SOME CONJECTURES ON ENDOSCOPIC REPRESENTATIONS IN ODD ORTHOGONAL GROUPS

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1. Introduction

Let π denote an irreducible generic cuspidal automorphic representation of $SO_{2m+1}(\mathbf{A})$. Here $\mathbf{A}$ is the ring of adeles of a number field F. We say that π is an endoscopic representation with respect to $SO_{2r+1}\times SO_{2(m-r)+1}$ if there are generic cuspidal automorphic representations σ_1 and σ_2 of $SO_{2r+1}(\mathbf{A})$ and $SO_{2(m-r)+1}(\mathbf{A})$, respectively, such that π is the Langlands functorial lift of $\sigma_1 \otimes \sigma_2$. This functorial lift corresponds to the L-group homomorphism $Sp_{2r}(\mathbf{C}) \times Sp_{2(m-r)}(\mathbf{C}) \to Sp_{2m}(\mathbf{C})$, which is given by the direct sum embedding. More generally, we say that π is an endoscopic representation with respect to $SO_{2r_1+1} \times \cdots \times SO_{2r_k+1}$ if there exists a generic cuspidal automorphic representation σ_i of each $SO_{2r_i+1}(\mathbf{A})$, such that π is the Langlands functorial lift of $\sigma_1\otimes\cdots\otimes\sigma_k$. Here $1 \le i \le k$ and $r_1+\ldots+r_k = m$. As above this lift corresponds to the L-group homomorphism

$$Sp_{2r_1}(\mathbf{C}) \times \cdots \times Sp_{2r_k}(\mathbf{C}) \mapsto Sp_{2m}(\mathbf{C}).$$

If an irreducible generic cuspidal automorphic representation π of $SO_{2m+1}(\mathbf{A})$ is not endoscopic, we say that π is stable. It follows from the Langlands conjecture that for any irreducible generic cuspidal automorphic representation π of $SO_{2m+1}(\mathbf{A})$, there exists a partition $m = \sum_{i=1}^k r_i$ and there exists an irreducible, stable, generic cuspidal automorphic representation σ_i of each $SO_{2r_i+1}(\mathbf{A})$, such that π is an endoscopy lift from $\sigma_1 \otimes \cdots \otimes \sigma_k$. In this case, we say that π has a stable endoscopy type $(\sigma_1, \cdots, \sigma_k)$. Hence if π has a stable endoscopy type $(\sigma_1, \cdots, \sigma_k)$, then π is also endoscopic with respect to $SO_{2r+1} \times SO_{2(m-r)+1}$ for a suitable values of r.

The Langlands conjectures suggest two important problems regarding this setup.

(1) *Existence.* Given a set of irreducible cuspidal automorphic representations σ_i defined on $SO_{2r_i+1}(\mathbf{A})$, with $i = 1, 2, \cdots, k$, prove the existence of an endoscopic representation π of $SO_{2m+1}(\mathbf{A})$ with respect to $SO_{2r_1+1} \times \cdots \times SO_{2r_k+1}$.

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(2) *Characterization.* Characterize the image of this lift. In other words, given an irreducible generic cuspidal automorphic representation π of $SO_{2m+1}(\mathbf{A})$, determine when it is an endoscopic representation. This characterization can be given in terms of poles of L-functions, or in terms of nonvanishing of certain period integrals, or both.

For Problem **(1)**, there are currently constructive methods to prove the existence of such lifts. The first construction is the so called descent method. This method uses a Fourier coefficient of a certain residue of an Eisenstein series defined on the even orthogonal groups. See [S] for some details. Another construction of endoscopic representation is given in [G], using an extension of the descent method. We should mention that both constructions use the functorial lift from odd orthogonal group to GL_n as established in [CKPSS]. The existence of such endoscopy lifts is in general expected to follow from the method of the Trace Formula.

For the characterization of the image, that is, Problem **(2)**, some partial results are known. First, regarding poles of L-functions, the result of [J] gives a relation between this lifting and the order of the pole of the partial L-function attached to the second fundamental representation of $Sp_{2m}(\mathbf{C})$. Regarding the characterization of the image of the lift in terms of period integrals, we have

Theorem 1 ([G], Theorem 6)**.** *An irreducible generic cuspidal automorphic representation π of $SO_{2m+1}(\mathbf{A})$ has a stable endoscopy type $(\sigma_1, \cdots, \sigma_k)$, with σ_i being an irreducible stable generic cuspidal automorphic representation of $SO_{2r_i+1}(\mathbf{A})$, $i = 1, 2, \cdots, k$, if and only if there exist k cuspidal representations τ_i of $GL_{2r_i}(\mathbf{A})$, respectively, such that all the period integrals $\mathcal{P}(\pi, \tau_i)$, defined in* [G]*, are nonzero for some choice of data.*

Even though the above theorem does give a characterization of the image, there are two problems with it. First, it requires a set of period integrals and not just one. Second, the period integral is not defined using the representation π only, but rather is defined also in terms of cuspidal representations of the group $GL_n(\mathbf{A})$.

In this paper we introduce a family of period integrals which we conjecture to give a certain characterization of the image of the lift. Our main conjecture is

Conjecture 1. *Assume that $2r \leq m$. An irreducible generic cuspidal automorphic representation π is endoscopic with respect to $SO_{2r+1} \times SO_{2(m-r)+1}$ if and only if the period integral*

$$(1) \qquad \mathcal{Q}_r(\pi) = \int\limits_{Sp_{2r}(F)\backslash Sp_{2r}(\mathbf{A})} \int\limits_{U_r(F)\backslash U_r(\mathbf{A})} \varphi_\pi(uh)\psi_{U_r}(u)dudh$$

is not zero for some choice of data.

The precise definition of the integrals $\mathcal{Q}_r(\pi)$ will be given in the next section. The difference between Conjecture 1 and Theorem 1 is that in Conjecture 1, the period integral involves only the representation π and only one period integral. On the other hand, Theorem 1 is more general, and also gives the stable endoscopy type of π.

The main result related to this conjecture is,

Theorem 2. *Conjecture 1 holds for* $r = 1$.

We prove this theorem in the next section.

