarrow} & & & \mathcal{L}^{\mathrm{Sp}_{4n}}(\frac{1}{2},\tau) \\
 & & & (\mathrm{A}) & & & \\
 & & \nwarrow^{\mathrm{lq}} & & {}^{\mathrm{lq}}\nearrow & & \\
\mathrm{gg}\big\downarrow & (\mathrm{B}) & & \tau & & (\mathrm{C}) & \big\downarrow\mathrm{fj} \\
 & & \nearrow_{\mathrm{lt}} & & {}_{\mathrm{lt}}\nwarrow & & \\
 & & & (\mathrm{D}) & & & \\
\sigma & & & \overset{\mathrm{tc}}{\longleftrightarrow} & & & \widetilde{\pi}
\end{array} \tag{1.2}
$$

Notation in Diagram (1.2) are explained as follows. σ is an irreducible generic supercuspidal representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$, which lifts to τ by the local Langlands functorial transfer from SO_{2n+1} to GL_{2n}. $\widetilde{\pi}$ is an irreducible ψ-generic supercuspidal representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$, which lifts to τ by the local Langlands functorial ψ-transfer from $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$ to GL_{2n}. Consider the maximal parabolic subgroup P of SO_{4n} with Levi subgroup GL_{2n}. Then the unitarily parabolic induction $\mathrm{I}^{\mathrm{SO}_{4n}}(1,\tau)$ has a unique Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1,\tau)$ and similarly, the unitarily parabolic induction $\mathrm{I}^{\mathrm{Sp}_{4n}}(\frac{1}{2},\tau)$ has a unique Langlands quotient $\mathcal{L}^{\mathrm{Sp}_{4n}}(\frac{1}{2},\tau)$. Finally the local Gelfand-Graev coefficient takes $\mathcal{L}^{\mathrm{SO}_{4n}}(1,\tau)$ from $\mathrm{SO}_{4n}(\mathcal{F})$ back to $\mathrm{SO}_{2n+1}(\mathcal{F})$ and the local Fourier-Jacobi coefficient takes $\mathcal{L}^{\mathrm{Sp}_{4n}}(\frac{1}{2},\tau)$ from $\mathrm{Sp}_{4n}(\mathcal{F})$ back to $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$, respectively. The detailed discussion of these mappings will be given in §2. The key point is

Theorem 1.2. *For an irreducible unitary symplectic supercuspidal representation* τ *of* $\mathrm{GL}_{2n}(\mathcal{F})$*, Diagram* (1.2) *is commutative.*

Now we explain the relation between Theorem 1.1 and Theorem 1.2, or the commutative diagrams (1.1) and (1.2).

First of all, it is proved in [JS03] that for a given irreducible unitary symplectic supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, there exists uniquely an irreducible generic supercuspidal representation σ of $\mathrm{SO}_{2n+1}(\mathcal{F})$ and an irreducible ψ-generic supercuspidal representation $\widetilde{\pi}$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$, respectively, such that the sub-diagram (D) is commutative. The characterization of the local Langlands functorial transfer property for τ is naturally given by the existence of the pole at $s=0$ of the local exterior square L-factor $L(s,\tau,\Lambda^2)$ or equivalently by definition the pole at $s=1$ of the local exterior square γ-factor $\gamma(s,\tau,\Lambda^2,\psi)$.

What makes it very interesting is the characterization in terms of the existence of nonzero Shalika model (or functional) and of nonzero linear model (or functional), respectively, following the idea of relative trace formula approach to the global Langlands functorial transfers. It was proved in [JNQ08] that for an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, the existence of nonzero Shalika model for τ is equivalent to the existence of nonzero linear model for τ, although it was expected for a quite while. In [JR96], it was proved that the existence of nonzero Shalika model for τ implies the existence of nonzero linear model for τ.

The key point is to explain that for an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, the existence of nonzero linear model for τ determines the local Langlands functorial transfer from $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$ to GL_{2n}, while the existence of nonzero Shalika model for τ determines the local Langlands functorial transfer from SO_{2n+1} to GL_{2n}. To this end, Ginzburg, Rallis, and Soudry ([GRS99]) show that if an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ has a nonzero linear model, i.e. a nonzero $\mathrm{GL}_n(\mathcal{F}) \times \mathrm{GL}_n(\mathcal{F})$-invariant functional, then the unique Langlands quotient $\mathcal{L}^{\mathrm{Sp}_{4n}}(\frac{1}{2}, \tau)$ of the unitarily parabolic induction $\mathrm{I}^{\mathrm{Sp}_{4n}}(\frac{1}{2}, \tau)$ (which is reducible) has a nonzero symplectic linear model, i.e. a nonzero $\mathrm{Sp}_{2n}(\mathcal{F}) \times \mathrm{Sp}_{2n}(\mathcal{F})$-invariant functional. Based on the existence of a nonzero symplectic linear model for $\mathcal{L}^{\mathrm{Sp}_{4n}}(\frac{1}{2}, \tau)$, they show that the ψ-local descent (the Fourier-Jacobi ψ-functor in this case) yields $\tilde{\pi}$ back to $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$. This proves that the sub-diagram (C) is commutative.

The local descent $\tau \mapsto \sigma$ from $\mathrm{GL}_{2n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$ was first obtained in [JS03] by combining the sub-diagrams (C) and (D) and by using the local converse theorem. In a more recent work of Jiang and Soudry, which has been announced in [S08] that the local descent $\tau \mapsto \sigma$ from $\mathrm{GL}_{2n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$ is obtained via the global theory of the automorphic descent ([GRS01]). The method in [S08] works for other classical groups.

Started in our previous work ([JQ07] and [JNQ08]), we are establishing the local descent $\tau \mapsto \sigma$ from $\mathrm{GL}_{2n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$ by means of the existence of a nonzero Shalika model for τ of $\mathrm{GL}_{2n}(\mathcal{F})$ and of a nonzero generalized Shalika model for the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$. We proved in [JNQ08], Theorem 3.1, that for an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model, the local Gelfand-Graev coefficient (a special type of twisted Jacquet functor) of the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$, which is a representation of $\mathrm{SO}_{2r+1}(\mathcal{F})$, vanishes for all

$r < n$, by means of purely local argument. In this paper, we show also by purely local argument that for an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model, the local Gelfand-Graev coefficient of the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1,\tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$ is an irreducible generic supercuspidal representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$, which is the main result of this paper (see Theorem 2.3 in §2). The idea of proving this result was suggested by the global argument as in [GRS01]. Our proof goes similarly to the case of symplectic linear models in [GRS99]. However, our proof is essentially based on the existence and uniqueness of the generalized Shalika model for the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1,\tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ and the technical details are of interest in their own right. We leave the detail in §3, 4 and 5.

One point remains here is to show that the local Gelfand-Graev coefficient on $\mathrm{SO}_{2n+1}(\mathcal{F})$ from $\mathcal{L}^{\mathrm{SO}_{4n}}(1,\tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ lifts to τ via the local Langlands functorial transfer. In [JS03] or [S08], the first named author and Soudry use the global argument to show that this is the case. However, it is desirable to also have a purely local argument to handle this point. One possibility to do this is to calculate explicitly that the local Rankin-Selberg integral for the tensor product L-functions for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_r$ by using the supercuspidal representation constructed explicitly by the local Gelfand-Graev coefficient from $\mathcal{L}^{\mathrm{SO}_{4n}}(1,\tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$. We decide to omit this point here. Hence, by combining Theorems 2.3 with the result in [JS03] or [S08], we have that the sub-diagram (B) is commutative.

Finally, we show that the sub-diagram (A) is also commutative by using a work of G. Muic ([M06]), which shows that the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1,\tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ and the Langlands quotient $\mathcal{L}^{\mathrm{Sp}_{4n}}(\frac{1}{2},\tau)$ of $\mathrm{Sp}_{4n}(\mathcal{F})$ correspond to each other via the local theta correspondence. By combining with Theorem 1.1, one deduces that the generalized Shalika model on $\mathrm{SO}_{4n}(\mathcal{F})$ and the symplectic linear model of $\mathrm{Sp}_{4n}(\mathcal{F})$ are related by the local theta correspondence. It remains interesting to check directly that the local theta correspondence relates the generalized Shalika model on $\mathrm{SO}_{4n}(\mathcal{F})$ and the symplectic linear model of $\mathrm{Sp}_{4n}(\mathcal{F})$ without using Theorem 1.1. This completes our explanation of Diagrams (1.1) and (1.2) and hence the conceptual reasons for Theorem 1.1 on the various characterizations of the symplectic property of an irreducible unitary supercuspidal representation of $\mathrm{GL}_{2n}(\mathcal{F})$.

We remark that it is a very interesting problem to understand the explicit relations between our Diagrams (1.1) and (1,2) and refined structures of the corresponding local Arthur packets. We prefer to consider this in a further coming work.

The first named author is grateful to David Soudry for sharing ideas and thoughts during their collaboration on the local descents for classical groups (as announced in [S08]), which stimulates his thoughts to establish the local descent in this case based on the theory of the generalized Shalika models for SO_{4n}. Finally we would like to thank the referee for valuable comments.

2. Main Result

We introduce definitions of various models and of the local descent in the case under consideration, and then state the main result for the local descent.

2.1. **Shalika models and generalized Shalika models.** Let $\mathcal{F}$ be a finite extension of the p-adic number field $\mathbb{Q}_p$ for some rational prime p. Take the maximal parabolic subgroup $P_{n,n} = M_{n,n}N_{n,n}$ of GL_{2n} with

$$M_{n,n} = \mathrm{GL}_n \times \mathrm{GL}_n,$$

and

$$N_{n,n} = \{n(X) = \begin{pmatrix} \mathrm{I}_n & X \\ 0 & \mathrm{I}_n \end{pmatrix} \in \mathrm{GL}_{2n}\}.$$

Let ψ be a nontrivial character of $\mathcal{F}$. Define a character

$$\psi_{N_{n,n}}(n(X)) = \psi(\mathrm{tr}(X)).$$

The stabilizer of $\psi_{N_{n,n}}$ in $M_{n,n}$ is GL_n^Δ, the diagonal embedding of GL_n into $M_{n,n}$. Denote by

$$\mathcal{S}_n = \mathrm{GL}_n^\Delta \rtimes N_{n,n} \tag{2.1}$$

the Shalika subgroup. Denote by $\psi_{\mathcal{S}_n}$ the extension of $\psi_{N_{n,n}}$ from $N_{n,n}$ to the Shalika subgroup $\mathcal{S}_n$, such that $\psi_{\mathcal{S}_n}$ is trivial on GL_n^Δ. The Shalika functionals of an irreducible admissible representation (τ, V_τ) of $\mathrm{GL}_{2n}(\mathcal{F})$ are nonzero elements of the following space

$$\mathrm{Hom}_{\mathcal{S}_n(\mathcal{F})}(V_\tau, \psi_{\mathcal{S}_n}).$$

By Frobenius reciprocity

$$\mathrm{Hom}_{\mathcal{S}_n(\mathcal{F})}(V_\tau, \psi_{\mathcal{S}_n}) \cong \mathrm{Hom}_{\mathrm{GL}_{2n}(\mathcal{F})}(V_\tau, \mathrm{Ind}_{\mathcal{S}_n(\mathcal{F})}^{\mathrm{GL}_{2n}(\mathcal{F})}(\psi_{\mathcal{S}_n})),$$

any nonzero Shalika functional ℓ_ψ in $\mathrm{Hom}_{\mathcal{S}_n(\mathcal{F})}(V_\tau, \psi_{\mathcal{S}_n})$ gives rise to an embedding of V_τ into the full induction $\mathrm{Ind}_{\mathcal{S}_n(\mathcal{F})}^{\mathrm{GL}_{2n}(\mathcal{F})}(\psi_{\mathcal{S}_n})$, the image of which is called a local Shalika model of V_τ. It is proved in [JR96] (and also in [N09-1] by different argument) that the local Shalika model is unique for any irreducible admissible representation of $\mathrm{GL}_{2n}(\mathcal{F})$.

The *generalized Shalika model* for $\mathrm{SO}_{4n}(\mathcal{F})$ was first introduced in [JQ07]. Let $\nu_1 = 1$ and inductively define

$$\nu_n = \begin{pmatrix} & 1 \\ \nu_{n-1} & \end{pmatrix}, \text{ for } n \geq 2, n \in \mathbb{N}. \tag{2.2}$$

Let SO_{4n} be the even special orthogonal group attached to the nondegenerate $4n$-dimensional quadratic vector space over $\mathcal{F}$ with respect to ν_{4n}. That is

$$\mathrm{SO}_{4n} = \{g \in \mathrm{GL}_{4n} | \ {}^t g \cdot \nu_{4n} \cdot g = \nu_{4n}\}.$$

Let $P_{2n} = M_{2n}V_{2n}$ be the Siegel parabolic subgroup of SO_{4n} consists of elements of the following form:

$$(g, X) = \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} \begin{pmatrix} \mathrm{I}_n & X \\ & \mathrm{I}_n \end{pmatrix}, \tag{2.3}$$

where $g \in \mathrm{GL}_{2n}$ and $g^* = \nu_{2n} \ {}^t g^{-1} \nu_{2n}$, and X satisfies ${}^t X = -\nu_{2n} X \nu_{2n}$.

The *generalized Shalika subgroup* $\mathcal{H}_{2n}$ of SO_{4n} was introduced in [JQ07], which is the subgroup of P consisting of elements (g, X) with $g \in \mathrm{Sp}_{2n}$. Here the symplectic group is given by

$$\mathrm{Sp}_{2n} = \{g \in \mathrm{GL}_{2n} | \ {}^t g \cdot J_{2n} \cdot g = J_{2n}\},$$

where J_{2n} is given by

$$J_{2n} = \begin{pmatrix} & \nu_n \\ \nu_n & \end{pmatrix}, \ n \in \mathbb{N}.$$

Define a character $\psi_{\mathcal{H}}$ of $\mathcal{H}_{2n}(\mathcal{F})$ (We write $\mathcal{H} = \mathcal{H}_{2n}$, when n is understood) by letting

$$\psi_{\mathcal{H}}((g, X)) = \psi(\mathrm{tr}(J_{2n} X \nu_{2n})) \tag{2.4}$$

$$= \psi(\mathrm{tr}(\begin{pmatrix} -\mathrm{I}_n & \\ & \mathrm{I}_n \end{pmatrix} X)). \tag{2.5}$$

It is well defined. The *generalized Shalika functional* or *$\psi_{\mathcal{H}}$-functional* of an irreducible admissible representation (σ, V_σ) of $\mathrm{SO}_{4n}(\mathcal{F})$ is a nonzero functional in the following space

$$\mathrm{Hom}_{\mathrm{SO}_{4n}(\mathcal{F})}(V_\sigma, \mathrm{Ind}_{\mathcal{H}_{2n}(\mathcal{F})}^{\mathrm{SO}_{4n}(\mathcal{F})}(\psi_{\mathcal{H}})) = \mathrm{Hom}_{\mathcal{H}_{2n}(\mathcal{F})}(V_\sigma, \psi_{\mathcal{H}}).$$

Nien has shown the uniqueness of the generalized Shalika model in [N09-2]. Hence one can use a nonzero generalized Shalika functional to define a generalized Shalika model for σ. In order to relate the Shalika model on GL_{2n} and the generalized Shalika model on SO_{4n}, we consider the following parabolic induction.

For an irreducible, unitary, supercuspidal representation (τ, V_τ) of $\mathrm{GL}_{2n}(\mathcal{F})$, we consider the unitary induced representation $\mathrm{I}(s, \tau)$ of

$\mathrm{SO}_{4n}(\mathcal{F})$ from the Siegel parabolic subgroup $P_{2n} = M_{2n}V_{2n}$, where the Levi part $M_{2n} \cong \mathrm{GL}_{2n}$ via the following bijection

$$a \in \mathrm{GL}_{2n} \mapsto m(a) := \begin{pmatrix} a & \\ & a^* \end{pmatrix} \in M_{2n}.$$

More precisely, a section $\phi_{\tau,s}$ in $\mathrm{I}(s,\tau)$ is a smooth function from $\mathrm{SO}_{4n}(\mathcal{F})$ to V_τ, such that

$$\phi_{\tau,s}(m(a)ng) = |\det a|^{\frac{s}{2}+\frac{2n-1}{2}}\tau(a)\phi_{\tau,s}(g),$$

where $m(a) \in M_{2n}$ with $a \in \mathrm{GL}_{2n}(\mathcal{F})$, $n \in V_{2n}$. In other words, one has

$$\mathrm{I}(s,\tau) = \mathrm{Ind}_{P_{2n}(\mathcal{F})}^{\mathrm{SO}_{4n}(\mathcal{F})}(|\det|^{\frac{s}{2}} \cdot \tau).$$

In the Introduction, we use notation $\mathrm{I}^{\mathrm{SO}_{4n}}(s,\tau)$ for $\mathrm{I}(s,\tau)$ to indicate that it is a representation of SO_{4n}. From now on , we simply use the notation $\mathrm{I}(s,\tau)$.