In the third section we state a conjecture involving a period integral, which characterizes when the representation π is an endoscopic functorial lift from $SO_3 \times SO_{2r+1} \times SO_{2(m-r)+1}$. Clearly, Theorem 1 covers this case. However, it requires more than one period integral. Our conjecture characterizes the image of this lift in terms of one period integral. However, in contrast to Conjecture 1, this period integral does not involve the representation π only, but also a cuspidal representation of $SO_3(\mathbf{A})$. The main result of that section is that this conjecture holds when $r = 1$, and for an arbitrary r, it holds if we assume Conjecture 1.

The last section is an appendix where we collect some known results which we use mainly in the third section.

2. On the Main Conjecture

We keep the notation of the introduction. Let π denote an irreducible generic cuspidal automorphic representation of $SO_{2m+1}(\mathbf{A})$. The orthogonal group is realized as the group of all matrices $g \in GL_{2m+1}$ such that $g^t J g = J$. Here J is the $2m+1$ matrix which has ones on the other diagonal and zeros elsewhere.

We now describe the groups used to define the period integral $\mathcal{Q}_r(\pi)$ as given in integral (1). Here $2r \le m$. Let U_r' denote the unipotent radical of the parabolic subgroup of SO_{2m+1} whose Levi part is $GL_1^{m-2r} \times GL_{2r}$. Let U_r be the subgroup of U_r' which consists of all $u = (u_{i,j}) \in U_r'$ such that $u_{i,m+1} = 0$ for all $m - 2r + 1 \le i \le m$. Let ψ denote a nontrivial additive character of $F\backslash\mathbf{A}$. The character ψ_{U_r} is defined as follows

$$\begin{aligned}\psi_{U_r}(u) &= \psi(u_{1,2} + u_{2,3} + \cdots + u_{m-2r-1,m-2r} + u_{m-2r,m+1} + \\ &\quad + u_{m-2r+1,m+2} + u_{m-2r+2,m+3} + \cdots + u_{m-r,m+r+1}).\end{aligned}$$

Then Sp_{2r} embedded inside GL_{2r} stabilizes the character ψ_{U_r}.

Proof of Theorem 2. Assume first that π is endoscopic with respect to $SO_3 \times SO_{2m-1}$. Thus π is the functorial lift of an irreducible generic cuspidal automorphic representation τ of $SO_3(\mathbf{A})$ and an irreducible generic cuspidal automorphic representation σ of $SO_{2m-1}(\mathbf{A})$.

We view τ as an irreducible representation of $GL_2(\mathbf{A})$ with a trivial central character. Then, as stated in Theorem 1, it follows from [G] that the integral

$$(2) \qquad \int\limits_{SO_4(F)\backslash SO_4(\mathbf{A})} \int\limits_{V_1(F)\backslash V_1(\mathbf{A})} \varphi_\pi(vg)\psi_{U_1}(v)E_\tau(g)dvdg$$

is not zero for some choice of data. Here V_1 is the unipotent radical of the parabolic subgroup of SO_{2m+1} whose Levi part is $GL_1^{m-2} \times SO_5$. Notice that V_1 is a subgroup of U_1, and we view ψ_{U_1} as a character of V_1 by restriction. Also, $E_\tau(g)$ denotes the residue of the Eisenstein defined on SO_4 which is induced from τ. More precisely, let $E_\tau(g,s)$ denote the Eisenstein series of $SO_4(\mathbf{A})$ associated with the induced representation $Ind_{P_2(\mathbf{A})}^{SO_4(\mathbf{A})}\tau\delta_{P_2}^s$. Here P_2 is the maximal parabolic subgroup of SO_4 whose Levi part is GL_2. Since τ has a trivial central character, then this Eisenstein series has a simple pole at $s = 1$. we denote the residue representation by $E_\tau(g)$.

Restricting $E_\tau(g)$ as a function to $SL_2(\mathbf{A}) \times SL_2(\mathbf{A})$, it follows from integral (2) that the integral

$$\int\limits_{(SL_2(F)\times SL_2(F))\backslash(SL_2(\mathbf{A})\times SL_2(\mathbf{A}))} \int\limits_{V_1(F)\backslash V_1(\mathbf{A})} \varphi_\pi(v(g_1,g_2))\psi_{U_1}(v)\varphi_\tau(g_1)dvdg_1dg_2$$

is not zero for some choice of data. This implies that as a function of $g_1 \in SL_2(\mathbf{A})$, the function

$$(3) \qquad f(g_1) = \int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \int\limits_{V_1(F)\backslash V_1(\mathbf{A})} \varphi_\pi(v(g_1,g_2))\psi_{U_1}(v)dvdg_2$$

is not zero, and moreover it is not the identity function. Therefore, it has a nonzero Fourier coefficient. Thus, there is $\alpha \in F^*$ such that the integral

$$\int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \int\limits_{F\backslash \mathbf{A}} \int\limits_{V_1(F)\backslash V_1(\mathbf{A})} \varphi_\pi(v(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g_1, g_2))\psi(\alpha x)\psi_{U_1}(v)dxdvdg_2$$

is not zero for some choice of data. However, conjugating by the torus element of SO_{2m+1} given by $\mathrm{diag}(I_{m_2}, \alpha, I_3, \alpha^{-1}, I_{m-2})$ we deduce that when $\alpha = 1$, the above integral is not zero for some choice of data. By definition the unipotent group we integrate over in the above integral is the group U_1. Also, the character defined on this group is exactly ψ_{U_1}. Thus, we conclude that integral (1) with $r = 1$ is not zero for some choice of data.

To prove the converse, we start with integral (1), and arguing backward we deduce that integral (3), is not zero for some choice of data. Changing variables in the g_2 variable in integral (3) we obtain that $f(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g_1) = f(g_1)$ for all $\alpha \in F^*$, and that $f(\begin{pmatrix} a & \\ & a \end{pmatrix} g_1) = f(g_1)$ for all $a \in \mathbf{A}^*$. Hence, the function $f(g_1)$ defined in integral (3) defines a nonzero automorphic function of $GL_2(\mathbf{A})$. Let τ' denote the representation of $GL_2(\mathbf{A})$ generated by the space of functions $f(g_1)$ given by (3). It follows from the above that τ' has a trivial central character. We claim that τ' is a cuspidal representation. Assuming that, let τ denote an irreducible summand of τ'. From the above we deduce that the integral

$$\int_{Z(\mathbf{A})GL_2(F)\backslash GL_2(\mathbf{A})}\int_{SL_2(F))\backslash SL_2(\mathbf{A})}\int_{V_1(F)\backslash V_1(\mathbf{A})}\varphi_\pi(v(g_1,g_2))\psi_{U_1}(v)\varphi_\tau(g_1)dvdg_1dg_2$$

is not zero for some choice of data, which implies that the integral (2) is not zero for some choice of data. Thus, it follows from [G] that π is endoscopic.

It remains to prove that the representation τ' is cuspidal. Thus, we need to prove that the integral

$$\int_{F\backslash\mathbf{A}} f(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g_1)dx$$

is zero for all choice of data. Let

$$w=\begin{pmatrix} & I_2 & & & \\ I_{m-2} & & & & \\ & & 1 & & \\ & & & & I_{m-2}\\ & & & I_2 & \end{pmatrix}.$$

Conjugating the argument of φ_π by w, we need to prove that the integral

$$(4)\quad \int\varphi_\pi\left[\begin{pmatrix}I_2 & x & y\\ & I_{2m-3} & x^*\\ & & I_2\end{pmatrix}\begin{pmatrix}I_2 & & \\ & v & \\ & & I_2\end{pmatrix}\begin{pmatrix}I_2 & & \\ z & I_{2m-3} & \\ & z^* & I_2\end{pmatrix}\begin{pmatrix}g & & \\ & I_{2m-3} & \\ & & g^*\end{pmatrix}\right]\psi_V(v)d(...)$$

is zero for all choice of data. Here, $x\in Mat^{f,m-1}_{2\times(2m-3)}$ is integrated over the group of matrices $Mat^{f,m-1}_{2\times(2m-3)}(F)\backslash Mat^{f,m-1}_{2\times(2m-3)}(\mathbf{A})$, where $f,m-1$ indicate that the first $m-1$ columns are zero. Also, y is a 2×2 matrix such that the first matrix in the argument of φ_π in integral (4) is a matrix in SO_{2m+1}. Let V denote the maximal unipotent subgroup of SO_{2m-3}. We embed it inside SO_{2m+1} as in the second matrix in integral (4). Then, v is integrated over $V(F)\backslash V(\mathbf{A})$. The character ψ_V is the Whittaker character of V. In other words, if $v=(v_{i,j})\in V$, then $\psi_V(v)=\psi(v_{1,2}+v_{2,3}+\cdots+v_{m-2,m-1})$. Next, z is integrated over $Mat^{l,m-1}_{2\times(2m-3)}(F)\backslash Mat^{l,m-1}_{2\times(2m-3)}(\mathbf{A})$, where $l,m-1$ indicate that the last $m-1$ rows are zero. Finally, g is integrated over $SL_2(F)\backslash SL_2(\mathbf{A})$.

Next we preform a sequence of Fourier expansions. We start by expanding integral (4) along the unipotent group $l(r_1,r_2)=I_{2m+1}+r_1(e_{1,m+1}-e_{m+1,2m+1})+r_1^2e_{1,2m+1}+r_2(e_{2,m+1}-e_{m+1,2m})+r_2^2e_{2,2m}$. Here, for all i,j, we denote by $e_{i,j}$ the $(2m+1)\times(2m+1)$ matrix which has one in the (i,j) entry and zero elsewhere. Thus, integral (4) is equal to

$$\sum_{\alpha_1,\alpha_2\in F}\int\varphi_\pi(l(r_1,r_2)(x,y,v,z,g))\psi_V(v)\psi(\alpha_1r_1+\alpha_2r_2)d(...).$$

Here, r_1 and r_2 are integrated over $F\backslash\mathbf{A}$, and all other variables as before. For fixed $\alpha_1, \alpha_2 \in F$ consider the matrix $s(\alpha_1, \alpha_2) = I_{2m+1} + \alpha_1(e_{m,1} - e_{2m+1,m+2}) + \alpha_2(e_{m,2} - e_{2m,m+2})$. Use the left invariant property of φ_π under matrices in $SO_{2m+1}(F)$, to write in the above integral $\varphi_\pi(h) = \varphi_\pi(s(\alpha_1, \alpha_2)h)$. Conjugating $s(\alpha_1, \alpha_2)$ to the right, changing variables in v, we obtain that the above integral is equal to

$$\int \varphi_\pi((x, y, v, z, g))\psi_V(v)d(...)$$

where now x is integrated over $Mat^{f,m-2}_{2\times(2m-3)}(F)\backslash Mat^{f,m-2}_{2\times(2m-3)}(\mathbf{A})$ and z is integrated over $Mat^{l,m-2}_{2\times(2m-3)}(F)\backslash Mat^{l,m-1}_{2\times(2m-3)}(\mathbf{A})$. All other variables are integrated as before. We continue this process and deduce that the following integral

$$\int \varphi_\pi \left[\begin{pmatrix} I_2 & x & y \\ & I_{2m-3} & x^* \\ & & I_2 \end{pmatrix} \begin{pmatrix} I_2 & & \\ & v & \\ & & I_2 \end{pmatrix} \begin{pmatrix} g & & \\ & I_{2m-3} & \\ & & g^* \end{pmatrix} \right] \psi_V(v)d(...)$$

is an inner integration to the integral (4). Here, x is integrated over the group of matrices $Mat^{f,1}_{2\times(2m-3)}(F)\backslash Mat^{f,1}_{2\times(2m-3)}(\mathbf{A})$, and all other variables as before. Thus it is enough to show that this integral is zero for all choice of data. To do that we expand it along the group $l(r_1, r_2) = I_{2m+1} + r_1(e_{1,3} - e_{2m-1,2m+1}) + r_2(e_{2,3} - e_{2m-1,2m})$. Thus the above integral is equal to

$$\sum_{\alpha_1, \alpha_2 \in F} \int \varphi_\pi(l(r_1, r_2)(x, y, v, g))\psi_V(v)\psi(\alpha_1 r_1 + \alpha_2 r_2)d(...).$$