The relation between the Shalika model on GL_{2n} and the generalized Shalika model on SO_{4n} is given by the following theorem, which is proved in [JQ07].

Theorem 2.1 (Theorem 3.1, [JQ07])**.** *The induced representation* $\mathrm{I}(s,\tau)$ *admits no nonzero generalized Shalika functionals except when* $s = 1$*. When* $s = 1$*,* $\mathrm{I}(1,\tau)$ *admits a nonzero generalized Shalika functional if and only if the supercuspidal datum* τ *admits a nonzero Shalika functional. In this case, the generalized Shalika functionals of* $\mathrm{I}(1,\tau)$ *are unique up to scalar and the nonzero generalized Shalika functionals of* $\mathrm{I}(1,\tau)$ *must factor through the unique Langlands quotient* $\mathcal{L}(1,\tau)$*.*

Note that we used in the Introduction $\mathcal{L}^{\mathrm{SO}_{4n}}(1,\tau)$ for $\mathcal{L}(1,\tau)$. Again from now on we simply use $\mathcal{L}(1,\tau)$.

2.2. **A family of degenerate Whittaker models.** Following [MW87], degenerate Whittaker models for a reductive group G can be defined for any given nilpotent orbit in the Lie algebra $\mathfrak{g}$ of G. For the purpose of this paper, we consider a family of nilpotent orbits $\mathcal{O}_{2n,2n-k}$ of SO_{4n} which correspond to a family of partitions $[2(2n-k)+1, 1^{2k-1}]$ for $k = 1, 2, \cdots, 2n$. This family of degenerate Whittaker models on $\mathrm{SO}_{4n}(\mathcal{F})$ are considered in [GPSR97] for construction of automorphic L-functions of orthogonal groups, and in [GRS99] for construction of the Ginzburg-Rallis-Soudry global descents. More precisely, we take a family of unipotent subgroups N_k of SO_{4n}, which consists of elements of following type

$$n = n(u,b,z) = \begin{pmatrix} u & b & z \\ & \mathrm{I}_{4n-2k} & b' \\ & & u' \end{pmatrix} \in \mathrm{SO}_{4n}, \tag{2.6}$$

where $u = (u_{i,j}) \in \mathrm{U}_k$, the maximal unipotent subgroup of GL_k consisting of all upper triangular unipotent matrices in GL_k, $b = (b_{i,j})$ is of size $(k) \times (4n-2k)$ and b', u' are determined by b, u such that n belongs to SO_{4n}. We define a character ψ_k on N_k

$$\psi_k(n) := \psi(u_{1,2} + \cdots + u_{k-1,k})\psi(b_{k,2n-k} + b_{k,2n-k+1}). \tag{2.7}$$

When $k = 2n-1$, N_{2n-1} coincides with the unipotent radical N of the Borel subgroup of SO_{4n}, and ψ_{2n-1} is the generic character of N. Let π be an irreducible admissible representation (π, V_π) of $\mathrm{SO}_{4n}(\mathcal{F})$. Then π has a nonzero ψ_k-functional if the following space

$$\mathrm{Hom}_{\mathrm{SO}_{4n}(\mathcal{F})}(V_\pi, \mathrm{Ind}_{N_k(\mathcal{F})}^{\mathrm{SO}_{4n}(\mathcal{F})}(\psi_k)) \cong \mathrm{Hom}_{N_k(\mathcal{F})}(V_\pi, \psi_k) \neq 0. \tag{2.8}$$

In this case, a nonzero element in $\mathrm{Hom}_{N_k(\mathcal{F})}(V_\pi, \psi_k)$ is called a ψ_k-functional of V_π, or more precisely, a ψ_k-degenerate Whittaker functional of V_π. For each ψ_k-functional ℓ_{ψ_k}, we define

$$\mathcal{W}_{\psi_k,v}(g) := \ell_{\psi_k}(\pi(g)(v)) \tag{2.9}$$

for $v \in V_\pi$, which yields a ψ_k-degenerate Whittaker model (also refer to as (N_k, ψ_k)-model) for V_π. In particular, when $k = 2n-1$, it produces a Whittaker model for V_π. Note that the different choices of the representatives in the $\mathcal{F}$-rational points of the unipotent orbit $\mathcal{O}_{2n,k}(\mathcal{F})$ produce different characters for $N_k(\mathcal{F})$, and hence different degenerate Whittaker models. However, the centralizers are all isomorphic, which is the $\mathcal{F}$-split $\mathrm{SO}_{4n-2k-1}(\mathcal{F})$. This is different from the case of odd orthogonal groups considered in [JS07].

We recall the definition of Jacquet functor and Jacquet module. Given a closed subgroup $\widetilde{P} = \widetilde{N} \rtimes \widetilde{M}$ of SO_{4n} with unipotent radical $\widetilde{N}$ and a character χ on $\widetilde{N}$ normalized by $\widetilde{M}$, then for a representation (V_π, π) of $\mathrm{SO}_{4n}(\mathcal{F})$, its Jacquet module with respect to $(\widetilde{N}, \chi)$ is defined by

$$\mathcal{J}_{\widetilde{N},\chi}(\pi) = V_\sigma/\mathrm{Span}\{\sigma(n)v - \chi(n)v | n \in \widetilde{N}, v \in V_\pi\},$$

viewed as a representation of $\widetilde{M}$. We call $\mathcal{J}_{\widetilde{N},\chi}$ the Jacquet functor with respect to $(\widetilde{N}, \chi)$. We write $\mathcal{J}_{\widetilde{N}}$ for $\mathcal{J}_{\widetilde{N},\chi}$, when χ is trivial. For the family of ψ_k-degenerate Whittaker models, the corresponding family of ψ_k-twisted Jacquet modules is abbreviated by

$$\mathcal{J}_{\psi_k}(V_\pi) := \mathcal{J}_{N_k,\psi_k}(V_\pi), \tag{2.10}$$

viewed as a representation of $\mathrm{SO}_{4n-2k-1}(\mathcal{F})$, following the definition of the ψ_k-twisted Jacquet module.

The following relation between the ψ_k-twisted Jacquet modules and generalized Shalika model for SO_{4n} was proved in [JNQ08].

Theorem 2.2 (Theorem 3.1, [JNQ08])**.** *Let (π, V_π) be an irreducible admissible representation of $\mathrm{SO}_{4n}(\mathcal{F})$. If π has a nonzero generalized Shalika model, then the ψ_k-twisted Jacquet modules $\mathcal{J}_{\psi_k}(V_\pi)$ are all zero for $n \le k \le 2n$.*

For an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model, we apply the family of the ψ_k-twisted Jacquet functors to the Langlands quotient $\mathcal{L}(1,\tau)$. By Theorem 2.2, the first interesting representation we get from $\mathcal{L}(1,\tau)$ is at $k = n-1$, i.e.

$$\sigma_{n-1} = \sigma_{n-1}(\tau) := \mathcal{J}_{\psi_{n-1}}(\mathcal{L}(1,\tau)), \tag{2.11}$$

which is an admissible representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$. We call σ_{n-1} to be the local descent of τ from GL_{2n} to SO_{2n+1}. The main result of this paper is

Theorem 2.3. *For an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model, the local descent σ_{n-1} is irreducible, generic and supercuspidal representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$.*

The proof of Theorem 2.3 takes §3, 4, and 5 below. In §3, we prove that the local descent σ_{n-1} as defined in (2.11) is quasi-supercuspidal, which means the (non-twisted) Jacquet module $\mathcal{J}_N(\sigma_{n-1})$ is trivial for the unipotent radical N of every standard proper parabolic group of SO_{2n+1}, see Theorem 3.1 for details. Hence we can write the local descent σ_{n-1} as a direct sum

$$\sigma_{n-1} = \sigma_{n-1}^1 \oplus \cdots \oplus \sigma_{n-1}^r \oplus \cdots ,$$

with σ_{n-1}^i's are irreducible supercuspidal representations of $\mathrm{SO}_{2n+1}(\mathcal{F})$. In §4, we show that the local descent σ_{n-1} has a nonzero Whittaker functional, which is unique up to a scalar (Part (2) of Theorem 4.1). Hence among the direct summands σ_{n-1}^i's, there exists one and only one irreducible summand has a nonzero Whittaker functional, i.e. it is generic. Finally, we prove in §5 that any irreducible supercuspidal summand in σ_{n-1} is in fact generic (Part (2) of Theorem 5.1). This implies that the local descent σ_{n-1} has only one irreducible summand, and therefore, σ_{n-1} is irreducible, generic, supercuspidal. Theorem 2.3 is proved.

3. Supercuspidality of the Local Descent

We start with the proof of the quasi-supercuspidality of

$$\sigma_{n-1} = \sigma_{n-1}(\tau) = \mathcal{J}_{\psi_{n-1}}(\mathcal{L}(1,\tau))$$

as defined in (2.11) for any irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model.

We relate any standard Jacquet module of σ_{n-1} to further descent σ_k of $\mathcal{L}(1,\tau)$ with $k \geq n$ in the tower of the local Gelfand-Graev models for the Langlands quotient $\mathcal{L}(1,\tau)$. Since $\mathcal{L}(1,\tau)$ has a nonzero generalized Shalika model, by Theorem 2,2, all standard Jacquet modules of σ_{n-1} must be zero. The same proof can be used to show that the local descents from $\mathcal{L}(1,\tau)$ satisfy the local tower property as in [GRS99], but we omit the details here.

First we have to fix notation. Consider the embedding of elements in SO_{2k-1} into SO_{2k} so that the embedding of unipotent elements are described explicitly.

Let $n = n(u,b,c)$ be an unipotent element of SO_{2k-1} of type

$$n = n(u,b,c) = \begin{pmatrix} u & b & c \\ & 1 & b' \\ & & u^* \end{pmatrix} \in \mathrm{SO}_{2k-1} \tag{3.1}$$

where $u \in \mathrm{U}_{k-1}$, which is the maximal upper triangular unipotent subgroup of GL_{k-1}. Then the embedding of n under the embedding from SO_{2k-1} into SO_{2k} is given by

$$n \mapsto \iota(n) = \begin{pmatrix} u & b & -b & c \\ & 1 & 0 & -b' \\ & & 1 & b' \\ & & & u^* \end{pmatrix} \in \mathrm{SO}_{2k}. \tag{3.2}$$

Theorem 3.1. *Let τ be an irreducible supercuspidal representation of $\mathrm{GL}_{2n}(\mathcal{F}), n \geq 2$ such that $L(s,\tau,\Lambda^2)$ has a pole at $s = 0$. Then $\sigma_{n-1} = \mathcal{J}_{\psi_{n-1}}(\mathcal{L}(1,\tau))$ is a quasi-supercuspidal representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$.*

Proof. For simplicity, we set $\sigma := \mathcal{L}(1,\tau)$, which is an admissible representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$. Denote by U_{n-1} be the maximal (upper triangular) unipotent subgroup of $\mathrm{GL}_{n-1}(\mathcal{F})$. Recall that N_{2n} is the unipotent radical of Siegel parabolic groups of SO_{4n}. For $x \in \mathcal{F}$, denote by $u_{i,j}(x)$ the unipotent matrix in SO_{4n} corresponding to $x(e_i - e_j)$, the x-multiple of root $e_i - e_j$ and let $U_{i,j} = \{u_{i,j}(x) | x \in \mathcal{F}\}$.

There are n unipotent radicals $Q_k, 1 \leq k \leq n$ corresponding to standard maximal parabolic subgroups of SO_{2n+1} given by

$$Q_k = \{ \begin{pmatrix} \mathrm{I}_k & C & D \\ & \mathrm{I}_{2n-2k+1} & C^* \\ & & \mathrm{I}_k \end{pmatrix} \} \subset \mathrm{SO}_{2n+1}.$$

Denote by ι the embedding of elements of SO_{2n+1} into SO_{2n+2} as in (3.2).

Let $H_1 = \iota(Q_k)N_{n-1}$, and denote its elements by

$$w(r,x,y,A,B) = \begin{pmatrix} r & & x & & y \\ & \begin{pmatrix} \mathrm{I}_k & A & B \\ & \mathrm{I}_{2n-2k+2} & A^* \\ & & \mathrm{I}_k \end{pmatrix} & & & x' \\ & & & & r^* \end{pmatrix}, r \in \mathrm{U}_{n-1}.$$

Write $r = (r_{i,j}), x = (x_{i,j}), \cdots$, etc.. Let ψ_{H_1} be the trivial extension of ψ_{n-1} to H_1, that is,

$$\psi_{H_1}(w(r,x,y,A,B)) = \psi(r_{1,2} + \cdots + r_{n-2,n-1})\psi(x_{n-1,n+1} + x_{n-1,n+2}).$$

To show that $\mathcal{J}_{\psi_{n-1}}(\sigma)$ is supercuspidal, it suffices to show the following:

$$\mathcal{J}_{\iota(Q_k)}(\mathcal{J}_{\psi_{n-1}}(\sigma)) = 0, \quad \text{for all} \quad 1 \leq k \leq n.$$

We begin by assuming on the contrast that

$$\mathcal{J}_{\iota(Q_k)}(\mathcal{J}_{\psi_{n-1}}(\sigma)) \neq 0, \quad \text{for some} \quad 1 \leq k \leq n.$$

Then there exists a non-zero functional Φ_1 on V_σ such that

$$\Phi_1(\sigma(g)v) = \psi_{H_1}(g)\Phi_1(v) \tag{3.3}$$

holds for $g \in H_1, v \in V_\sigma$.

Let H_2 be the compliment of $\prod_{i=1}^{n-1} U_{i,n}$ in H_1, and define a character ψ_{H_2} on H_2 by

$$\psi_{H_2} = \psi_{H_1}|_{H_2}.$$

Then

$$\Phi_1(\sigma(g)v) = \psi_{H_2}(g)\Phi_1(v) \tag{3.4}$$

holds for $g \in H_2, v \in V_\sigma$. Denote by η the permutation matrix in SO_{4n} corresponding to the permutation product of two cycles

$$(1, \cdots, n-1, n)(3n+1, \cdots, 4n).$$

Let $H_3 = \eta H_2 \eta^{-1}$ and $\psi_{H_3}(g) = \psi_{H_2}(\eta^{-1}g\eta), g \in H_3$. Now we have a nontrivial functional Φ_3 on V_σ such that

$$\Phi_3(\sigma(g)v) = \psi_{H_3}(g)\Phi_3(v) \tag{3.5}$$

holds for $g \in H_3, v \in V_\sigma$. Note that the functional Φ_3 is given by $\Phi_3(v) = \Phi_2(\eta v), v \in V_\sigma$.

Let H_4 be a subgroup of $H_3 \bigcap N_n$, consisting of elements in the form of

$$h = (h_{i,j}) = \begin{pmatrix} \mathrm{I}_n & (0_{n\times(k-1)}|*) & \overbrace{*}^{} \\ & \mathrm{I}_{2n} & \frac{*}{\underbrace{0}_{}} \\ & & \mathrm{I}_n \end{pmatrix}, \text{ with } h_{1,2n} = -h_{1,2n+1}.$$

Let $\psi_{H_4} = \psi_{H_3}|_{H_4}$. That is

$$\psi_{H_4}(h) = \psi(h_{n,2n} + h_{n,2n+1}).$$

Let $H_5 = U_{1,2n}H_4$ and ψ_{H_5} be the character of H_5 extending ψ_{H_4} with trivial value on $U_{1,2n}$. For $u_{1,2n}(x) \in U_{1,2n}$, the adjoint action $\mathrm{ad}(u_{1,2n}(x))$ preserves H_4 and ψ_{H_4}. Therefore there exists a character χ on $U_{1,2n}$ and a functional Φ_4 on V_σ such that

$$\Phi_4(\sigma(ug)v) = \chi(u)\psi_{H_4}(g)\Phi_4(v) \tag{3.6}$$

holds for $u \in U_{1,2n}, g \in H_4, v \in V_\sigma$.

Assume that $\chi(x) = \psi(ax)$ for some $a \in \mathcal{F}$. Note that

$$\mathrm{ad}(u_{n,1}(-a))u_{1,2n}(x) = u_{1,2n}(x)u_{n,2n}(-ax).$$

Moreover, $\mathrm{ad}(u_{n,1}(-a))$ preserves both H_4 and ψ_{H_4}. Define

$$\Phi_5(v) = \Phi_4(u_{n,1}(-a)v).$$

Then

$$\Phi_5(\sigma(g)v) = \psi_{H_5}(g)\Phi_5(v) \tag{3.7}$$

holds for $g \in H_5, v \in V_\sigma$.

Let $X_0 = H_5$ and $\psi^{(0)} = \psi_{H_5}$. For $1 \le m \le n$, let

$$X_m = U_{m,m+1} \cdots U_{m,n+k-1}$$

and write its elements by

$$X_m(\vec{x}) = \begin{pmatrix} r & & \\ & \mathrm{I}_2 & \\ & & r^* \end{pmatrix}, r = (r_{i,j}) \in \mathrm{U}_{2n-1}, \vec{x} \in \mathcal{F}^{n+k-m-1},$$

where the m-th row of r is $(0_{m-1}, 1, \vec{x}, 0_{n-k+1})$ and $r_{i,j} = \delta_{i,j}$, for $i \neq m$.