Recall that g is integrated along $SL_2(F)\backslash SL_2(\mathbf{A})$. From the embedding of this group inside SO_{2m+1}, as described in integral (4), it follows that this group acts on the above expansion with two orbits. The trivial orbit contributes zero to the above integral. Indeed, in this case the integration over the x and y variable produces the integral of φ_π along the unipotent radical of the maximal parabolic subgroup of SO_{2m+1} whose Levi part is $GL_2 \times SO_{2m-3}$. By the cuspidality of π, it follows that it is zero. When considering the second orbit of the above expansion, the stabilizer inside SL_2 is the group N which consists of all upper unipotent matrices inside SL_2. If we combine this group with the x and y integration, we obtain as inner integration, an integration over a unipotent radical of a parabolic subgroup of SO_{2m+1}. This time it is the parabolic subgroup whose Levi part is $GL_1 \times SO_{2m-1}$. Thus it is also zero. Thus integral (4) is zero which implies that the representation τ' is cuspidal.

The proof of Theorem 2 is complete.

3. On the Lift from Three Orthogonal Groups

Let $1 \leq r$ and assume that $m \geq 2r + 1$. In this section we state our conjecture on when an irreducible generic cuspidal automorphic representation π of $SO_{2m+1}(\mathbf{A})$ is an endoscopic representation with respect to $SO_3 \times SO_{2r+1} \times SO_{2(m-r)-1}$. Notice that since $m \geq 2r+1$ then $2(m-r)-1 \geq 2r+1$.

To state the conjecture we first fix some notation. Let $E(g,s)$ denote the Eisenstein series on $SO_{4(r+1)}$ associated with the induced representation $Ind_{P(\mathbf{A})}^{SO_{4(r+1)}(\mathbf{A})}\delta_P^s$. Here P is the maximal parabolic subgroup of $SO_{4(r+1)}$ whose Levi part is $GL_{2(r+1)}$. For $Re(s) > 1/2$, the poles of this Eisenstein series were studied in [KR]. Let $\Theta_{[2^{2r}1^4]}$ denote the residual representation of $E(g,s)$ at the point $s_0 = \frac{r+1}{2r+1}$. Then one can show that the unipotent orbit attached to this residual representation is $[2^{2r}1^4]$.

With the above data, we define the following family of period integrals. First, if $m = 2r+1$, define

$$\mathcal{P}_r(\pi,\tau) = \int\limits_{SO_{4r+3}(F)\backslash SO_{4r+3}(\mathbf{A})} \varphi_\pi(g)\theta_{\tau,4(r+1)}(g)\theta_{[2^{2r}1^4]}(g)dg.$$

Here $\theta_{[2^{2r}1^4]}$ is a vector in the representation $\Theta_{[2^{2r}1^4]}$, and $\theta_{\tau,4(r+1)}$ is a vector in the residual representation $\Theta_{\tau,4(r+1)}$, as defined in [G].

When $m > 2r+1$, we define $\mathcal{P}_r(\pi,\tau)$ to be equal to

$$\int\limits_{SO_{4(r+1)}(F)\backslash SO_{4(r+1)}(\mathbf{A})} \int\limits_{V^m_{m-2r-2}(F)\backslash V^m_{m-2r-2}(\mathbf{A})} \varphi_\pi(vg)\psi_{V^m_{m-2r-2}}(v)\theta_{\tau,4(r+1)}(g)\theta_{[2^{2r}1^4]}(g)dvdg.$$

Here, for $1 \leq k \leq m$, we denote by V_k^m the unipotent subgroup of SO_{2m+1} defined as follows. Consider the standard parabolic subgroup of SO_{2m+1} whose Levi part is $GL_1^k \times SO_{2m-2k+1}$. We denote its unipotent radical by V_k^m. To define $\psi_{V_k^m}$, let $v = (v_{i,j}) \in V_k^m$. Then we set $\psi_{V_k}(v) = \psi(v_{1,2} + v_{2,3} + \ldots + v_{k-1,k} + v_{k,m+1})$. We have

Conjecture 2. *With the above notation, the representation π is endoscopic with respect to $SO_3 \times SO_{2r+1} \times SO_{2(m-r)-1}$ if and only if the period integral $\mathcal{P}_r(\pi,\tau)$ is not zero for some choice of data.*

The relation between the two conjectures is

Theorem 3. *Conjecture 1 implies Conjecture 2.*

Corollary 1. *Conjecture 2 holds for $r = 1$.*

3.1. **Proof of Theorem 3.** Assume first that π is endoscopic with respect to $SO_3 \times SO_{2r+1} \times SO_{2(m-r)-1}$. Thus there exist irreducible generic cuspidal automorphic representations τ, μ, and ϵ of $SO_3(\mathbf{A})$, $SO_{2r+1}(\mathbf{A})$, and $SO_{2(m-r)-1}(\mathbf{A})$, respectively, such that π is endoscopic with respect to (τ,μ,ϵ).

Consider the automorphic function on $SO_{2m-1}(\mathbf{A})$ defined by integral (26), and let σ denote the representation of $SO_{2m-1}(\mathbf{A})$ generated by these functions. Since π is endoscopic with respect to (τ, μ, ϵ), it follows from Theorem 1 that integral (2) is not zero for some choice of data. Applying Theorem 5 in the Appendix, we deduce that σ is a nonzero generic cuspidal representation. Moreover, it is endoscopic with respect to (μ, ϵ) as defined above. Since we assume Conjecture 1, we deduce that the integral

$$\mathcal{Q}_r(\sigma) = \int_{Sp_{2r}(F)\backslash Sp_{2r}(\mathbf{A})} \int_{U_r^{m-1}(F)\backslash U_r^{m-1}(\mathbf{A})} \varphi_\sigma(uh)\psi_{U_r^{m-1}}(u)dudh \tag{5}$$

is not zero for some choice of data. Notice that in integral (1), the representation π is defined on SO_{2m+1}, and in integral (5) the representation σ is defined on SO_{2m-1}. Also, we wrote U_r^{m-1} in integral (5) instead of U_r as written in integral (1). This means that here we view the groups Sp_{2r} and U_r^{m-1} as subgroups of SO_{2m-1}. Plugging integral (26) in integral (5) we deduce that the integral

$$\int_{Sp_{2r}(F)\backslash Sp_{2r}(\mathbf{A})} \int_{U_r^{m-1}(F)\backslash U_r^{m-1}(\mathbf{A})} \int_{SO_{2m+1}(F)\backslash SO_{2m+1}(\mathbf{A})} \varphi_\pi(g)\theta_{\tau,4m}((g,uh))\psi_{U_r^{m-1}}(u)dgdudh \tag{6}$$

is not zero for some choice of data. This integral converges, but not absolutely. In fact, because σ is a cuspidal representation, it follows that the integral

$$\int_{Sp_{2r}(F)\backslash Sp_{2r}(\mathbf{A})} \left| \int_{U_r^{m-1}(F)\backslash U_r^{m-1}(\mathbf{A})} \int_{SO_{2m+1}(F)\backslash SO_{2m+1}(\mathbf{A})} \varphi_\pi(g)\theta_{\tau,4m}((g,uh))\psi_{U_r^{m-1}}(u)dgdu \right| dh$$

converges. Thus, as long as we dont change the order of integration involving the h variable, we can preform a series of Fourier expansions.

The embedding of the matrices (g, uh) inside SO_{4m} is as follows. First we embed

$$(g,h) \mapsto \mathrm{diag}(I_{m-2r-1}, h, g, h^*, I_{m-2r-1}).$$

Then, the embedding of the group U_r^{m-1} is as follows

$$u = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \\ & I_{2r} & & y & u_4^* \\ & & 1 & & u_3^* \\ & & & I_{2r} & u_2^* \\ & & & & u_1^* \end{pmatrix} \mapsto (1,u) = \begin{pmatrix} u_1 & u_2 & 0 & u_3 & 0 & u_4 & u_5 \\ & I_{2r} & 0 & 0 & 0 & y & u_4^* \\ & & I_{2m} & & & 0 & 0 \\ & & & I_2 & & 0 & u_3^* \\ & & & & I_{2m} & 0 & 0 \\ & & & & & I_{2r} & u_2^* \\ & & & & & & u_1^* \end{pmatrix}. \tag{7}$$

Here, the variables are defined as follows. First, we have $u_1 \in Z_{m-2r-1}$ defined to be the standard maximal unipotent subgroup of GL_{m-2r-1}. Thus, Z_{m-2r-1} consists of all upper unipotent matrices. Next, we have $u_2 \in Mat_{(m-2r-1)\times 2r}$ and $u_3 \in Mat_{(m-2r-1)\times 1}$. The matrix $\begin{pmatrix} u_4 & u_5 \\ y & u_4^* \end{pmatrix} \in Mat_{(m-1)\times(m-1)}$ is such that the above matrix u is in SO_{2m-1}. Finally, all u_i^* are such that the above matrix is in SO_{4m}.

To proceed we shall apply an inductive argument. In other words, we shall relate integral (6) for the values (m,r) when $m > 2r+1$ with integral (6) for the values $(m-1,r)$.

Let Y_1 denote the unipotent subgroup of SO_{4m} which consists of all matrices of the form

$$y_1 = I_{4m} + \sum_{i=1}^{2m+2} [r_i(e_{1,m+i-1} - e_{m+i-1,4m})] + r^* e_{1,4m}. \tag{8}$$

Here r^*, which depends on $r_1, \ldots, r_{2m+2}$, is such that the above matrix is in SO_{4m}. Expand the function $\theta_{\tau,4m}$ in integral (6) along the group Y_1 with points in $F\backslash\mathbf{A}$. The group SO_{2m+2}, embedded in SO_{4m} as $t \mapsto \mathrm{diag}(I_{m-1}, t, I_{m-1})$ acts on this expansion with three type of orbits. Each term of the expansion corresponds to a vector of size $2m+2$. Consider the contribution to the expansion from the orbits corresponding to vectors of nonzero length. For $1 \le i \le 2m-1$, let L_i denote the unipotent radical of the standard parabolic subgroup of SO_{4m} whose Levi part is $GL_1^i \times SO_{2(2m-i)}$. Thus, when considering, in the above expansion, the terms corresponding to vectors with nonzero length, we get, when combining the integration over Y_1 and U_r^{m-1}, the integral

$$\int\limits_{L_1(F)\backslash L_1(\mathbf{A})} \theta_{\tau,4m}(l)\psi_L(l)dl$$

as inner integration. Here ψ_L is defined as follows. For $l = (l_{i,j})$ we have $\psi_L(l) = \psi(l_{1,2} + l_{1,2m} + \alpha l_{1,2m+1})$ for some $\alpha \in F^*$. It follows from the description of Fourier coefficients corresponding to unipotent orbits as described in [G2], that this Fourier coefficient corresponds to the unipotent orbit (31^{4m-3}). From Proposition 1 in the Appendix, it follows that the representation $\Theta_{\tau,4m}$ does not support this Fourier coefficients. Thus the contribution to the expansion from vectors of nonzero length is zero.

Similarly, the contribution to the expansion from the constant term along Y_1 also contributes zero. To see that we further expand the integral, this time along the unipotent group Y_2 given by

$$y_2 = I_{4m} + \sum_{i=1}^{2m+2} [r_i(e_{2,m+i-1} - e_{m+i-1,4m-1})] + r^* e_{2,4m-1}.$$

Acting on this expansion by the same copy of SO_{2m+2} we obtain two types of Fourier coefficients. The first type of Fourier coefficient corresponds to the unipotent orbit (51^{4m-5}) which contributes zero by Proposition 1 in the Appendix. In the second type of Fourier coefficient we obtain the integral

$$\int\limits_{L_2(F)\backslash L_2(\mathbf{A})} \theta_{\tau,4m}(l)\psi_{L_2}(l)dl$$

as inner integration. Here ψ_{L_2} is defined as follows. For $l = (l_{i,j})$ we have $\psi_{L_2}(l) = \psi(l_{1,2} + l_{2,3})$. Further expansions show that $\Theta_{\tau,4m}$ cannot support this Fourier coefficient.