Let $\psi^{(m)}$ be the restriction of the character ψ_n of N_n to the subgroup $X_m \cdots X_1 H_5$.

For each $0 \le m \le n$, we claim in general that there exists a functional Φ_m on V_σ such that

$$\Phi_m(uv) = \psi^{(m)}(u)\Phi_m(v) \tag{3.8}$$

holds for $u \in X_m \cdots X_1 H_5, v \in V_\sigma$.

We proceed by induction. For $m = 0$, the claim is true because of Equation (3.7). Assume that the claim is true for $0 \le j-1 \le n-2$.

Note that X_j is abelian and $\mathrm{ad}(X_j(\vec{x}))$ preserves $X_{j-1} \cdots X_1 H_5$ and $\psi^{(j-1)}$. Therefore there exists a character χ_j on X_j such that

$$\Phi_{j-1}(\sigma(ug)v) = \chi_j(u)\psi^{(j-1)}(g)\Phi_{j-1}(v) \tag{3.9}$$

holds for $u \in X_j, g \in X_{j-1} \cdots X_1 H_5, v \in V_\sigma$.

Assume that

$$\chi_j(X_j(t_1, \cdots, t_{n+k-j-1})) = \psi(a_1 t_1 + \cdots + a_{n+k-j-1} t_{n+k-j-1}), a_i \in \mathcal{F}.$$

If $a_i = 0$ for all $1 \le i \le n+k-j-1$, then Equation (3.9) induces a nontrivial functional on V_τ, which is invariant under $\tau(u)$,

$$u \in \{\begin{pmatrix} \mathrm{I}_j & * \\ & \mathrm{I}_{2n-j} \end{pmatrix} \in |\mathrm{GL}_{2n}\}.$$

It contradicts to the supercuspidality of τ. Hence there exists nonzero a_i. Let

$$m_0 = \min\{1 \le i \le n+k-j-1 | a_i \neq 0\}.$$

Case (1): If $m_0 = 1$, define $\Phi_j(v) = \Phi_{j-1}(\tilde{\lambda} v)$, where $\tilde{\lambda} = \mathrm{diag}(\lambda, \lambda^*) \in \mathrm{SO}_{4n}$,

$$\lambda = \begin{pmatrix} \mathrm{I}_j & & & & & \\ & a_1 & & & & \\ & a_2 & 1 & & & \\ & \vdots & & \mathrm{I}_{n+k-j-3} & & \\ & a_{n+k-j-1} & & & 1 & \\ & & & & & \mathrm{I}_{n-k} \end{pmatrix} \in \mathrm{GL}_{2n}.$$

Note that

$$\mathrm{ad}(\tilde{\lambda}) X_j(t_1, \cdots, t_{n+k-j-1}) =$$
$$X_j(-a_1^{-1} t_1 - a_1^{-1} a_2 t_2 \cdots - a_1^{-1} a_{n+k-j-1} t_{n+k-j-1}, t_2, \cdots, t_{n+k-j-1}).$$

Moreover, $\mathrm{ad}(\tilde{\lambda})$ preserves both $X_{j-1} \cdots X_1 H_5$ and $\psi^{(j-1)}$. Hence

$$\Phi_j(\sigma(u)v) = \psi^{(j)}(u)\Phi_j(v), u \in X_j \cdots X_1 H_5. \tag{3.10}$$

Case (2): If $m_0 > 1$, take $\theta = u_{j+1,m_0}(1)$ and $\tilde{\theta} = \mathrm{diag}(\theta, \theta^*) \in \mathrm{SO}_{4n}$, and then define

$$\Phi_j''(v) = \Phi_{j-1}(\tilde{\theta} v).$$

Then, for $u \in X_j$, $g \in X_{j-1} \cdots X_1 H_5$, and $v \in V_\sigma$, the following

$$\Phi_j''(\sigma(ug)v) = \chi'(u)\psi^{(j-1)}(g)\Phi_j''(v)$$

holds for some character χ' on X_j satisfying

$$\chi'(X_j(x_1, \cdots, x_{n+k-j-1})) = \psi(b_1 x_1 + \cdots + b_{n+k-j-1} x_{n+k-j-1}),$$

with $b_1 \neq 0$. By repeating the same procedure as in the first case, again we reach the conclusion Equation (3.10).

By induction, we have shown that

$$\Phi_{n-1}(\sigma(u)v) = \psi^{(n-1)}(u)\Phi_{n-1}(v), u \in X_{n-1} \cdots X_1 H_5.$$

By similar argument, we also obtain that

$$\Phi_n'(\sigma(ug)v) = \chi''(u)\psi^{(n-1)}(g)\Phi_n'(v),$$

where $u \in X_n, g \in X_{n-1} \cdots X_1 H_5, v \in V_\sigma$, holds for some character χ'' on X_n satisfying

$$\chi''(X_n(x_1, \cdots, x_{k-1})) = \psi(d_1 x_1 + \cdots + d_{k-1} x_{k-1}).$$

Finally, we take $\Phi_n(v) = \Phi_n'(\mathrm{diag}(\gamma, \gamma^*)v), v \in V_\sigma$, where

$$\gamma = \begin{pmatrix} \mathrm{I}_n & & \\ & \mathrm{I}_{k-1} & \begin{pmatrix} 0, \cdots, d_1 \\ \vdots \\ 0, \cdots, d_{k-1} \end{pmatrix} \\ & & \mathrm{I}_{n-k+1} \end{pmatrix} \in \mathrm{GL}_{2n},$$

and obtain that

$$\Phi_n(\sigma(u)v) = \psi^{(n)}(u)\Phi_n(v) \tag{3.11}$$

holds for $u \in X_n \cdots X_1 H_5, v \in V_\sigma$.

Since $N_n = X_n \cdots X_1 H_5$, Equation (3.11) gives a nontrivial ψ_n-functional on V_σ. This conclusion contradicts to Theorem 2.2 that generalized Shalika models and (N_n, ψ_n)-models are disjoint. The assumption at the beginning must be false, so

$$\mathcal{J}_{\iota(Q_k)}(\mathcal{J}_{\psi_{n-1}}(\sigma)) = 0, \quad \text{for all } \ 1 \leq k \leq n$$

and $\mathcal{J}_{\psi_{n-1}}(\sigma)$ is quasi-supercuspidal. □

4. Genericity of the Local Descent

By Theorem 3.1, the local descent

$$\sigma_{n-1} = \sigma_{n-1}(\tau) = \mathcal{J}_{\psi_{n-1}}(\mathcal{L}(1,\tau))$$

as defined in (2.11) is a quasi-supercuspidal representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$. We may write

$$\sigma_{n-1} = \sigma^1_{n-1} \oplus \cdots \oplus \sigma^r_{n-1} \oplus \cdots ,$$

with σ^i_{n-1}'s are irreducible supercuspidal representations of $\mathrm{SO}_{2n+1}(\mathcal{F})$. Note that τ is an irreducible unitary supercuspidal representation of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model. We are going to prove Theorem 4.1, part (2) of which asserts that σ_{n-1} has a nonzero Whittaker functional, which is unique up to a scalar. In particular, σ_{n-1} is generic. It follows that among the summands σ^i_{n-1}'s, there exists one and only one (without multiplicity) irreducible summand which is generic.

In order to consider the Whittaker functional of $\sigma_{n-1} = \mathcal{J}_{\psi_{n-1}}(\mathcal{L}(1,\tau))$, we recall from (2.6) and (2.7) that

$$N_{n-1} = \{n(z,x,y) = \begin{pmatrix} z & x & y \\ & \mathrm{I}_{2n+2} & x' \\ & & z' \end{pmatrix} | z \in \mathrm{U}_{n-1}\} \subset \mathrm{SO}_{4n} \tag{4.1}$$

and the character ψ_{n-1} of N_{n-1} is given by

$$\psi_{n-1}(n(z,x,y)) = \psi(z_{1,2} + \cdots + z_{n-2,n-1})\psi(x_{n-1,n+1} + x_{n-1,n+2}).$$

Recall from (2.10) that the twisted Jacquet module $\sigma_{n-1} = \mathcal{J}_{\psi_{n-1}}(\mathcal{L}(1,\tau))$ is a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$. Let Z_k be the standard maximal unipotent subgroup of the split special orthogonal group SO_k consisting of upper-triangular matrix with 1 along the diagonal. That is

$$Z_{2n+1} = \{z(u,b,w) = \begin{pmatrix} u & b & w \\ & 1 & b' \\ & & u' \end{pmatrix} \in \mathrm{SO}_{2n+1} |\ u = (u_{i,j}) \in \mathrm{U}_n\}. \tag{4.2}$$

We may write $b = (b_1, \cdots, b_n)^t \in \mathcal{F}^n$. The Whittaker character $\psi_{Z_{2n+1}}$ of Z_{2n+1} is defined by

$$\psi_{Z_{2n+1}}(z(u,b,w)) = \psi(u_{1,2} + \cdots + u_{n-1,n} - b_n). \tag{4.3}$$

By the Frobenius reciprocity law, in order to show that σ_{n-1} has a nonzero Whittaker functional, it is enough to show that the following twisted Jacquet module

$$\mathcal{J}_{Z_{2n+1},\psi_{Z_{2n+1}}}(\sigma_{n-1}) = \mathcal{J}_{Z_{2n+1},\psi_{Z_{2n+1}}}(\mathcal{J}_{\psi_{n-1}}(\mathcal{L}(1,\tau)))$$

is nonzero.

To compose the two twisted Jacquet functors $\mathcal{J}_{Z_{2n+1},\psi_{Z_{2n+1}}}$ and $\mathcal{J}_{\psi_{n-1}}$, we set $E_1 = \tilde{\iota}(Z_{2n+1})N_{n-1}$ and let ψ_{E_1} be the character of E_1 defined by

$$\psi_{E_1}(vn) = \psi_{Z_{2n+1}}(v)\psi_{n-1}(n), \text{ for } v \in Z_{2n+1}, n \in N_{n-1},$$

where $\tilde{\iota} : \mathrm{SO}_{2k+1} \hookrightarrow \mathrm{SO}_{4n}$ is given by

$$g \in \mathrm{SO}_{2k+1} \mapsto \tilde{\iota}(g) = \begin{pmatrix} \mathrm{I}_{2n-k-1} & & \\ & \iota(g) & \\ & & \mathrm{I}_{2n-k-1} \end{pmatrix}$$

for any $k = 0, 1, \cdots, 2n-1$, and the embedding ι is defined in (3.2). Hence we have

$$\mathcal{J}_{E_1,\psi_{E_1}}(V_\pi) = \mathcal{J}_{Z_{2n+1},\psi_{Z_{2n+1}}} \circ \mathcal{J}_{\psi_{n-1}}(V_\pi)$$

for any irreducible admissible representation (π, V_π) of $\mathrm{SO}_{4n}(\mathcal{F})$.

We consider the maximal unipotent subgroup of SO_{4n} as defined in (2.6) with $k = 2n$, which is

$$N_{2n} = \{n(z, y) = \begin{pmatrix} z & y \\ & z' \end{pmatrix} | z \in \mathrm{U}_{2n}\}. \tag{4.4}$$

Define a degenerate character $\tilde{\psi}$ of N_{2n} by

$$\tilde{\psi}(n(z, y)) = \psi(z_{1,2} + \cdots + z_{2n-1,2n}).$$

We define the twisted Jacquet module $\mathcal{J}_{N_{2n},\tilde{\psi}}(V_\pi)$ for any irreducible admissible representation (π, V_π) of $\mathrm{SO}_{4n}(\mathcal{F})$.

The main result of this section is

Theorem 4.1. *Let π be an irreducible smooth representation of* SO_{4n}, *admitting a nonzero generalized Shalika model. Then the following hold.*

(1) *There exists an isomorphism as vector spaces between the two twisted Jacquet modules:*

$$\mathcal{J}_{E_1,\psi_{E_1}}(V_\pi) \simeq \mathcal{J}_{N_{2n},\tilde{\psi}}(V_\pi).$$

(2) *The local descent σ_{n-1} has a nonzero Whittaker functional, which is unique up to a scalar.*

Proof. The proof of Part (1) needs to use the local version of Fourier expansion for representations, in particular, the General Lemma in [GRS99], in many cases and will be carried out in Subsections 4.1–4.4.

We show here that Part (2) follows from Part (1). Take π to be $\mathcal{L}(1, \tau)$ and consider $\mathcal{J}_{N_{2n},\tilde{\psi}}(V_\pi) = \mathcal{J}_{N_{2n},\tilde{\psi}}(\mathcal{L}(1, \tau))$. We may write

$N_{2n} = \mathrm{U}_{2n} \ltimes V_{2n}$, where V_{2n} is the unipotent radical of the Siegel parabolic subgroup P_{2n} of SO_{4n} as defined in (2.3). Then we decompose the twisted Jacquet functor as

$$\mathcal{J}_{N_{2n},\tilde{\psi}} = \mathcal{J}^{\mathrm{GL}_{2n}}_{\mathrm{U}_{2n},\psi_{\mathrm{U}_{2n}}} \circ \mathcal{J}_{V_{2n}}$$

where $\mathcal{J}^{\mathrm{GL}_{2n}}_{\mathrm{U}_{2n},\psi_{\mathrm{U}_{2n}}}$ is the Whittaker functor of GL_{2n} and $\mathcal{J}_{V_{2n}}$ is the non-twisted Jacquet functor (i.e. the constant term functor along V_{2n}).

We first consider $\mathcal{J}_{V_{2n}}(\mathcal{L}(1,\tau))$. By the Geometric Lemma of Beinstein and Zelevinsky ([BZ77]), we obtain that

$$\mathcal{J}_{V_{2n}}(\mathcal{L}(1,\tau)) \simeq \tau \otimes |\det|^{-\frac{1}{2}}$$

as representations of $\mathrm{GL}_{2n}(\mathcal{F})$. By the local uniqueness of Whittaker model of τ, we obtain that the space

$$\mathcal{J}^{\mathrm{GL}_{2n}}_{\mathrm{U}_{2n},\psi_{\mathrm{U}_{2n}}} \circ \mathcal{J}_{V_{2n}}(\mathcal{L}(1,\tau))$$

is one-dimensional. Therefore, by Part (1), the space $\mathcal{J}_{E_1,\psi_{E_1}}(\mathcal{L}(1,\tau))$ is one-dimensional, in particular, the local descent σ_{n-1} has a unique Whittaker functional. □

4.1. We start to prove Part (1) of Theorem 4.1 by constructing a few intermediate twisted Jacquet modules relating both $\mathcal{J}_{E_1,\psi_{E_1}}(V_\pi)$ and $\mathcal{J}_{N_{2n},\tilde{\psi}}(V_\pi)$. The relations are explained in terms of the local versions of Fourier expansions for representations, which has been summarized as a General Lemma[1].

In this subsection and Subsection 4.2, we consider the general case when (π, V_π) is any smooth representation of $\mathrm{SO}_{4n}(\mathcal{F})$.

Let

$$C_1 = \{\tilde{\imath}(v)n | v \in Z_{2n+1}, n = n(z,x,y) \text{ such that } x_{n-1,1} = 0\}.$$

Denote by $\psi_{C_1} = \psi_{E_1}|_{C_1}$. For $i = 1, \cdots, n$, let

$$X_i = \{\begin{pmatrix} \mathrm{I}_{n-1} & x & 0 \\ & \mathrm{I}_{2n+2} & x' \\ & & \mathrm{I}_{n-1} \end{pmatrix} \in N_{n-1} | x_{s,t} \in \delta_{s,n-1}\delta_{t,i} \cdot \mathcal{F}\},$$

where $\delta_{i,j}$ is defined by that $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ if $i \neq j$. For $i = 1, \ldots, n-1$, set

$$Y_i = \{\mathrm{I}_{4n} + \lambda E_{n+i-1,2n+1} - \lambda E_{2n,3n+2-i} | \lambda \in \mathcal{F}\} \subset \mathrm{SO}_{4n},$$

[1]Since the general lemma in [GRS99] appears numerous times throughout this paper, we will refer to as "The General Lemma" for short.

where $E_{i,j} = (e_{k,l}), e_{k,l} = \delta_{k,i}\delta_{l,j}$, and set

$$Y_n = \{\begin{pmatrix} \mathrm{I}_{2n-2} & & \\ & h & \\ & & \mathrm{I}_{2n-2} \end{pmatrix} | h = \begin{pmatrix} 1 & x & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & -x \\ & & & 1 \end{pmatrix}\} \subset \mathrm{SO}_{4n}.$$

Then X_i, Y_i both normalize C_i and ψ^i.