Thus, in the expansion along Y_1 we are left with the term corresponding to nonzero vectors of zero length. Under the action of $SO_{2m+2}(F)$ this is one orbit, and hence integral (6) is equal to

$$\int \sum_{\gamma \in P^0(SO_{2m})(F)\backslash SO_{2m+2}(F)} \int_{Y_1(F)\backslash Y_1(\mathbf{A})} \varphi_\pi(g)\theta_{\tau,4m}(y_1\gamma(g,uh))\psi_{Y_1}(y_1)\psi_{U_r^{m-1}}(u)dy_1dgdudh.$$

Here $P^0(SO_{2m}) = SO_{2m}S_1$ is the subgroup of the standard maximal parabolic subgroup of SO_{2m+2} whose Levi part is $GL_1 \times SO_{2m}$. The character ψ_{Y_1} is given by $\psi_{Y_1}(y_1) = \psi(r_1)$ where we use the coordinates given in (8).

Consider the space of double cosets $P^0(SO_{2m})(F)\backslash SO_{2m+2}(F)/SO_{2m+1}(F)$. To choose a set of representatives, we first denote the following Weyl elements of SO_{4m} by

$$w_j = \text{diag}(I_{j-1}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, I_{4m-2j-2}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, I_{j-1})$$

Here $1 \le j \le 2m-1$. With this notation, representatives of the double cosets can be chosen as e and $w_m w_{m+1} \cdots w_{2m-2} w_{2m-1} t$ where t is a certain torus element. We claim that if $m+1 > 2r+1$ then the contribution to the integral from the second representative is zero. Indeed, consider the unipotent subgroup of SO_{2m-1} given by all matrices of the form $x(r) = I_{2m-1} + r(e_{1,m} - e_{m,2m-1}) + r^* e_{1,2m-1}$. Here r^* is such that $x(r)$ is a matrix in SO_{2m-1}. This is a subgroup of U_r^{m-1}, and assuming that $m+1 > 2r+1$, the character $\psi_{U_r^{m-1}}$ is trivial on this subgroup. By the parametrization given in (7), we have

$$(1, x(r)) = I_{4m} + r(e_{1,2m} - e_{1,2m+1} + e_{2m,4m} - e_{2m+1,4m}) + r^* e_{1,4m}.$$

Notice that the group $(1, x(r))$ is a subgroup of Y_1. Further more, after conjugation by $w_m w_{m+1} \cdots w_{2m-2} w_{2m-1} t$, this group is conjugated to the subgroup of Y_1 given by all matrices of the form $I_{4m} + r(e_{1,m} - e_{1,3m+1} + e_{m,4m} - e_{3m+1,4m}) + r^* e_{1,4m}$. It follows from the definition of ψ_{Y_1} that this character is not trivial on this group, and after a change of variables in Y_1, we obtain $\int\limits_{F\backslash\mathbf{A}} \psi(\alpha r)dr$ as inner integration. Here $\alpha \in F^*$ is a factor which is obtained from the torus element t. Thus, this integral is zero.

We are left with the representative e. Thus, the above integral is equal to

$$\int \int_{P^0(SO_{2m-1})(F)\backslash SO_{2m+1}(\mathbf{A})} \int_{Y_1(F)\backslash Y_1(\mathbf{A})} \varphi_\pi(g)\theta_{\tau,4m}(y_1(g,uh))\psi_{Y_1}(y_1)\psi_{U_r^{m-1}}(u)dy_1dgdudh.$$

Here $P^0(SO_{2m-1}) = SO_{2m-1}V_1^{m-1}$ is the subgroup of the maximal parabolic subgroup of SO_{2m+1} whose Levi part is $GL_1 \times SO_{2m-1}$. The group V_1^{m-1} was defined right after the definition of $\mathcal{P}_r(\pi, \tau)$. The other variables are integrated as before.

As subgroups of SO_{4m}, we have $V_1^{m-1}Y_1 = L_1$ where L_1 was defined right after (8). Also, V_1^{m-1} is a subgroup of U_r^{m-1}. Hence, if we factor the integration over that group, the above integral is equal to

$$\int\limits_{U_r^{m-1}(F)V_1^{m-1}(\mathbf{A})\backslash U_r^{m-1}(\mathbf{A})} \int \int\limits_{L_1(F)\backslash L_1(\mathbf{A})} \varphi_\pi(g)\theta_{\tau,4m}(l(g,uh))\psi'_{L_1}(l)\psi_{U_r^{m-1}}(u)dldgdudh.$$

Here, the character ψ'_{L_1} is defined as follows. For $l=(l_{i,j})\in L_1$ we have $\psi'_{L_1}(l)=\psi(l_{1,2}+l_{1,m})$. Notice that $V_1^{m-1}\backslash U_r^{m-1}$ can be identified with U_r^{m-2}. Denote $z_1 = w_2w_3\ldots w_{m-1}x_{\gamma_1}(1)$ where $x_{\gamma_1}(1) = I_{4m}+e_{2,m}-e_{3m+1,4m-1}$. Then $z_1\in SO_{4m}(F)$. Hence $\theta_{\tau,4m}(l(g,uh)) = \theta_{\tau,4m}(z_1 l(g,uh))$. Conjugating z_1 to the right, changing variables, we obtain

$$\int\limits_{P^0(SO_{2m-1})(F)\backslash SO_{2m+1}(\mathbf{A})} \int\limits_{U_r^{m-2}(F)\backslash U_r^{m-2}(\mathbf{A})} \varphi_\pi(g)\theta_{\tau,4m}^{L_1,\psi}(z_1(g,uh))\psi_{U_r^{m-2}}(u)dudgdh \tag{9}$$

where h is integrated over $Sp_{2r}(F)\backslash Sp_{2r}(\mathbf{A})$. Also, we denote

$$\theta_{\tau,4m}^{L_1,\psi}(z_1(g,uh)) = \int\limits_{L_1(F)\backslash L_1(\mathbf{A})} \theta_{\tau,4m}(lz_1(g,uh))\psi_{L_1}(l)dl \tag{10}$$

where ψ_{L_1} is defined as follows. For $l=(l_{i,j})\in L_1$ we define $\psi_{L_1}(l)=\psi(l_{1,2})$. In fact we have

$$\theta_{\tau,4m}^{L_1,\psi}(z_1(g,uh)) = \theta_{\tau,4m}^{L_2,\psi}(z_1(g,uh))$$

where the character ψ_{L_1} is extended trivially from L_1 to L_2. Indeed, to derive this identity, we expand integral (10) along the group $L_1\backslash L_2$ with points in $F\backslash\mathbf{A}$. The group $SO_{4m-4}(F)$ acts on this expansion with three type of orbits. In a similar way as in the expansion along Y_1 and then Y_2, as was done right after (8), we show that only the constant term gives a nonzero contribution. Thus the above identity holds.