Note that X_1 is the complement of C_1 in E_1, i.e. $E_1 = C_1 \rtimes X_1$. Let $D_1 = C_1 \rtimes Y_1$ and ψ_{D_1} be the trivial extension of ψ_{C_1} to D_1. This forms a setting which The General Lemma of [GRS99] applies. Hence we have

$$\mathcal{J}_{E_1,\psi_{E_1}}(V_\pi) \simeq \mathcal{J}_{D_1,\psi_{D_1}}(V_\pi).$$

For $i = 2, \ldots, n$, define a series of subgroups C_i of $Z_{2n+2}N_{n-1}$ by

$$C_i = \{vn | v = \begin{pmatrix} u & t & w \\ & \iota(h) & t' \\ & & u' \end{pmatrix} \in Z_{2n+2}, u \in \mathrm{U}_{i-1}, h \in Z_{2n+3-2i},$$

$$n = n(z,x,y) \in N_{n-1}, \text{ such that } x_{n-1,1} = x_{n-1,2} = \cdots = x_{n-1,i} = 0\},$$

where Z_{2n+2} is identified with its embedding in the middle diagonal part of SO_{4n}. Let ψ^i be the character of C_i defined by

$$\psi^i(vn) = \psi_{n-1}(n)\psi(u_{1,2} + \cdots + u_{i-2,i-1} + t_{i-1,1})\psi_{Z_{2n+3-2i}}(h).$$

The trivial extensions of ψ^i to $C_i \rtimes X_i$ and $C_i \rtimes Y_i$ are still denoted by ψ^i. Let $D_i := C_i \rtimes Y_i$. Then

$$D_{i-1} \simeq C_i \rtimes X_i, \quad (i = 2, \ldots n)$$

and the characters ψ^{i-1} and ψ^i of D_{i-1} are equal. Again, this forms the setting of The General Lemma, and we obtain

$$\mathcal{J}_{D_{i-1},\psi^{i-1}}(V_\pi) \simeq \mathcal{J}_{D_i,\psi^i}(V_\pi), \quad i = 2, \ldots, n.$$

Hence we obtain an isomorphism of vector spaces between the following two twisted Jacquet modules:

$$\mathcal{J}_{E_1,\psi_{E_1}}(V_\pi) \simeq \mathcal{J}_{D_n,\psi^n}(V_\pi).$$

Note that

$$D_n = \{\begin{pmatrix} z & y & w \\ & h & y' \\ & & z' \end{pmatrix} | h = \begin{pmatrix} 1 & \tilde{f} & -f & w \\ & 1 & 0 & f \\ & & 1 & -\tilde{f} \\ & & & 1 \end{pmatrix} \in Z_4,$$

$$z \in \mathrm{U}_{2n-2} \text{ with } z_{n-1,i} = 0, \text{ for } i \geq n\} \subset Z_{4n}.$$

Then we also have the following isomorphism of vector spaces:

$$\mathcal{J}_{D_n,\psi^n}(V_\pi) \simeq \mathcal{J}_{D_n,\psi_{D_n}}(V_\pi),$$

where the character ψ_{D_n} of D_n is given by

$$\psi_{D_n}(v) = \psi(z_{1,2} + z_{2,3} + \cdots + z_{2n-3,2n-2} + y_{n-1,2} + y_{n-1,3} - f).$$

4.2. Let ν be the permutation matrix in GL_{2n} given by

$$\begin{pmatrix} 1 & 2 & \dots & n-1 & n & n+1 & \dots & 2n-1 & 2n \\ 2 & 4 & \dots & 2(n-1) & 1 & 3 & \dots & 2n-1 & 2n \end{pmatrix},$$

and is identified with its embedding $m(\nu)$, where

$$m : g \in \mathrm{GL}_{2n} \mapsto \begin{pmatrix} g & \\ & g^* \end{pmatrix} \in \mathrm{SO}_{4n}.$$

Let $E = \nu\ D_n\ \nu^{-1}$, and define a character ψ_E of E by

$$\psi_E(n) := \psi_{D_n}(\nu^{-1}\ n\ \nu), \text{ for } n \in E.$$

Let $T(n)$ be the subgroup of GL_{2n} consisting of elements $t = (t_{i,j})$ satisfying

(1) $t_{i,i} = 1, 1 \leq i \leq 2n$;

(2) For $j \leq n-2$, $\bar{t}_{2j-1} = \begin{pmatrix} * \\ 0 \\ * \\ 0 \\ \vdots \\ * \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; $\bar{t}_{2n-3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; $\bar{t}_{2n-1} = 0$;

(3) For $j \leq n$, $\bar{t}_{2j} = (0, \cdots, 0)^t$;

(4) For $i \leq n$, $t_{2i-1} = \begin{pmatrix} 0 & * & 0 & * & \dots & * & 0 & * & * \end{pmatrix}$;

(5) $t_{2(n-1)} = (0, *)$,

where $\bar{t}_j = (t_{j+1,j}, \dots, t_{2n,j})^t$, $\quad t_i = (t_{i,i+1}, \dots, t_{i,2n})$ for $i, j \leq 2n-1\}$.
Then

$$E = \{n = \begin{pmatrix} t & X \\ & t' \end{pmatrix} | t \in T(n)\}$$

and the character ψ_E is given by

(4.5) $\psi_E(n) = \psi(t_{1,3} + t_{2,4} + \cdots + t_{2n-3,2n-1} + t_{2n-2,2n} + x_{2n-2,1} + x_{2n-1,1}).$

Example 4.2. *In the case of* $n = 4$,

$$T(4) = \{\begin{pmatrix} 1 & 0 & * & 0 & * & 0 & * & * \\ * & 1 & * & * & * & * & * & * \\ 0 & 0 & 1 & 0 & * & 0 & * & * \\ * & 0 & * & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\} \subset \mathrm{GL}_8.$$

Since $\psi_E(n) = \psi_{D_n}(\nu^{-1}\ n\ \nu)$ for all $n \in E$, we have the following isomorphism of vector spaces:

$$\mathcal{J}_{D_n,\psi_{D_n}}(V_\pi) \simeq \mathcal{J}_{E,\psi_E}(V_\pi).$$

Next, we will apply The General Lemma to "fill the zeroes of t_{2i-1} from the right to the left, using $\bar{t}_{2i-1}$". Thus "obtain E_2 from E and ψ_{E_2} from ψ_E".

Let

$$\begin{aligned} Y^{i,1} &:= \{m(\mathrm{I}_{2n} + yE_{2i,1}) | y \in \mathcal{F}\}, i = 1, \dots, n-2, \\ X^{1,j} &:= \{m(\mathrm{I}_{2n} + xE_{1,2j}) | x \in \mathcal{F}\}, j = 2, \dots, n-1, \\ E^{i,1} &:= \{n \in E | n_{j,1} = 0, \forall j > 2i\} \cdot \prod_{j=i+2}^{n} X^{1,j}, i \leq n-3, \\ E^{n-2,1} &:= E \\ C^{i,1} &:= \{n \in E^{i,1} | n_{2i,1} = 0\}, \\ D^{1,i+1} &:= C^{i,1} X^{1,i+1}, \\ A^{1,i+1} &:= D^{1,i+1} Y^{i,1}. \end{aligned}$$

Define a series of characters

$$\psi^{i,1} = \psi_E|_{C^{i,1}}.$$

Extend $\psi^{i,1}$ trivially to $D^{1,i+1}$ (and $E^{i,1}$ respectively), and denote the extension by $\psi^{i,1}_{D^{1,i+1}}$ (and $\psi^{i,1}_{E^{i,1}}$ respectively). Note that

$$D^{1,i+1} = E^{i-1,1}, \quad \psi^{i,1}_{D^{1,i+1}}|_{C^{i-1,1}} = \psi^{i-1,1}.$$

By The General Lemma, we conclude the following isomorphisms of vector spaces:

$$\mathcal{J}_{E^{i,1},\psi^{i,1}_{E^{i,1}}}(V_\pi) \simeq \mathcal{J}_{D^{1,i+1},\psi^{i,1}_{D^{1,i+1}}}(V_\pi) \simeq \mathcal{J}_{E^{i-1,1},\psi^{i-1,1}_{E^{i-1,1}}}(V_\pi),$$

for $i = n-2, \dots, 2$. In particular, we have the following isomorphism of vector spaces:

$$\mathcal{J}_{E,\psi_E}(V_\pi) \simeq \mathcal{J}_{D^{1,2},\psi^{1,1}_{D^{1,2}}}(V_\pi).$$

Note that the GL_{2n} part of $D^{1,2}$ looks like

$$\begin{pmatrix} \mathrm{I}_2 & * \\ 0 & T' \end{pmatrix}$$

with $T' \in T(n-1)$. Furthermore, let

$$Y^{r,s} = \{m(\mathrm{I}_{2n} + yE_{2r,2s-1}) | y \in \mathcal{F}\}, \text{ for } 1 \leq r, s \leq n-2$$

and let

$$X^{r,s} = \{m(\mathrm{I}_{2n} + xE_{2r-1,2s}) | x \in \mathcal{F}\}, \text{ for } 1 \leq r \leq n-2, 1 \leq s \leq n-1.$$

For $1 \leq j \leq i \leq n-2$, we define

$$E^{i,j} = \tilde{E}^{i,j} \prod_{s=i+2}^{n-1} X^{j,s},$$

where $\tilde{E}^{i,j} = \{\begin{pmatrix} t & X \\ & t' \end{pmatrix} \in \mathrm{SO}_{4n}\}$, among which t is defined by

$$t = \begin{pmatrix} \mathrm{I}_2 & & * & * \\ & \cdots & & \\ & & \mathrm{I}_2 & * \\ & & & Z \end{pmatrix}, Z \in T(n-j+1), t_{\ell,2j-1} = 0, \forall \ell > 2i.$$

We further define

$$\begin{aligned} C^{i,j} &= \{n = \begin{pmatrix} t & X \\ & t' \end{pmatrix} \in E^{i,j} | t_{2i,2j-1} = 0\}, \\ D^{j,i+1} &= C^{i,j} X^{j,i+1}, \\ A^{j,i+1} &= D^{j,i+1} Y^{i,j}. \end{aligned}$$

We define that $\psi^{i,j} = \psi_E|_{C^{i,j}}$. Note that $D^{j,i+1} \simeq A^{i-1,j}$ for $i \geq j+1$ and $D^{j,j+1} \simeq A^{n-1,j+1}$. The relations among those $\psi^{i,j}$ and their trivial extensions (denoted by $\psi^{i,j}_{D^{j,i+1}}$ and $\psi^{i,j}_{A^{i,j}}$ respectively) to $D^{j,i+1}$, $A^{i,j}$ are compatible in the sense of the requirement of The General Lemma. We conclude the following isomorphisms of vector spaces:

$$\begin{aligned} \mathcal{J}_{E,\psi}(V_\pi) &\simeq \mathcal{J}_{D^{1,2},\psi^{1,1}_{D^{1,2}}}(V_\pi) \simeq \cdots \simeq \mathcal{J}_{D^{j,j+1},\psi^{j,j}_{D^{j,j+1}}}(V_\pi) \\ &\simeq \cdots \simeq \mathcal{J}_{D^{n-2,n-1},\psi^{n-2,n-2}_{D^{n-2,n-1}}}(V_\pi). \end{aligned}$$

Denote by B_n the standard Borel subgroup of GL_n. The subgroup $D^{n-2,n-1}$ consists of elements of the following form

$$\begin{pmatrix} t & X \\ & t' \end{pmatrix} \in \mathrm{SO}_{4n}, \text{ with } t = \begin{pmatrix} \mathrm{I}_2 & y_1 & * & \cdots & * \\ & \mathrm{I}_2 & y_2 & \cdots & * \\ & & \cdots & & \\ & & & \mathrm{I}_2 & y_{n-1} \\ & & & & z \end{pmatrix},$$

where $y_1, \ldots, y_{n-2} \in \mathrm{Mat}_2$, $y_{n-1} \in \mathrm{B}_2$ and $z \in \mathrm{U}_2$. The character $\psi^{n-2,n-1}_{D^{n-2,n-1}}$ is given by

$$\psi^{n-2,n-2}_{D^{n-2,n-1}}(n) = \psi(\mathrm{tr}(y_1 + \cdots + y_{n-1}))\psi(x_{2n-2,1} + x_{2n-1,1}).$$

The detailed discussion above can be summarized as

Proposition 4.3. *Let π be a smooth representation of SO_{4n}. Then there exists an isomorphism of vector spaces between the two twisted Jacquet modules:*

$$\mathcal{J}_{E_1,\psi_{E_1}}(V_\pi) \simeq \mathcal{J}_{D^{n-2,n-1},\psi^{n-2,n-1}_{D^{n-2,n-2}}}(V_\pi).$$

Note that so far we only assume π to be a smooth representation of $\mathrm{SO}_{4n}(\mathcal{F})$.

4.3. The next step is to eliminate the character place $x_{2n-2,1}$ appearing in the above formula, and we need two auxiliary propositions: Proposition 4.4 and 4.8. To this end, we have to assume that V_π is an irreducible admissible representation of $\mathrm{SO}_{4n}(\mathcal{F})$ with a nonzero generalized Shalika model.

We define

(4.6)

$$D = \{\begin{pmatrix} T & X \\ & T' \end{pmatrix} | T = \begin{pmatrix} t_1 & z_1 & \ldots & \ldots & * \\ & t_2 & z_2 & \ldots & * \\ & & \cdots & & \cdots \\ & & & t_{n-1} & z_{n-1} \\ & & & & t_n \end{pmatrix}, t_i \in \mathrm{U}_2,\ z_i \in \mathrm{B}_2\},$$

and define a character ψ_D of D by

$$\psi_D(n) = \psi(\mathrm{tr}(z_1 + \cdots + z_{n-1}) + x_{2n-2,1} + x_{2n-1,1}).$$

Proposition 4.4. *Let π be an irreducible smooth representation of SO_{4n}, admitting a nonzero generalized Shalika model. Then there exists an isomorphism of vector spaces:*

$$\mathcal{J}_{D^{n-2,n-1},\psi^{n-1,n-1}_{D^{n-2,n-1}}}(V_\pi) \simeq \mathcal{J}_{D,\psi_D}(V_\pi).$$

Proof. After applying The General Lemma $n-2$ times, we have the following isomorphism of vector spaces:

$$\mathcal{J}_{D^{n-2,n-1},\psi^{n-1,n-1}_{D^{n-2,n-1}}}(V_\pi) \simeq \mathcal{J}_{H_1,\psi_{H_1}}(V_\pi),$$

where

$$H_1 = \{\begin{pmatrix} T & X \\ & T' \end{pmatrix} | T = \begin{pmatrix} \mathrm{I}_2 & z_1 & \dots & \dots & * \\ & t_2 & z_2 & \dots & * \\ & & \dots & \dots & \\ & & & t_{n-1} & z_{n-1} \\ & & & & t_n \end{pmatrix}, t_i \in \mathrm{U}_2,\ z_i \in \mathrm{B}_2\}$$

and ψ_{H_1} is defined by

$$\psi_{H_1}(n) = \psi(\mathrm{tr}(z_1 + \cdots + z_{n-1}) + x_{2n-2,1} + x_{2n-1,1}),\ n \in H_1.$$

Note that the following group

$$m(\{\left(\begin{array}{cc|c} 1 & * & 0 \\ & 1 & 0 \\ \hline & & \mathrm{I}_{2n-2} \end{array}\right)\}) \subset m(\mathrm{GL}_{2n}) \subset \mathrm{SO}_{4n}$$

normalizes H_1 and ψ_{H_1}.

For $\lambda \in \mathcal{F}^*$, define a character $\psi'_{D,\lambda}$ of D by

$$\psi'_{D,\lambda}(n) = \psi(\mathrm{tr}(z_1 + \cdots + z_{n-1}) + x_{n-2,1} + x_{n-1,1})\psi(\lambda t),$$

where $t_1 = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$ as in H_1. We will show in the following lemma that

$$\mathcal{J}_{D,\psi'_{D,\lambda}}(V_\pi) = 0 \text{ for all } \lambda \in \mathcal{F}^*.$$

Hence the only twisted Jacquet module remains is the one corresponding to $\lambda = 0$. In this case we have $\psi'_{D,0} = \psi_D$, and therefore, we have

$$\mathcal{J}_{E_1,\psi_{E_1}}(V_\pi) \simeq \mathcal{J}_{D,\psi_D}(V_\pi).$$

We reach the proof. □

Lemma 4.5. *Assume that π is an irreducible representation of* SO_{4n}*, admitting a nonzero generalized Shalika model. Then*

$$\mathcal{J}_{D,\psi'_{D,\lambda}}(V_\pi) = 0, \ \textit{for all } \lambda \in \mathcal{F}^*.$$

Proof. First we consider the case of $\lambda = 1$. Let $\psi'_D := \psi'_{D,1}$. Then for

$$n = \begin{pmatrix} T & X \\ & T' \end{pmatrix} \in D$$

we have

$$\psi'_D(n) = \psi(T_{1,2} + T_{1,3} + \sum_{i=2}^{n-2} T_{i,i+2})\psi(x_{2n-2,1} + x_{2n-1,1}).$$

Denote by

$$z_1 = \begin{pmatrix} Z & 0 \\ & \mathrm{I}_{2n-3} \end{pmatrix} \in \mathrm{GL}_{2n}, \text{ with } Z = \begin{pmatrix} 1 & & \\ & 1 & 0 \\ & 1 & 1 \end{pmatrix}.$$

Then z_1 normalizes D. Let $\psi_{D,1}$ be the character of D defined by
(4.7)

$$\psi_{D,1}(n) = \psi'_D(z_1\ n\ z_1^{-1}) = \psi(T_{1,2}+T_{2,4}+\sum_{i=2}^{n-2} T_{i,i+2})\psi(x_{2n-2,1}+x_{2n-1,1}).$$

It is clear that there exists an isomorphism of vector spaces between the following two twisted Jacquet modules:

$$\mathcal{J}_{D,\psi'_D}(V_\pi) \simeq \mathcal{J}_{D,\psi_{D,1}}(V_\pi). \tag{4.8}$$

For $i = 2, \ldots, n-1$, let $z_i = \mathrm{I}_{2n} + E_{2i+1,2i} \in \mathrm{GL}_{2n}$, and $\psi_{D,i}$ be the character of D defined by

$$\psi_{D,i} := \psi_{D,i-1}(z_i n z_i^{-1}).$$

Then we have
(4.9)

$$\psi_{D,i}(n) = \psi(T_{1,2}+T_{2i,2i+3}+\sum_{j=2}^{n-2} T_{j,j+2})\psi(x_{2n-2,1}+x_{2n-1,1}), 2 \le i \le n-2,$$

and

$$\psi_{D,n-1}(n) = \psi(T_{1,2} + \sum_{j=2}^{n-2} T_{j,j+2})\psi(x_{2n-1,1} + 2x_{2n-2,1}).$$

It is clear that

$$\mathcal{J}_{D,\psi_{D,i}}(V_\pi) \simeq \mathcal{J}_{D,\psi_{D,i+1}}(V_\pi), \quad i = 2, \ldots, n-2. \tag{4.10}$$

From (4.8), (4.10) we conclude that the following isomorphism of vector spaces

$$\mathcal{J}_{D,\psi'_D}(V_\pi) \simeq \mathcal{J}_{D,\psi_{D,n-1}}(V_\pi)$$

holds.

Now we assume on the contrary that

$$\mathcal{J}_{D,\psi_{D,n-1}}(V_\pi) \neq 0. \tag{4.11}$$

Then by Frobenius reciprocity law, there exists a nonzero functional ℓ on V_π such that

$$\ell(\pi(n)v) = \psi_{D,n-1}(n)\ell(v), \quad n \in D, v \in V_\pi. \tag{4.12}$$

Note that such a functional ℓ on V_π factors through the twisted Jacquet module $\mathcal{J}_{D,\psi_{D,n-1}}(V_\pi)$. Hence the nonvanishing of $\mathcal{J}_{D,\psi_{D,n-1}}(V_\pi)$ is equivalent to the nonvanishing of such ℓ.

Let μ be the permutation matrix in GL_{2n} given by

$$\begin{aligned} \mu(1) &= 1; \\ \mu(2i-2) &= i, \quad i = 2, \dots, n; \\ \mu(2i-1) &= n+i-1, \quad i = 2, \dots, n; \\ \mu(2n) &= 2n, \end{aligned}$$

and be identified with its embedding $m(\mu)$ in SO_{4n}. Denoted by Ni_k the set of nilpotent elements in GL_k. Then

$$F := \mu\ D\mu^{-1} = \{\begin{pmatrix} T & X \\ & T' \end{pmatrix} | T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

$$\alpha, \delta, \gamma \in \mathrm{B}_n \cap \mathrm{Ni}_n, \beta \in \mathrm{B}_n, \text{ and } \gamma_{i,i+1} = 0, \text{ for } i = 1, \dots, n-1\}.$$

Example 4.6. *When $n = 4$, those T in F is of the following form:*

$$\left(\begin{array}{cccc:cccc} 1 & * & * & * & * & * & * & * \\ 0 & 1 & * & * & 0 & * & * & * \\ 0 & 0 & 1 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & * \\ \hdashline 0 & 0 & * & * & 1 & * & * & * \\ 0 & 0 & 0 & * & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Let ψ_F be the character of F defined by

$$\psi_F(n) = \psi_{D,n-1}(\mu^{-1}\ n\ \mu) = \psi(\sum_{i=1, i\neq n}^{2n-2} T_{i,i+1} + T_{n,2n} + 2X_{n,1} + X_{2n-1,1}).$$

Define a linear functional on V_π by

$$\ell_F(v) = \ell(\pi(\mu^{-1})v), \quad v \in V_\pi.$$

Then ℓ_F is a nonzero functional ℓ_F on V_π satisfying

$$\ell_F(\pi(n)v) = \psi_F(n)\ell_F(v), \quad n \in F.$$

Since the functional ℓ_F factors through the twisted Jacquet module $\mathcal{J}_{F,\psi_F}(V_\pi)$, we obtain that $\mathcal{J}_{F,\psi_F}(V_\pi)$ must be nonzero.

Again, by The General Lemma, we get

$$\mathcal{J}_{F,\psi_F}(V_\pi) \simeq \mathcal{J}_{F',\psi_{F'}}(V_\pi),$$

where

$$F' = \{\begin{pmatrix} T & X \\ & T' \end{pmatrix} | T \in \mathrm{U}_{2n}, T_{n,n+i} = 0 \text{ for } i = 1, \dots, n-1\}$$

and the character $\psi_{F'}$ is given by

$$\psi_{F'}(n) = \psi(\sum_{i=1, i\neq n}^{2n-2} T_{i,i+1} + T_{n,2n} + 2X_{n,1} + X_{2n-1,1}). \tag{4.13}$$

Example 4.7. *Those T in F' is of the form*

$$\left(\begin{array}{cccc:cccc} 1 & * & * & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & * \\ \hdashline 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

(Compare this form with that in Example 4.6 to see how The General Lemma works.)

Since $\mathcal{J}_{F,\psi_F}(V_\pi) \simeq \mathcal{J}_{F',\psi_{F'}}(V_\pi) \neq 0$, there is a nonzero linear functional $\ell_{F'}$ on V_π such that

$$\ell_{F'}(\pi(n)\ v) = \psi_{F'}(n)\ell_{F'}(v), \quad n \in F'.$$

Next, we consider the intersection $F'_n := F' \cap N_n$. Then

(4.14)

$$F'_n = \{\begin{pmatrix} \alpha & \beta & x & y \\ & \mathrm{I}_n & 0 & x' \\ & & \mathrm{I}_n & \beta' \\ & & & \alpha' \end{pmatrix} | \alpha \in \mathrm{U}_n, \beta \in \mathrm{B}_n, \beta_{n,i} = 0, \text{ for } i = 1, \ldots, n-1\}.$$

and $\ell_{F'}$ is a nonzero linear functional on V_π such that

$$\ell_{F'}(\pi(n)\ v) = \psi_{F'}(n)\ell_{F'}(v), \quad n \in F'_n.$$

Note that F'_n is different from N_n on the entries of β of (4.14). (In F'_n, $\beta_{n,i} = 0$ for $i = 1, \ldots, n-1$.) Now we will apply the local version of Fourier expansion to "fill the zeroes of β".

Define a series of subgroups $F'_n \subset F'_{n-1} \subset \cdots \subset F'_1 = N_n$ as follows.

(4.15)

$$F'_i = \{\begin{pmatrix} \alpha & \beta & x & y \\ & \mathrm{I}_n & 0 & x' \\ & & \mathrm{I}_n & \beta' \\ & & & \alpha' \end{pmatrix} \in N_n | \alpha \in \mathrm{U}_n, \beta_{n,j} = 0, \text{ for } j = 1, \ldots, i-1\}.$$

Let $\psi_{F'_i}$ be the character of F'_i defined by the same formula of (4.13), more explicitly

$$\psi_{F'_i}(n) = \psi(\alpha_{1,2} + \cdots + \alpha_{n-1,n} + \beta_{n,2n} + 2x_{n,1}).$$

Now we use induction on reversed order. The case of $i = n$ is shown in (4.14). Assume for some $2 \le i \le n$ that we have a nonzero linear functional ℓ_i on V_π satisfying the following quasi-invariant property:

$$\ell_i(\pi(n)\ v) = \psi_{F'_i}(n)\ell_i(v), \quad n \in F'_i. \tag{4.16}$$

We show that the functional ℓ_{i-1} is an extension of ℓ_i with i replaced by $i - 1$.

Note that the root group of $e_n - e_{i-1}$ normalizes the character $\psi_{F'_i}$. There are two possibilities:

(1) The ℓ_i with $(F'_i, \psi_{F'_i})$-quasi-invariant property can be trivially extended to ℓ_{i-1} with $(F'_{i-1}, \psi_{F'_{i-1}})$-quasi-invariant property. In this case, we are done.

(2) If ℓ_i can be non-trivially extended to a nonzero linear functional ℓ'_{i-1} with $(F'_{i-1}, \psi_{F'_{i-1}})$-quasi-invariant property, such that

$$\ell'_{i-1}(\pi(n)\ v) = \tilde{\psi}_{F'_{i-1}}(n)\ell'_{i-1}(v), \quad n \in F'_{i-1},$$

then

$\tilde{\psi}_{F'_i}(n) = \psi(\alpha_{1,2} + \cdots + \alpha_{n-1,n} + \beta_{n,2n} + 2x_{n,1})\psi(c\ \beta_{n,i}),$

for some $c \in \mathcal{F}^*$. Let

$$z = \mathrm{I}_{2n} + \alpha\ E_{n+i,2n} \in \mathrm{GL}_{2n}.$$

Then we can choose a certain $\alpha \in \mathcal{F}^\times$ such that z normalizes F'_i and changes $\tilde{\psi}_{F'_{i-1}}$ back to the character $\psi_{F'_{i-1}}$. Hence we get (4.16) for ℓ_{i-1}.

By induction, we obtain a nonzero linear functional ℓ_1 on V_π, which factors through the twisted Jacquet module $\mathcal{J}_{N_n,\psi_n}(V_\pi)$.

By the assumption, the representation V_π has a nonzero generalized Shalika model. It follows from Theorem 2.2 that such a representation V_π has no nonzero twisted Jacquet module $\mathcal{J}_{N_n,\psi_n}(V_\pi)$. Hence the linear functional ℓ_1 must be zero.

Therefore, the assumption (4.11) must be wrong and the twisted Jacquet module $\mathcal{J}_{D,\psi_{D,n-1}}(V_\pi)$ must be zero. The proves the case when $\lambda = 1$.

If $\lambda \neq 1$, the conjugation by $m(a)$ with

$$a = \mathrm{diag}(\lambda^{-1}, 1, \lambda^{-1}, 1, \ldots, \lambda^{-1}, 1) \in \mathrm{GL}_{2n}$$

will give an isomorphism of vector spaces

$$\mathcal{J}_{D,\psi'_{D,\lambda}}(V_\pi) \simeq \mathcal{J}_{D,\psi_{D,\lambda}}(V_\pi),$$

where $\psi_{D,\lambda}$ is almost the same with the character of D defined in (4.7) except that the coefficient of $x_{2n-1,1}$ is λ^{-1}. In the proof of the case when $\lambda = 1$, we see that the coefficients of $x_{2n-1,1}$ and $x_{2n-2,1}$ play no roles and a similar argument applies. This completes the proof. $\square$

Proposition 4.8. *Let π be a smooth representation of* SO_{4n}. *Then*

$$\mathcal{J}_{D,\psi_D}(V_\pi) \simeq \mathcal{J}_{D,\tilde{\psi}_D}(V_\pi),$$

where $\tilde{\psi}$ is the character of D defined by (in the notation of (4.6)*)*

$$\tilde{\psi}_D(n) = \psi(\mathrm{tr}(z_1 + \cdots + z_{n-1}) + x_{2n-1,1}).$$

Proof. The proof of the proposition is almost the same as that of Lemma 4.5. We only give a sketch.

(1) Let $\bar{\mathrm{B}}_n$ denote the opposite standard Borel subgroup of GL_n. By The General Lemma, we see the following isomorphism of vector spaces:

$$\mathcal{J}_{D,\tilde{\psi}_D}(V_\pi) \simeq \mathcal{J}_{\tilde{D},\tilde{\psi}_{\tilde{D}}}(V_\pi),$$

where
(4.17)

$$\tilde{D} = \{\begin{pmatrix} T & X \\ & T' \end{pmatrix}, |T = \begin{pmatrix} 1 & * & * & \dots & & \dots & * \\ & t_1 & z_1 & * & \dots & & * \\ & & t_2 & z_2 & * & \dots & * \\ & & & \vdots & & \vdots & \\ & & & & t_{n-2} & z_{n-2} & * \\ & & & & & \mathrm{I}_2 & * \\ & & & & & & 1 \end{pmatrix}, t_i \in \mathrm{U}_2, z_i \in \bar{\mathrm{B}}_2\}$$

and the character $\tilde{\psi}_{\tilde{D}}$ of $\tilde{D}$ is given by

$$\tilde{\psi}_{\tilde{D}}(n) = \psi(\sum_{i_1}^{2n-2} T_{i,i+2})\psi(x_{2n-2,1} + x_{2n-1,1}).$$

(2) Let

$$z = \begin{pmatrix} \mathrm{I}_{2n-3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It normalizes $\tilde{D}$ and change $\tilde{\psi}_{\tilde{D}}$ to $\tilde{\psi}'_{\tilde{D}}$, defined by (in the notation of (4.17)):

$$\tilde{\psi}'_{\tilde{D}}(n) = \psi(\sum_{i_1}^{2n-2} T_{i,i+2})\psi(x_{2n-1,1}).$$

(3) Use The General Lemma to transfer $\bar{\mathrm{B}}_2$ into B_2 in (1), and the proposition follows. $\square$

4.4. Now, we are ready to prove Part (1) of Theorem 4.1. The proof is similar to that of Theorem 4.2.1 in [GRS99], with the help of the local version of Fourier expansions for representations. Let ν be the permutation matrix in GL_{4n} given by

$$\begin{cases} \nu_{i,2i-1} = 1, & \text{for i=1,\ldots, 2n;} \\ \nu_{2n+i,2i} = 1, & \text{for i=1,\ldots, 2n;} \\ \nu_{i,j} = 0, & \text{otherwise.} \end{cases}$$

Let $B = \nu D \nu^{-1}$, and define a character ψ_B of B by

$$\psi_B(e) = \tilde{\psi}_D(\nu^{-1} e \nu), \text{ for } e \in B.$$

Then the following isomorphism of vector spaces

$$\mathcal{J}_{D,\tilde{\psi}_D}(V_\pi) \simeq \mathcal{J}_{B,\psi_B}(V_\pi)$$

holds. Note that

$$B = \{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} | \alpha, \delta \in \mathrm{U}_{2n}, \beta \in \mathrm{B}_{2n}, \tag{4.18}$$

$$\gamma \in \mathrm{B}_{2n} \cap \mathrm{Ni}_{2n} \text{ and } \gamma_{i,i+1} = 0, \text{ for } i = 1, \ldots, 2n\},$$

and the character ψ_B is given by

$$\psi_B(e) = \psi(\alpha_{1,2} + \cdots + \alpha_{n,n+1} - \alpha_{n+1,n+2} - \cdots - \alpha_{2n-1,2n}).$$

Example 4.9. *For $n = 4$, elements in B are of the form*

$$\left(\begin{array}{cccccccc|cccccccc}
1 & \boxed{*} & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
 & 1 & \boxed{*} & * & * & * & * & * & 0 & * & * & * & * & * & * & * \\
 & & 1 & \boxed{*} & * & * & * & * & 0 & 0 & * & * & * & * & * & * \\
 & & & 1 & \boxed{*} & * & * & * & 0 & 0 & 0 & * & * & * & * & * \\
 & & & & 1 & \boxed{*} & * & * & 0 & 0 & 0 & 0 & * & * & * & * \\
 & & & & & 1 & \boxed{*} & * & 0 & 0 & 0 & 0 & 0 & * & * & * \\
 & & & & & & 1 & \boxed{*} & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
 & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
\hline
0 & 0 & * & * & * & * & * & * & 1 & * & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & 0 & 1 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 1 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 1 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 1 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right),$$

where the boxes indicate the nontrivial character position of ψ_B.