In integral (9), we conjugate the matrices $(1,u)$, where $u\in U_r^{m-2}$, across z_1 and we obtain the matrices $(1,u)_1$ given by

$$u\mapsto (1,u)_1 = \begin{pmatrix} I_2 & & \\ & (1,u) & \\ & & I_2 \end{pmatrix}.$$

Here we view $(1,u)$ as an element in SO_{4m-4} embedded in SO_{4m} as above, and is parameterized as in (7). In this way we see that we reduced the computations to the case with $m-1$ instead of m.

We repeat this process $m-2r-2$ times. We deduce that integral (6) is not zero for some choice of data if and only if the integral

$$\int\limits_{P^0(SO_{4r+5})(F)\backslash SO_{2m+1}(\mathbf{A})} \int\limits_{U_r^{2r+1}(F)\backslash U_r^{2r+1}(\mathbf{A})} \varphi_\pi(g)\theta_{\tau,4m}^{L_{2(m-2r-2)},\psi}(\widetilde{z}(g,uh))\psi_{U_r^{2r+1}}(u)dudgdh \tag{11}$$

is not zero for some choice of data. Here h is integrated as in integral (9), and $\widetilde{z}$ is defined as follows. For $1 \le i \le m-2r-2$ let $x_{\gamma_i}(1) = I_{4m} + e_{2i,m+i-1} - e_{3m-i+2,4m-2i+1}$. Define $z_i = w_{2i}w_{2i+1}\ldots w_{m+i-2}x_{\gamma_i}(1)$. Then $\widetilde{z} = z_{m-2r-2}\ldots z_2 z_1$. Finally, we let $\theta_{\tau,4m}^{L_{2(m-2r-2)},\psi}$ be defined as in (10) where we integrate over $L_{2(m-2r-2)}(F)\backslash L_{2(m-2r-2)}(\mathbf{A})$, and $\psi_{L_{2(m-2r-2)}}(l)$ is defined as follows. For $l = (l_{i,j}) \in L_{2(m-2r-2)}$ we define

$$\psi_{L_{2(m-2r-2)}}(l) = \psi(l_{1,2} + l_{3,4} + l_{5,6} \cdots + l_{2(m-2r-2)-1,2(m-2r-2)}).$$

Next we expend integral (11) along the unipotent group $Y_{2(m-2r-2)+1}$ defined as the group of all matrices of the form

$$y = I_{4m} + \sum_{i=1}^{4r+6}[r_i(e_{2m-4r-3,2m-2r+i-3} - e_{2m+2r-i+4,2m+4r+4})] + r^* e_{2m-4r-3,2m+4r+4}. \tag{12}$$

Here r^* is defined in a similar way as in (8). The group $SO_{4r+6}(F)$, embedded as $g \mapsto \mathrm{diag}(I_{2m-2r-3}, g, I_{2m-2r-3})$, acts on this expansion with three types of orbits. One type of the orbits is the constant term, the second type corresponds to nonzero vectors with zero length, and the third type corresponds to all vectors which have nonzero length. As in the case of the expansion of Y_1, we show that the contributions to the integral from the constant term and from the nonzero length vectors, are zero. We are left with the orbit of all nonzero vectors which have zero length. Thus, integral (11) is equal to

$$\int \sum_{\gamma \in P^0(SO_{4r+4})(F)\backslash SO_{4r+6}(F)} \int_{Y_{2(m-2r-2)+1}(F)\backslash Y_{2(m-2r-2)+1}(\mathbf{A})} \varphi_\pi(g) \times$$
$$\theta_{\tau,4m}^{L_{2(m-2r-2)},\psi}(y\widetilde{z}\gamma(g,uh))\psi_{Y_{2(m-2r-2)+1}}(y)\psi_{U_r^{2r+1}}(u)\,dydudgdh.$$

Here $\psi_{Y_{2(m-2r-2)+1}}$ is defined as follows. In the coordinates of this group, as given in (12), we have $\psi_{Y_{2(m-2r-2)+1}}(y) = \psi(r_1)$. All other variables are integrated as in (11).

Next we consider the space of double cosets $P^0(SO_{4r+4})(F)\backslash SO_{4r+6}(F)/SO_{4r+5}(F)$. The space contains two types of elements, and as representatives we may choose e, and the elements $w_{2(m-r-1)}w_{2m-2r-1}\ldots w_{2m-2}w_{2m-1}h(\zeta)$. Here $h(\zeta) = \mathrm{diag}(I_{2m-1}, \zeta, \zeta^{-1}, I_{2m-1})$ where $\zeta \in F^*$. The contribution to the above integral from the identity element is zero. This follows from the definition of the character $\psi_{U_r^{2r+1}}$. As for the other representatives, it also follows from the definition of $\psi_{U_r^{2r+1}}$ that if $\zeta \ne 1$ then the contribution to the above integral is zero. Indeed, conjugating the element $u \in U_r^{2r+1}$ across $w_{2(m-r-1)}w_{2m-2r-1}\ldots w_{2m-2}w_{2m-1}h(\zeta)$, and then changing variables in $Y_{2(m-2r-2)+1}$, we obtain the integral $\int \psi((1-\zeta)r)dr$ as inner integration. Here we integrate over $F\backslash\mathbf{A}$, and hence this integral is zero unless $\zeta = 1$. Thus we are left with the Weyl element $w_0 = w_{2(m-r-1)}w_{2m-2r-1}\ldots w_{2m-2}w_{2m-1}$. The stabilizer of w_0 inside SO_{4r+5} is SO_{4r+4}, and hence we may collapse summation with integration in the above integral. It follows that g is now integrated over $V_{m-2r-2}^m(F)SO_{4(r+1)}(F)\backslash SO_{2m+1}(\mathbf{A})$.