Our goal is to "fatten" β in (4.18), "using" the entries of γ, by successive applications of The General Lemma until we transfer $\mathcal{J}_{B,\psi_B}$ into $\mathcal{J}_{V_{2n},\tilde{\psi}}$. Let

$$\mathcal{X} = \{x \in \mathrm{Mat}_{2n}(\mathcal{F}) | \begin{pmatrix} \mathrm{I}_{2n} & x \\ & \mathrm{I}_{2n} \end{pmatrix} \in \mathrm{SO}_{4n}\}.$$

For $x \in \mathcal{X}$, write

$$\epsilon(x) = \begin{pmatrix} \mathrm{I}_{2n} & x \\ & \mathrm{I}_{2n} \end{pmatrix}, \quad \overline{\epsilon}(x) = \begin{pmatrix} \mathrm{I}_{2n} & 0 \\ x & \mathrm{I}_{2n} \end{pmatrix}.$$

For a subspace $S \subset \mathcal{X}$, define

$$\epsilon(S) = \{\epsilon(x) |\ x \in S\}, \quad \overline{\epsilon}(S) = \{\overline{\epsilon}(x) |\ x \in S\}.$$

Put

$$\begin{aligned} \mathcal{X}_0 &= \{x \in \mathcal{X} |\ x \in \mathrm{B}_{2n}\}, \\ \mathcal{Y}_0 &= \{x \in \mathcal{X} |\ x \in \mathrm{B}_{2n} \cap \mathrm{Ni}_{2n}, x_{i,i+1} = 0, i = 1, \ldots, n-1\}. \end{aligned}$$

For $1 \leq i < j-1$, define

$$\begin{aligned} \mathcal{Y}_{i,j} &= \{x \in \mathcal{X}_0 |\ x_{r,l} = 0 \text{ for } r, l < j-1 \text{ and } x_{r,j} = 0 \text{ for } r \geq i\}, \\ \mathcal{Y}^{i,j} &= \mathrm{I} + \mathcal{F}(E_{i,j} - E_{2n+1-j,2n+1-i}). \end{aligned}$$

Then elements in B can be written in the form

$$v = \epsilon(x) m(z) \overline{\epsilon}(y) \tag{4.19}$$

with $x \in \mathcal{X}_0, y \in \mathcal{Y}_0$ and $z \in \mathrm{U}_{2n}$. Let

$$\mathcal{Y}_{1,3} = \{x \in \mathcal{X}_0 | x_{1,3} = 0\}.$$

Let $C^{1,3}$ be the subgroup of the form (4.19) such that $y \in \mathcal{Y}_{1,3}$. Thus

$$C^{1,3} = \epsilon(\mathcal{X}_0) m(\mathrm{U}_{2n}) \overline{\epsilon}(\mathcal{Y}_{13}).$$

Let

$$Y^{1,3} = \overline{\epsilon}(\mathcal{Y}^{1,3}).$$

Denote by $X^{2,1} = \epsilon(\mathcal{X}^{2,1})$, where $\mathcal{X}^{2,1} = \mathcal{F}(e_{2,1} - e_{2n,2n-1})$. Let

$$\psi_B^{1,3} = \psi_B|_{C^{1,3}}, \quad B^{1,3} = B, \quad D^{1,3} = C^{1,3} X^{2,1}$$

Put

$$\mathcal{X}_{2,1} = \mathcal{X}_0 \oplus \mathcal{X}^{2,1}.$$

Then

$$D^{1,3} = \epsilon(\mathcal{X}_{2,1}) m(\mathrm{U}_{2n}) \overline{\epsilon}(\mathcal{Y}_{1,3}).$$

By The General Lemma, we conclude that

$$\mathcal{J}_{B^{1,3},\psi^{1,3}_{B^{1,3}}}(V_\pi) \simeq \mathcal{J}_{D^{1,3},\psi^{1,3}_{D^{1,3}}}(V_\pi),$$

where $\psi^{1,3}_{D^{1,3}}$ is the character of $D^{1,3}$ which is trivial on $\epsilon(\mathcal{X}_{2,1}) \cdot \overline{\mathcal{Y}}_{1,3}$.

Define

$$\mathcal{X}^{r,s} = \mathrm{I} + \mathcal{F}(E_{r,s} - E_{2n+1-s,2n+1-r}), \text{ for } 1 \le s < r \le 2n.$$

Let

$$\mathcal{X}_{r,s} = \mathcal{X}_0 \oplus (\bigoplus_{q<l\le r-1} \mathcal{X}^{l,q}) \oplus (\bigoplus_{q=s}^{r-1} \mathcal{X}^{r,q}), \text{ for } 1 \le s < r \le n.$$

For $1 \le i < j-1, j \le n+1$, let

$$C^{i,j} = \epsilon(\mathcal{X}_{j-1,i+1}) m(\mathrm{U}_{2n}) \overline{\epsilon}(\mathcal{Y}_{i,j}), \text{ if } i+1 \le j-1.$$

For $1 \le i < j \le n+1$, we define $Y^{i,j} = \overline{\epsilon}(\mathcal{Y}^{i,j})$ and $X^{j,i} = \epsilon(\mathcal{X}^{j,i})$, and also define

$$B^{i,j} = C^{i,j} Y^{i,j}, \quad D^{i,j} = C^{i,j} X^{j-1,i}, \quad A^{i,j} = D^{i,j} Y^{i,j}.$$

Let $\psi^{i,j}$ be the character of $C^{i,j}$, which is trivial on $\epsilon(\mathcal{X}_{j-1,i+1}) \cdot \overline{\epsilon}(\mathcal{Y}_{i,j})$. Then by The General Lemma, we have the following isomorphism of vector spaces:

$$\mathcal{J}_{B^{i,j}, \psi^{i,j}_{B^{i,j}}}(V_\pi) \simeq \mathcal{J}_{D^{i,j}, \psi^{i,j}_{D^{i,j}}}(V_\pi).$$

for all $1 \le i < j-1, j \le n+1$.

Note that for $2 \le i < j-1, j \le n+1$, we have

$$D^{i,j} = B^{i-1,j} \text{ and } \psi^{i,j}_{D^{i,j}} = \psi^{i-1,j}_{B^{i-1,j}},$$

and for $j = 3, \ldots, n+1$, we have

$$D^{1,j} = B^{j-1,j+1} \text{ and } \psi^{1,j}_{D^{1,j}} = \psi^{j-1,j+1}_{B^{j-1,j+1}}.$$

We conclude by The General Lemma again that

$$\mathcal{J}_{B^{1,3}, \psi^{1,3}_{B^{1,3}}}(V_\pi) \simeq \mathcal{J}_{D^{1,n+1}, \psi^{1,n+1}_{D^{1,n+1}}}(V_\pi) \tag{4.20}$$

as vector spaces. Note that

$$D^{1,n+1} = \epsilon(\mathcal{X}_{n,1}) m(\mathrm{U}_{2n}) \overline{\epsilon}(\mathcal{Y}_{1,n+1}).$$

We remark that so far in this proof, we have not used any particular property of V_π. In the following we are going to use the property that V_π has a nonzero generalized Shalika model.

For $n+1 \le r \le 2n-1$ and $1 \le s \le 2n-r$, define

$$\mathcal{X}_{r,s} = \mathcal{X}_{n,1} \oplus (\bigoplus_{n+1 \le l \le r-1, 1 \le q \le 2n-l} \mathcal{X}_{l,q}) \oplus (\bigoplus_{q=s}^{2n-r} \mathcal{X}^{r,q}).$$

Then $X^{n+1,n-1}$ normalizes $D^{1,n+1}$ and $\psi^{1,n+1}_{D^{1,n+1}}$. We consider the action of $X^{n+1,n-1}$ on the right hand side of (4.20), and claim that for any nontrivial character ξ of $X^{n+1,n-1}$,

$$\mathcal{J}_{X^{n+1,n-1}, \xi}(\mathcal{J}_{D^{1,n+1}, \psi^{1,n+1}_{D^{1,n+1}}}(V_\pi)) = 0.$$

Hence we must have the trivial character when $X^{n+1,n-1}$ acts on the right hand side of (4.20).

To justify our claim, we assume on the contrary, by the Frobenius reciprocity law, that there exists ℓ a non-zero linear functional on V_π such that

$$\ell(\pi(xn)v) = \psi_{1,n+1}^{1,n+1}(n)\xi(x)\ell(v),$$

for all $x \in X^{n+1,n-1}, n \in D^{1,n+1}$ and $v \in V_\pi$.

We may assume that there is a $\lambda \in \mathcal{F}^*$ such that

$$\xi(x(t)) = \psi(\lambda\ t),$$

where $x(t) = \mathrm{I}_{4n} + t(E_{n+1,3n-1} - E_{n+2,3n})$. Then ℓ is a nonzero linear functional on V_π such that

$$\ell(\pi(n)v) = \psi_{D^{1,n+1}}^{1,n+1}(n)\ell(v),$$

for $n \in X^{n+1,n-1}D^{1,n+1} \cap N_{n+1}$. Note that $X^{n+1,n-1}D^{1,n+1} \cap N_{n+1}$ consists of elements of the following form

$$\begin{pmatrix} z & y & w \\ & \mathrm{I}_{2n-2} & y' \\ & & z' \end{pmatrix} \in \mathrm{SO}_{4n}, \tag{4.21}$$

with $z \in \mathrm{U}_{n+1}$, $y \in \mathrm{Mat}_{n+1,2n-2}$ such that $y_{n+1,n+i} = 0$ for $i = 1, \ldots, n-1$.

Now the situation is similar to that of (4.14). The same argument shows that ℓ can be extended trivially to N_{n+1} such that

$$\ell(\pi(n)v) = \psi_{N_{n+1}}^{1,n+1}(n)\ell(v), \quad \text{for } n \in N_{n+1},$$

where $\psi_{N_{n+1}}^{1,n+1}$ is the trivially extension of restriction of $\psi_{D^{1,n+1}}^{1,n+1}$ on $D^{1,n+1} \cap N_{n+1}$.

Note that for element $n \in N_{n+1}$ of the form (4.21)

$$\psi_{D^{1,n+1}}^{1,n+1}(n) = \psi(z_{1,2} + \cdots + z_{n,n+1})\psi(y_{n+1,1} + y_{n+1,2n-2})$$

Let ν' be the permutation matrix in GL_{2n} defined by

$$\begin{cases} \nu'(i) = i, & i = 1, \ldots, n+1, \\ \nu'(n+2) = 2n, \\ \nu'(n+2+i) = n+1+i, & i = 1, \ldots, n-2, \end{cases}$$

which is identified with its embedding $m(\nu')$ in SO_{4n}. Then ν' normalizes N_{n+1} and transfer $\psi_{N_{n+1}}^{1,n+1}$ into ψ_{n+1}. Hence we obtain a nonzero linear functional which factors through the twisted Jacquet module $\mathcal{J}_{\psi_{n+1}}(V_\pi)$. In particular, we have

$$\mathcal{J}_{\psi_{n+1}}(V_\pi) \neq 0.$$

On the other hand, V_π has a nonzero generalized Shalika model by assumption. Following Theorem 2.2, the twisted Jacquet module $\mathcal{J}_{\psi_{n+1}}(V_\pi)$ must be zero. We get a contradiction. Hence $X^{n+1,n-1}$ must act trivially on $\mathcal{J}_{D^{1,n+1},\psi^{1,n+1}_{D^{1,n+1}}}(V_\pi)$.

Next we continue this process. Define

$$B^{n-2,n+2} = D^{1,n+1}X^{n+1,n-1},$$

and extend $\psi^{1,n+1}_{D^{1,n+1}}$ to a character $\psi^{n-2,n+2}_{B^{n-2,n+2}}$ on $B^{n-2,n+2}$ by making it trivial on $X^{n+1,n-1}$. Thus we have

$$\mathcal{J}_{B^{n-2,n+2},\psi^{n-2,n+2}_{B^{n-2,n+2}}}(V_\pi) \simeq \mathcal{J}_{D^{1,n+1},\psi^{1,n+1}_{D^{1,n+1}}}(V_\pi).$$

Now we can repeat the argument as before, by "replacing the $n-2$ coordinates of $\oplus_{i=1}^{n-2}\mathcal{Y}_{i,n+2}$ with $\oplus_{i=1}^{n-2}\mathcal{X}_{n+1,i}$". For $1 \le i \le n-2$ and $j \ge n+2$, define $C^{i,j} = \epsilon(\mathcal{X}_{j-1,i+1})m(\mathrm{U}_{2n})\bar{\epsilon}(\mathcal{Y}_{i,j})$ and

$$B^{i,j} = C^{i,j}Y^{i,j}, \quad D^{i,j} = C^{i,j}X^{j-1,i}, \quad A^{i,j} = D^{i,j}Y^{i,j}.$$

Let $\psi^{i,n+2}$ be the character of $C^{i,n+2}$, which is trivial on $\ell(C_{n+1,i+1})\bar{\ell}(Y_{i,n+2})$. By The General Lemma, we conclude that

$$\mathcal{J}_{D^{1,n+1},\psi^{1,n+1}_{D^{1,n+1}}}(V_\pi) \simeq \mathcal{J}_{D^{1,n+2},\psi^{1,n+2}_{D^{1,n+2}}}(V_\pi) \tag{4.22}$$

as vector spaces. Then, by using the property that V_π has a nonzero generalized Shalika model, we show that $X^{n+2,n-2}$ acts trivially on the right hand side of (4.22). As before, we get

$$\mathcal{J}_{D^{1,n+2},\psi^{1,n+2}_{D^{1,n+2}}}(V_\pi) \simeq \cdots \simeq \mathcal{J}_{D^{1,2n-1},\psi^{1,2n-1}_{D^{1,2n-1}}}(V_\pi)$$

as vector spaces. Note that $D^{1,2n-1} = N_{2n}$ and $\psi^{1,2n-1}_{1,2n-1} = \tilde{\psi}$. We conclude that

$$\mathcal{J}_{D^{1,2n-1},\psi^{1,2n-1}_{D^{1,2n-1}}}(V_\pi) = \mathcal{J}_{N_{2n},\tilde{\psi}}(V_\pi).$$

This completes the proof of Part (1) of Theorem 4.1.

5. Irreducibility of the Local Descent

In order to finish the proof of Theorem 2.3, it remains to show that σ_{n-1} is irreducible. From Sections 3 and 4, we proved that the local descent (as defined in (2.11))

$$\sigma_{n-1} = \mathcal{J}_{\psi_{n-1}}(\mathcal{L}(1,\tau)),$$

as representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$, is quasi-supercuspidal and has a unique nonzero Whittaker functional. Hence it is enough to show that any irreducible summand of σ_{n-1} is generic, i.e. has a nonzero Whittaker functional. This is proved in Part (2) of Theorem 5.1 below. The proof of Theorem 5.1 is standard, which may be viewed as a generalization of the Geometric Lemma of Bernstein and Zelevinsky ([BZ77]) for the twisted Jacquet functor $\mathcal{J}_{\psi_{n-1}}$ applied to $\mathcal{L}(1,\tau)$. See [GRS99] for a similar discussion for the metaplectic and symplectic groups.

For a given irreducible supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, recall that $\mathrm{I}(s,\tau)$ is the induced representation of $\mathrm{SO}_{4n}(\mathcal{F})$ from the supercuspidal datum (P_{2n},τ) as defined in Subsection 2.1. The unique Langlands quotient of $\mathrm{I}(s,\tau)$ at $s=1$ is $\mathcal{L}(1,\tau)$.

Theorem 5.1. *Let (V_σ,σ) be an irreducible supercuspidal representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$.*

(1) *If, for any $s\in\mathbb{C}$, the space*

$$\mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}_{\psi_{n-1}}(\mathrm{I}(s,\tau)),V_\sigma)$$

is nonzero, then σ is generic.

(2) *If the space*

$$\mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}_{\psi_{n-1}}(\mathcal{L}(1,\tau)),V_\sigma)$$

is nonzero, then σ is generic.

It is clear that in Theorem 5.1, Part (2) follows from Part (1) by the exactness of the twisted Jacquet functors. Part (1) is proved Subsection 5.2 below.

We start with an investigation of the structure of the twisted Jacquet module $\mathcal{J}_{\psi_{n-1}}(\mathrm{I}(s,\tau))$ with aim at the genericity of σ. We realize the irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ in its Whittaker model $\mathcal{W}(\tau,\psi)$, and realize the induced representation $\mathrm{I}(s,\tau)$ as $\mathrm{I}(s,\mathcal{W}(\tau,\psi))$. Then we consider the twisted Jacquet module $\mathcal{J}_{\psi_{n-1}}(\mathrm{I}(s,\mathcal{W}(\tau,\psi)))$.

5.1. The twisted Jacquet module $\mathcal{J}_{\psi_{n-1}}(\mathrm{I}(s,\mathcal{W}(\tau,\psi)))$. We consider first the orbital structure of the closed subgroup $\mathrm{SO}_{2n+2}\cdot N_{n-1}$ acting on the generalized flag variety $P_{2n}\backslash\mathrm{SO}_{4n}$ over the p-adic field $\mathcal{F}$, and then consider the semisimplification of the twisted Jacquet module $\mathcal{J}_{\psi_{n-1}}(\mathrm{I}(s,\mathcal{W}(\tau,\psi)))$ as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$.

For $j=1,\dots,2n$, let

$$P_j=\{\begin{pmatrix} h & * & * \\ & g & * \\ & & h^* \end{pmatrix} | h\in\mathrm{GL}_j, g\in\mathrm{SO}_{4n-2j}\}$$

be the standard maximal parabolic subgroup of SO_{4n}. Then the generalized Bruhat decomposition

$$P_{2n}\backslash\mathrm{SO}_{4n}/P_{n-1}$$

can have a complete set of representatives given by $\{\gamma_i \mid i \in 2\mathbb{N}, n \leq i \leq 2n\}$, where for $i \in 2\mathbb{N}, n \leq i \leq 2n$,

$$\gamma_i = \begin{pmatrix} & & \nu_{2n-i} \\ & \mathrm{I}_{2i} & \\ \nu_{2n-i} & & \end{pmatrix},$$

where ν_j is defined in (2.2). For $k = 0, 1, \ldots, n-1$, let M_k be the standard maximal parabolic subgroup of GL_{n-1} corresponding to the partition $(k, n-k-1)$ of $n-1$ such that the Levi part is $\mathrm{GL}_k \times \mathrm{GL}_{n-k-1}$ and the unipotent radical is

$$L_k = \{\begin{pmatrix} \mathrm{I}_k & \\ A & \mathrm{I}_{n-1-k} \end{pmatrix} \in \mathrm{GL}_{n-1} \mid A \in \mathrm{Mat}_{n-1-k,k}\}.$$

Lemma 5.2. *The orbits of the closed subgroup* $\mathrm{SO}_{2n+2} \cdot N_{n-1}$ *acting on the generalized flag variety* $P_{2n}\backslash\mathrm{SO}_{4n}$ *are represented by elements of the form* $\gamma_i w$*, where* $n \leq i \leq 2n$ *is even and* w*'s are elements of* $W(\mathrm{GL}_{n-1})$ *given by:*

$$\begin{cases} w \in [W(\mathrm{GL}_{2n-i}) \times W(\mathrm{GL}_{i-n-1})]\backslash W(\mathrm{GL}_{n-1}), & \text{if } i \neq n; \\ w = \mathrm{id}, & \text{if } i = n. \end{cases}$$

Here $W(\mathrm{GL}_m)$ *denotes the Weyl group of* GL_m*.*

Proof. Clearly, we have $\mathrm{SO}_{2n+2}N_{n-1} \subset P_{n-1}$. Hence we can choose $\gamma_i w$ to be the representative of any double cosets in $P_{2n}\backslash\mathrm{SO}_{4n}/[\mathrm{SO}_{2n+2}N_{n-1}]$, for some

$$w \in [\gamma_i^{-1}P_{2n}\gamma_i \cap P_{n-1}]\backslash P_{n-1}/[\mathrm{SO}_{2n+2}N_{n-1}].$$

Since $M_{2n-i} \subset \gamma_i^{-1}P_{2n}\gamma_i \cap P_{n-1}$, we may choose a set of representatives for $[\gamma_i^{-1}P_{2n}\gamma_i \cap P_{n-1}]\backslash P_{n-1}/[\mathrm{SO}_{2n+2}N_{n-1}]$ from $M_{2n-i}\backslash\mathrm{GL}_{n-1}/N_{n-1}$. Then a complete set of representatives for $M_{2n-i}\backslash\mathrm{GL}_{n-1}/N_{n-1}$ can be chosen from $[W(\mathrm{GL}_{2n-i}) \times W(\mathrm{GL}_{i-n-1})]\backslash W(\mathrm{GL}_{n-1})$. □

Let $\alpha_1, \ldots, \alpha_{n-2}$ denote the simple roots of GL_{n-1} with respect to N_{n-1}. Let

$$\{x_{\alpha_j}(t) = \mathrm{I}_{n-1} + tE_{j,j+1} | t \in \mathcal{F}\}$$

denote the one parameter unipotent subgroup of N_{n-1} corresponding to the root α. We will take $w = \mathrm{id}$ to be the representative of the coset $W(\mathrm{GL}_k) \times W(\mathrm{GL}_{n-1-k})$ in $W(\mathrm{GL}_{n-1})$.

Lemma 5.3 (Lemma 4.3, [GRS99])**.** *If a Weyl group element*

$$w \in [W(\mathrm{GL}_k) \times W(\mathrm{GL}_{n-1-k})]\backslash W(\mathrm{GL}_{n-1}),$$

which is not id*, then there exists a simple root* α_j *such that* $wx_{\alpha_j}(t)w^{-1} \in L_k, \forall\ t \in \mathcal{F}$*.*

Next we consider the semisimplification of the twisted Jacquet module $\mathcal{J}_{\psi_{n-1}}(\mathrm{I}(s,\mathcal{W}(\tau,\psi)))$ as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$. It is a standard process to decompose the representation as spaces of functions on $\mathrm{SO}_{4n}(\mathcal{F})$ according the orbital decomposition obtained in Lemma 5.2.

It is clear that among the orbits

$$\mathcal{O}_{i,w} = [P_{2n}]\gamma_i w[\mathrm{SO}_{2n+2}N_{n-1}],$$

for $i \in 2\mathbb{N}, n \leq i \leq 2n$, the orbit $\mathcal{O}_{2[\frac{n+1}{2}],\mathrm{id}}$ is the unique open orbit. Let E be a union of orbits $\mathcal{O}_{i,\omega}$. We denote by $\mathcal{S}(E,\tau_s)$ the space of smooth functions ϕ on E, which are compactly supported modulo P_{2n} and have values in the Whittaker model $\mathcal{W}(\tau,\psi)$, such that

$$\phi(\begin{pmatrix} a & * \\ & a^* \end{pmatrix} g, r) = |\det a|^{\frac{s}{2}+n-\frac{1}{2}}\phi(g, ra),$$

for $g \in \mathrm{SO}_{4n}$, and $a, r \in \mathrm{GL}_{2n}$. We may arrange the orbits in a sequence

$$P_{2n}\mathrm{SO}_{2n+2}N_{n-1} = \Omega_1,\ \ldots,\ \Omega_l = \mathcal{O}_{2[\frac{n+1}{2}],\mathrm{id}}$$

such that $F_i = \cup_{j=1}^{i}\Omega_j$ is closed in SO_{4n}. It is clear that Ω_i is open in F_i and F_{i-1} is closed in F_i. We obtain the following exact sequence

$$0 \to \mathcal{S}(\Omega_{i+1},\tau_s) \xrightarrow{e} \mathcal{S}(F_{i+1},\tau_s) \xrightarrow{r} \mathcal{S}(F_i,\tau_s) \to 0, \tag{5.1}$$

where the map e is the natural embedding and r is the restriction to F_i. Apply the twisted Jacquet functor $\mathcal{J}_{\psi_{n-1}}$ to the exact sequence (5.1). Since the Jacquet functors are exact, we obtain another exact sequence

$$0 \to \mathcal{J}_{\psi_{n-1}}(\mathcal{S}(\Omega_{i+1},\tau_s)) \to \mathcal{J}_{\psi_{n-1}}(\mathcal{S}(F_{i+1},\tau_s)) \to \mathcal{J}_{\psi_{n-1}}(\mathcal{S}(F_i,\tau_s)) \to 0.$$

We obtain the semisimplification of $\mathcal{J}_{\psi_{n-1}}(\mathrm{I}(s,\mathcal{W}(\tau,\psi)))$ as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$ as

$$\oplus_{i=1}^{l}\mathcal{J}_{\psi_{n-1}}(\mathcal{S}(\Omega_i,\tau_s)).$$

In the following, we are going to study the space $\mathcal{J}_{\psi_{n-1}}(\mathcal{S}(\Omega_i,\tau_s))$ for $i = 1, 2, \cdots, l$. Also we assume, for the rest of this section, unless state otherwise, that all inductions are un-normalized.

As $\mathrm{SO}_{2n+2}N_{n-1}$ module, we have

$$\mathcal{S}(\mathcal{O}_{i,w},\tau_s) \simeq \mathrm{c}-\mathrm{Ind}_{P_{2n}^{\gamma_{i,w}}}^{\mathrm{SO}_{2n+2}N_{n-1}}(\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\gamma_{i,w}},$$

where $\mathrm{c}-\mathrm{Ind}$ denotes the compact induction, and

$$R_{i,w} = P_{2n}^{\gamma_{i,w}} := (\gamma_i w)^{-1}P_{2n}\gamma_i w \cap \mathrm{SO}_{2n+1}N_{n-1}.$$

Lemma 5.4. *With notation above, the following vanishing properties hold.*

(1) *For* $w \neq \mathrm{id}$*,*

$$\mathcal{J}_{\psi_{n-1}}(\mathrm{c}-\mathrm{Ind}_{R_{i,w}}^{\mathrm{SO}_{2n+2}N_{n-1}}(\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\gamma_{i,w}})=0,$$

for $i \geq 2[\frac{n+1}{2}]$.

(2) *For* $w = \mathrm{id}$*,*

$$\mathcal{J}_{\psi_{n-1}}(\mathrm{c}-\mathrm{Ind}_{R_{i,\mathrm{id}}}^{\mathrm{SO}_{2n+2}N_{n-1}}(\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\gamma_{i,\mathrm{id}}})=0$$

for all $i > 2[\frac{n+1}{2}]$.

Proof. When $w \neq \mathrm{id}$, by Lemma 5.3, there is a simple root subgroup $x(t)$ inside N_{n-1} such that $\gamma_i w x(t)(\gamma_i w)^{-1}$ lies in the unipotent radical of P_{2n}. This shows that $x(t) \in R_{i,w} \cap N_{n-1}$, $(\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\gamma_{i,w}}(x(t)) = \mathrm{id}$, while $\psi_{n-1}(x(t)) = \psi(t)$.

When $w = \mathrm{id}$, if $i > 2[\frac{n+1}{2}]$, the root subgroup $x_\alpha(t)$ of SO_{4n} for $\alpha = e_{n-1} + e_{2n}$ is invariant under the conjugation by $\gamma_i'^{-1}$. Hence $x(t) \in R_{i,w} \cap N_{n-1}$, $(\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\gamma_{i,w}}(x(t)) = \mathrm{id}$, while $\psi_{n-1}(x(t)) = \psi(t)$. □

Therefore, we are left with $\mathcal{J}_{\psi_{n-1}}(\mathcal{S}(\Omega_l, \tau_s))$ for the Zariski open orbit $\Omega_l = \mathcal{O}_{2[\frac{n+1}{2}],\mathrm{id}}$. We summarize what we proved as

Proposition 5.5. *The following*

$$\mathcal{J}_{\psi_{n-1}}(\mathrm{I}(s, \mathcal{W}(\tau,\psi))) \simeq \mathcal{J}_{\psi_{n-1}}(\mathcal{S}(\mathcal{O}_{2[\frac{n+1}{2}],\mathrm{id}}, \tau_s))$$

holds for all $s \in \mathbb{C}$ *as representations of* $\mathrm{SO}_{2n+1}(\mathcal{F})$*.*

5.2. Proof of Part (1) of Theorem 5.1. Keep the notation as before. By Proposition 5.5, the space

$$\mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}_{\psi_{n-1}}(\mathrm{I}(s,\tau)), V_\sigma)$$

is isomorphic to the space

$$\mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}_{\psi_{n-1}}(\mathcal{S}(\mathcal{O}_{2[\frac{n+1}{2}],\mathrm{id}}, \tau_s)), V_\sigma).$$

This reduces to a further understanding of $\mathcal{J}_{\psi_{n-1}}(\mathcal{S}(\mathcal{O}_{2[\frac{n+1}{2}],\mathrm{id}}, \tau_s))$ as representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$.

It is more convenient to choose the representative ν_{4n} for the orbit $\mathcal{O} = \mathcal{O}_{2[\frac{n+1}{2}],\mathrm{id}}$ than the original $\gamma_{2[\frac{n+1}{2}],\mathrm{id}}$. Then

$$\mathcal{S}(\mathcal{O}, \tau_s) \simeq \mathrm{c}-\mathrm{Ind}_{P_{2n}^{\nu_{4n}}}^{\mathrm{SO}_{2n+2}N_{n-1}}(\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\nu_{4n}},$$

where $P_{2n}^{\nu_{4n}} = \nu_{4n}^{-1} P_{2n} \nu_{4n} \cap \mathrm{SO}_{2n+2} N_{n-1}$. Let $\mathcal{Q}_{n+1}$ be the maximal standard parabolic subgroup of SO_{2n+2} whose Levi component is isomorphic to GL_{n+1}, and $\mathcal{Q}_{n+1}^-$ be the opposite parabolic subgroup. Then we have

$$\begin{aligned} P_{2n}^{\nu_{4n}} &= \{m(\begin{pmatrix} z & c \\ & \mathrm{I}_{n+1} \end{pmatrix}) \in \mathrm{SO}_{4n} | z \in \mathrm{U}_{n-1}\} \cdot \overline{\mathcal{Q}}_{n+1} \\ &:= m(\mathrm{U}_{2n,n-1}) \cdot \mathcal{Q}_{n+1}^-, \qquad (5.2) \end{aligned}$$

where $\mathrm{U}_{2n,j}$ is the subgroup of the unipotent radical U_{2n} of the standard Borel subgroup of GL_{2n} consisting of elements of type

$$\begin{pmatrix} z & c \\ 0 & \mathrm{I}_{2n-j} \end{pmatrix} \in \mathrm{U}_{2n}$$

with $z \in \mathrm{U}_j$.

For $\phi \in \mathrm{c}-\mathrm{Ind}_{P_{2n}^{\nu_{4n}}}^{\mathrm{SO}_{2n+2}N_{n-1}}(\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\nu_{4n}}$ and for $q = \begin{pmatrix} a & 0 \\ * & a^* \end{pmatrix} \in \mathcal{Q}_{n+1}^-$, we have

$$(\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\nu_{4n}}(\begin{pmatrix} \mathrm{I}_{n-1} & & \\ & q & \\ & & \mathrm{I}_{n-1} \end{pmatrix})(\phi)(g,r) = |\det a|^{-(\frac{s}{2}+n-\frac{1}{2})}\phi(g, r\begin{pmatrix} \mathrm{I}_{n-1} & \\ & a \end{pmatrix}),$$

and for $\begin{pmatrix} z & c \\ 0 & \mathrm{I}_{n+1} \end{pmatrix}) \in \mathrm{U}_{2n,n-1}$, we have

$$(\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\nu_{4n}}(m(\begin{pmatrix} z & c \\ & \mathrm{I}_{n+1} \end{pmatrix}))(\phi)(g,r) = \phi(g, r\begin{pmatrix} z & c \\ 0 & \mathrm{I}_{n+1} \end{pmatrix}).$$

In order to understand $\mathcal{J}_{\psi_{n-1}}(\mathcal{S}(\mathcal{O},\tau_s))$ as representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$, we consider the double coset decomposition

$$P_{2n}^{\nu_{4n}} \backslash \mathrm{SO}_{2n+2} \cdot N_{n-1} / \mathrm{SO}_{2n+1} \cdot N_{n-1},$$

which reduces to the computation of the following double cosets

$$\mathcal{Q}_{n+1}^- \backslash \mathrm{SO}_{2n+2} / \mathrm{SO}_{2n+1}.$$

Next proposition shows that it has only one orbit.

Proposition 5.6. *Over any field k of characteristic zero, the generalized flag variety $\mathcal{Q}_{n+1}^-(k)\backslash \mathrm{SO}_{2n+2}(k)$ has only one orbit under the action of $\mathrm{SO}_{2n+1}(k)$.*

Proof. Let $X = k^{2n+2}$ be a k-vector space, written its elements as column vector, with a quadratic form q defined by $\frac{1}{2}\nu_{2n+2}$. Then $\mathrm{SO}(X) \simeq \mathrm{SO}_{2n+2}$ Let $e_1, \ldots, e_{2n+2}$ be the standard basis of X, $v_0 = e_{n+1} + e_{n+2}$. Let $Y = (k \cdot v_0)^\perp$. Then $\dim Y = 2n+1$ and $\mathrm{SO}(Y) = \mathrm{SO}_{2n+1}$. Note that Y has a basis:

$$e_{n+1} - e_{n+2}, e_1, e_2 \ldots, e_n, e_{n+3}, \ldots e_{2n+1}, e_{2n+2}. \qquad (5.3)$$

Then a basis of X can be chosen to be

(5.4) $\quad e_{n+1}+e_{n+2}, e_{n+1}-e_{n+2}, e_1, e_2\ldots, e_n, , e_{n+3}, \ldots e_{2n+1}, e_{2n+2}$

Let $g \in \mathrm{SO}(X)$ such that $g(v_0)=v_0$, then $g(Y)=Y$ and assume that the matrix of $g|Y$ under the basis (5.3) is A_g, then g under the basis (5.4) is

$$\begin{pmatrix} 1 & \\ & A_g \end{pmatrix}$$

As $\det g = 1$, we must have $\det(A_g)=1$, hence $g \in \mathrm{SO}(Y)$. That is, the stabilizer of v_0 is $\mathrm{SO}(Y)$.

Note that $q(v_0)=1$. Let $Z=\{v \in X|\ q(v)=1\}$. Then SO_{2n+2} acts transitively on Z. To show the proposition, we only need to show that $\mathcal{Q}^-_{n+1}$ acts on Z transitively. In fact, if $\mathcal{Q}^-_{n+1}$ acts transitively on Z, let $h \in \mathrm{SO}(X)$, then there exists $t \in \mathcal{Q}^-_{n+1}$ such that

$$h \cdot v_0 = t \cdot v_0.$$

Hence $(t^{-1}h)\cdot v_0 = v_0$, and then $t^{-1}h \in \mathrm{SO}_{2n+1}, h \in \mathcal{Q}^-_{n+1}\mathrm{SO}_{2n+1}$. This means that

$$\mathrm{SO}_{2n+2} = \mathcal{Q}^-_{n+1}\mathrm{SO}_{2n+1}.$$

Now we show that $\mathcal{Q}^-_{n+1}$ acts transitively on Z. We only need to show that any element of Z can be moved to v_0 under the action of some element in $\mathcal{Q}^-_{n+1}$. Let $v=(v_1,v_2)\in X$ with $v_1, v_2 \in k^{n+1}$. Take $g \in \mathcal{Q}^-_{n+1}$ be

$$g=\begin{pmatrix} a & 0 \\ b & a^* \end{pmatrix} \text{ with } a \in \mathrm{GL}_{n+1}.$$

Then the action of g on v is given by $g \cdot v = (av_1, bv_1 + a^*v_2)^t$.

Assume now $q(v)=1$. Then $v_1 \neq 0$, otherwise $q(v)=0$. Then there is $a \in \mathrm{GL}_{n+1}$ such that

$$av_1 = (0,\cdots,0,1)^t.$$

For this a, there exists $b \in \mathrm{Mat}_{n+1}(k)$ such that

$$bv_1 = (1,0,\cdots,0)^t - a^*v_2, \text{ since } v_1 \neq 0.$$

Now

$$g=\begin{pmatrix} a & 0 \\ b & a^* \end{pmatrix} \in \mathcal{Q}^-_{n+1} \text{ and } g\cdot v = v_0,$$

which completes the proof of the proposition. □

It follows that

$$P^{\nu_{4n}}_{2n}\backslash \mathrm{SO}_{2n+2}N_{n-1} = [P^{\nu_{4n}}_{2n} \cap \mathrm{SO}_{2n+1}N_{n-1}]\backslash \mathrm{SO}_{2n+1}N_{n-1}.$$

By restriction to $\mathrm{SO}_{2n+1}N_{n-1}$, we have

$$\mathrm{c}-\mathrm{Ind}_{P_{2n}^{\nu_{4n}}}^{\mathrm{SO}_{2n+2}N_{n-1}}((\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\nu_{4n}}) \simeq \mathrm{c}-\mathrm{Ind}_{P_{2n}^{\nu_{4n}}\cap \mathrm{SO}_{2n+1}N_{n-1}}^{\mathrm{SO}_{2n+1}N_{n-1}}((\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\nu_{4n}})$$

as representations of $\mathrm{SO}_{2n+1}(\mathcal{F})\ltimes N_{n-1}(\mathcal{F})$. Hence the twisted Jacquet module

$$\mathcal{J}_{\psi_{n-1}}(\mathrm{c}-\mathrm{Ind}_{P_{2n}^{\nu_{4n}}}^{\mathrm{SO}_{2n+2}N_{n-1}}((\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\nu_{4n}}))$$

is isomorphic to the twisted Jacquet module

$$\mathcal{J}_{\psi_{n-1}}(\mathrm{c}-\mathrm{Ind}_{P_{2n}^{\nu_{4n}}\cap \mathrm{SO}_{2n+1}N_{n-1}}^{\mathrm{SO}_{2n+1}N_{n-1}}((\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\nu_{4n}}))$$

as representations of $\mathrm{SO}_{2n+1}(\mathcal{F})$.

Define $\psi_{\mathrm{U}_{2n,n-1}}(u(z,c)) := \psi_{n-1}|_{\mathrm{U}_{2n,n-1}}(u(z,c))$. Then we have

Proposition 5.7. *With notation above, the twisted Jacquet module*

$$\mathcal{J}_{\psi_{n-1}}(\mathrm{c}-\mathrm{Ind}_{P_{2n}^{\nu_{4n}}\cap \mathrm{SO}_{2n+1}N_{n-1}}^{\mathrm{SO}_{2n+1}N_{n-1}}((\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\nu_{4n}}))$$

is isomorphic to

$$\mathrm{c}-\mathrm{Ind}_{\mathcal{P}_n^-}^{\mathrm{SO}_{2n+1}}(\mathcal{J}_{\psi_{\mathrm{U}_{2n,n-1}}}(\tau')|\det|^{-\frac{s}{2}-\frac{1}{2}})$$

as representations of $\mathrm{SO}_{2n+1}(\mathcal{F})$*, where* $\mathcal{P}_n^- := \mathcal{Q}_{n+1}^- \cap \mathrm{SO}_{2n+1}$*, the representation* τ' *is obtained by restriction to* $\mathcal{P}_n^-(\mathcal{F})$ *of the representation of* $\mathcal{Q}_{n+1}^-(\mathcal{F})$ *given by the formula right before Proposition 5.6, and* $\mathcal{J}_{\psi_{\mathrm{U}_{2n,n-1}}}(\tau')$ *denotes the twisted Jacquet module of* τ' *along* $(\mathrm{U}_{2n,n-1},\psi_{\mathrm{U}_{2n,n-1}})$.

Proof. Let f be a section in $\mathrm{c}-\mathrm{Ind}_{P_{2n}^{\nu_{4n}}\cap \mathrm{SO}_{2n+1}N_{n-1}}^{\mathrm{SO}_{2n+1}N_{n-1}}((\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\nu_{4n}})$. Consider the restriction of f to $\mathrm{SO}_{2n+1}(\mathcal{F})$. It is clear that this restriction map factors through the twisted Jacquet module

$$\mathcal{J}_{\psi_{n-1}}(\mathrm{c}-\mathrm{Ind}_{P_{2n}^{\nu_{4n}}\cap \mathrm{SO}_{2n+1}N_{n-1}}^{\mathrm{SO}_{2n+1}N_{n-1}}((\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\nu_{4n}})),$$

which is still denoted by $f\mapsto f|_{\mathrm{SO}_{2n+1}(\mathcal{F})}$. By the formulas displayed right before Proposition 5.6, the restriction $f|_{\mathrm{SO}_{2n+1}(\mathcal{F})}$ belongs to the space

$$\mathrm{c}-\mathrm{Ind}_{\mathcal{P}_n^-}^{\mathrm{SO}_{2n+1}}(\mathcal{J}_{\psi_{\mathrm{U}_{2n,n-1}}}(\tau')|\det|^{-\frac{s}{2}-\frac{1}{2}}).$$

By using the orbital decomposition in Proposition 5.6 and the formulas right before Proposition 5.6, it is not hard to check that $f\mapsto f|_{\mathrm{SO}_{2n+1}(\mathcal{F})}$ is in fact injective. The argument is the same as in the proof of Formula (6.5) in [GRS99] and similar to that of Lemma 5.3 in [K86]. We omit the details.

The surjectivity can be verified as follows. Assume that we have a smooth $\mathcal{J}_{\psi_{\mathrm{U}_{2n,n-1}}}(V_{\tau'})$-valued function g on SO_{2n+1}, compactly supported modulo $\mathcal{P}_n^-$, satisfying

$$g(qy) = \mathcal{J}_{\psi_{\mathrm{U}_{2n,n-1}}}(\tau')|\det|^{-\frac{s}{2}-\frac{1}{2}}(q)g(y)$$

for $q \in \mathcal{P}_n^-$ and $y \in \mathrm{SO}_{2n+1}$.

Since g is locally constant, we may pull back g to a smooth $V_{\tau'}$-valued function g' on SO_{2n+1}, compactly supported modulo $\mathcal{P}_n^-$, satisfying

$$g'(qy) = \tau'(q)|\det|^{-(\frac{s}{2}+n-\frac{1}{2})}g'(y),$$

for $q \in \mathcal{P}_n^-$ and $y \in \mathrm{SO}_{2n+1}$.

Note that the unipotent subgroup N_{n-1} can be written as

$$N_{n-1} = m(\mathrm{U}_{2n,n-1}) \ltimes N''$$

where N'' is the intersection of N_{n-1} with the unipotent radical V_{2n} of P_{2n}. Then we have

$$\mathrm{SO}_{2n+1}N_{n-1} = \mathrm{SO}_{2n+1}\mathrm{U}''_{2n,n-1}$$

which is in fact a homeomorphism. In fact, let $z'y'x' = zyx$ with $x, x' \in \mathrm{SO}_{2n+1}$, $z, z' \in B_{n-1}, y, y' \in N''$. Then

$$y = (z^{-1}z')y'(x'x^{-1}) \in N''$$

Hence $x = x', z = z', y = y'$.

Then we can pull back g to a section f in

$$\mathcal{J}_{\psi_{n-1}}(\mathrm{c}-\mathrm{Ind}_{P_{2n}^{\nu_{4n}}\cap\mathrm{SO}_{2n+1}N_{n-1}}^{\mathrm{SO}_{2n+1}N_{n-1}}((\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\nu_{4n}})),$$

which is defined as follows.

We choose a compactly supported smooth function ϕ on N'', which has a nonzero projection under the twisted Jacquet functor with respect to $(N'', \psi_{n-1}|_{N''})$ and define

$$f'(uyx, r) := \phi(y)g'(x, ru),$$

for all $x \in \mathrm{SO}_{2n+1}$, $u \in \mathrm{U}_{2n,n-1}$, $y \in N''$, and $r \in \mathrm{GL}_{2n}$. It is clear that f' is a nonzero section in

$$\mathrm{c}-\mathrm{Ind}_{P_{2n}^{\nu_{4n}}\cap\mathrm{SO}_{2n+1}N_{n-1}}^{\mathrm{SO}_{2n+1}N_{n-1}}((\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\nu_{4n}}).$$

By checking the action of N_{n-1} on f', it is clear that f' factors through the twisted Jacquet module

$$\mathcal{J}_{\psi_{n-1}}(\mathrm{c}-\mathrm{Ind}_{P_{2n}^{\nu_{4n}}\cap\mathrm{SO}_{2n+1}N_{n-1}}^{\mathrm{SO}_{2n+1}N_{n-1}}((\delta_{P_{2n}}^{\frac{1}{2}}\tau_s)^{\nu_{4n}})),$$

whose image f has the restriction to $\mathrm{SO}_{2n+1}(\mathcal{F})$ equal to g. The proof is then completed. □

Note that the elements of $\mathcal{P}_n^-$ has the form

$$\begin{pmatrix} b & & & \\ x & 1 & 0 & \\ -x & 0 & 1 & \\ y' & -x' & x' & b^* \end{pmatrix} \in \mathrm{SO}_{2n+2}(\mathcal{F}),$$

which is identified (following the embedding we assumed as before) with

$$\begin{pmatrix} b & & \\ x & 1 & \\ y & x' & b^* \end{pmatrix} \in \mathrm{SO}_{2n+1}(\mathcal{F}).$$

Following the above discussions, we deduce that the space

$$\mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}_{\psi_{n-1}}(\mathrm{I}(s,\tau)), V_\sigma)$$

is isomorphic to the space

$$\mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathrm{c}-\mathrm{Ind}_{\mathcal{P}_n^-(\mathcal{F})}^{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}_{\psi_{\mathrm{U}_{2n,n-1}}}(\tau')|\det|^{-\frac{s}{2}-\frac{1}{2}}), V_\sigma).$$

By the Frobenius Reciprocity law, the last space is isomorphic to the following space:

$$\mathrm{Hom}_{\mathcal{P}_n^-(\mathcal{F})}(\mathcal{J}_{\psi_{\mathrm{U}_{2n,n-1}}}(\tau')|\det|^{-\frac{s}{2}-\frac{1}{2}}, V_\sigma).$$

By the assumption of Theorem 5.1, the last space is nonzero. Since the argument below only uses the genericity of τ' and the supercuspidality of σ, and does not depend on the value s, for simplicity, we may consider only the nonzero space

$$\mathrm{Hom}_{\mathcal{P}_n^-(\mathcal{F})}(\mathcal{J}_{\psi_{\mathrm{U}_{2n,n-1}}}(\tau'), V_\sigma).$$

We take any nonzero element ξ in the above space, which is a $\mathcal{P}_n^-(\mathcal{F})$-equivariant, linear mapping from $\mathcal{J}_{\psi_{\mathrm{U}_{2n,n-1}}}(\tau')$ to V_σ. In particular, for any $v \in V_{\tau'}$, we have

$$\sigma(\begin{pmatrix} a & & \\ x & 1 & \\ y & x' & a^* \end{pmatrix})(\xi(v)) = \xi(\mathcal{J}_{\psi_{\mathrm{U}_{2n,n-1}}}(\tau')(\begin{pmatrix} \mathrm{I}_{n-1} & & \\ & a & \\ & x & 1 \end{pmatrix})(v)). \tag{5.5}$$

Take $a = \mathrm{I}_n$ and consider the action of the unipotent radical of $\mathcal{P}_n^-(\mathcal{F})$, which is denoted by $\mathcal{V}_n^-(\mathcal{F})$ and consists of elements of the following form

$$v^-(x,y) := \begin{pmatrix} \mathrm{I}_n & & \\ x & 1 & \\ y & x' & \mathrm{I}_n \end{pmatrix}.$$

Then (5.5) implies that the center (the elements of type $v^-(0,y)$) of $\mathcal{V}_n^-(\mathcal{F})$ acts on V_σ trivially. Since V_σ is supercuspidal, there is a nonzero vector $v \in \mathcal{J}_{\psi_{\mathrm{U}_{2n,n-1}}}(V_{\tilde{\tau}})$ such that the unipotent radical of $\mathcal{V}_n^-(\mathcal{F})$ acts on $\xi(v)$ by a nontrivial character. Since the $\mathrm{GL}_n(\mathcal{F})$ acts on the x-part (more precisely, the quotient of $\mathcal{V}_n^-(\mathcal{F})$ modulo the center) with two orbits, we may assume that

$$\sigma(\overline{v}(x,y))(\xi(v)) = \psi_{\mathcal{V}_n^-}(v^-(x,y))\xi(v) = \psi(x_n)\xi(v),$$

where $\psi(x_n)$ is a nonzero character of $\mathcal{V}_n^-(\mathcal{F})$. In other words, the mapping ξ descents to a mapping from $\mathcal{J}_{\psi_{\mathrm{U}_{2n,n-1}}}(\tau')$ to $\mathcal{J}_{\psi_{\mathcal{V}_n^-}}(V_\sigma)$.

By (5.5), we have

$$\xi(\mathcal{J}_{\psi_{n-1}}(\tau')(\begin{pmatrix} \mathrm{I}_{n-1} & & \\ & a & \\ & x & 1 \end{pmatrix})(v)) = \psi(x_n)\xi(v).$$

Now consider the subgroup $B_{2n,n}$ of GL_{2n} consisting of elements of form

$$b(z,c,e,y,d) := \begin{pmatrix} z & c & e \\ 0 & 1 & y \\ 0 & 0 & d \end{pmatrix}$$

with $d \in \mathrm{GL}_n(\mathcal{F})$ and $z \in \mathrm{U}_{n-1}$. Let μ be the Weyl element of GL_{2n} corresponding to the elementary matrix

$$\begin{pmatrix} \mathrm{I}_{n-1} & 0 \\ 0 & \nu_{n+1} \end{pmatrix}$$

Then it is easy to obtain that

$$\xi(\mathcal{J}_{\psi_{n-1}}(\tau')(b(z,c,e,y,\mathrm{I}_n))(\mu v)) = \psi_{U_{n-1}}(z)\psi(c_{n-1})\psi(y_1)\xi(\mu v).$$

This means that the mapping ξ factors through the n-th derivative $\tilde{\tau}^{(n)}$ in the sense of Bernstein and Zelevinsky ([BZ76]). Therefore, we can view ξ as a mapping from the n-th derivative $\tilde{\tau}^{(n)}$ to the twisted Jacquet module $\mathcal{J}_{\psi_{\mathcal{V}_n^-}}(V_\sigma)$, which has the following equivalent property, for $a \in \mathrm{GL}_{n-1}$,

$$\mathcal{J}_{\psi_{\mathcal{V}_n^-}}(\sigma)(\begin{pmatrix} a & 0 \\ x & 1 \end{pmatrix})\xi(v) = \xi((\tau')^{(n)}(\begin{pmatrix} \mathrm{I}_n & & \\ & 1 & x^* \\ & 0 & \nu_{n-1}a\nu_{n-1} \end{pmatrix}))(\mu v),$$

where $x^* = (x_{n-1}, x_{n-2}, \cdots, x_1)$ if $x = (x_1, \cdots, x_{n-1})$.

Now we come back to the situation of (5.5) with $a \in \mathrm{GL}_{n-1}$. We repeat the same process with the supercuspidality of σ and the genericity of τ. Eventually, we arrive at the $2n$-th derivative of τ', which is the twisted Jacquet module of Whittaker type. The equivalent property in this last case shows that V_σ has a nonzero Whittaker functional. Hence it is generic. This finishes the proof of Part (1) of Theorem 5.1.

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