We have the group identity

$$L_{2(m-2r-2)+1} = L_{2(m-2r-2)}Y_{2(m-2r-2)+1}w_0U_r^{2r+1}w_0^{-1}.$$

After a suitable conjugation, the above integral is equal to

$$(13)\qquad \int \int\limits_{U_r^{2r}(F)\backslash U_r^{2r}(\mathbf{A})} \varphi_\pi(g)\theta_{\tau,4m}^{L_{2(m-2r-2)+1},\psi}(z_0(g,uh))\psi_{U_r^{2r}}(u)dudgdh.$$

Here h is integrated over $Sp_{2r}(F)\backslash Sp_{2r}(\mathbf{A})$, and g is integrated over the quotient given by $V_{m-2r-2}^m(F)SO_{4(r+1)}(F)\backslash SO_{2m+1}(\mathbf{A})$. The element $z_0 \in SO_{4m}(F)$ is equal to $z'\widetilde{z}w_0$ where $z' = w_{2m-4r-3}w_{2m-4r-2}\ldots w_{2m-2r-3}$. Finally, $\theta_{\tau,4m}^{L_{2(m-2r-2)+1},\psi}$ is defined in a similar way as in (10) where the character of the group $L_{2(m-2r-2)+1}$ is defined as follows. If $l = (l_{i,j})$ then the character is defined by $\psi(l_{1,2}++l_{3,4}+l_{5,6}\cdots+l_{2(m-2r-2)-1,2(m-2r-2)}+l_{2(m-2r-2)+1,2(m-2r-2)+2})$. Also, in a similar way as is shown right after (10) we have $\theta_{\tau,4m}^{L_{2(m-2r-2)+1},\psi} = \theta_{\tau,4m}^{L_{2(m-2r-2)+2},\psi}$, where the character is extended trivially.

Next we factor the integration $V_{m-2r-2}^m(F)SO_{4(r+1)}(F)\backslash SO_{2m+1}(\mathbf{A})$ into

$$\int\limits_{V_{m-2r-2}^m(\mathbf{A})SO_{4(r+1)}(\mathbf{A})\backslash SO_{2m+1}(\mathbf{A})} \quad \int\limits_{SO_{4(r+1)}(F)\backslash SO_{4(r+1)}(\mathbf{A})} \quad \int\limits_{V_{m-2r-2}^m(F)\backslash V_{m-2r-2}^m(\mathbf{A})}.$$

A matrix multiplication, and a change of variables implies that for all $v \in V_{m-2r-2}^m(\mathbf{A})$ we have

$$\theta_{\tau,4m}^{L_{2(m-2r-2)+2},\psi}(z_0(vg,uh)) = \psi_{V_{m-2r-2}^m}(v)\theta_{\tau,4m}^{L_{2(m-2r-2)+2},\psi}(z_0(g,uh)).$$

We denote

$$\varphi_\pi^{V,\psi}(g) = \int\limits_{V_{m-2r-2}^m(F)\backslash V_{m-2r-2}^m(\mathbf{A})} \varphi_\pi(vg)\psi_{V_{m-2r-2}^m}(v)dv.$$

Using the same notation, and conjugating by z_0, integral (13) is equal to

$$(14)\qquad \int \int\limits_{U_r^{2r,z_0}(F)\backslash U_r^{2r,z_0}(\mathbf{A})} \varphi_\pi^{V,\psi}(g_0g)\theta_{\tau,4m}^{L_{2(m-2r-1)},\psi}(\widetilde{y}(g_0,h)_1z_0(g,1))\widetilde{\psi}(y)dydg_0dgdh.$$

Here

$$\widetilde{y}(g_0,h)_1 = \begin{pmatrix} I_a &&&& \\ & I_{2r} && y & \\ && I_{4(r+1)} && \\ &&& I_{2r} & \\ &&&& I_a \end{pmatrix}\begin{pmatrix} I_a &&&& \\ & h &&& \\ && g_0 && \\ &&& h^* & \\ &&&& I_a \end{pmatrix}$$

where $a = 2(m-2r-1)$ and $g_0 \in SO_{4(r+1)}$. In the above integral, g_0 is integrated over $SO_{4(r+1)}(F)\backslash SO_{4(r+1)}(\mathbf{A})$, the variable g over $V_{m-2r-2}^m(\mathbf{A})SO_{4(r+1)}(\mathbf{A})\backslash SO_{2m+1}(\mathbf{A})$ and all other variables as before. We also denote $U_r^{2r,z_0} = z_0U_r^{2r}z_0^{-1}$, and define the character $\widetilde{\psi}$ by $\widetilde{\psi}(y) = \psi(y_{1,1}+y_{2,2}+\cdots+y_{r,r})$.

Consider the group SO_{8r+4} embedded inside SO_{4m} as $t \mapsto \mathrm{diag}(I_a, t, I_a)$ where $t \in SO_{8r+4}$. It follows from the above that the matrices $\widetilde{y}(g_0, h)_1$ are embedded inside the maximal parabolic subgroup of SO_{8r+4} whose Levi part is $GL_{2r} \times SO_{4r+4}$. Then, the unipotent radical of this parabolic group, which we shall denote by W_r, has a structure of a generalized Heisenberg group. Its center is the group U_r^{2r,z_0}. To proceed with the computations, we now apply the theory of Fourier Jacobi coefficients, as developed in [I]. In our context, this theory asserts that as a function of g_0 and h, the space of functions

$$(15) \qquad \theta_{\phi,\psi}^{Sp_{8r(r+1)}}((g_0, h)) \int\limits_{W_r(F)\backslash W_r(\mathbf{A})} \theta_{\tau,4m}^{L_{2(m-2r-1)},\psi}(w(g_0, h)_1 z_0(g, 1))\theta_{\phi_1,\psi}^{Sp_{8r(r+1)}}(l(w)(g_0, h))dw$$

is a dense subspace inside the space of functions

$$(g_0, h) \mapsto \int\limits_{U_r^{2r,z_0}(F)\backslash U_r^{2r,z_0}(\mathbf{A})} \theta_{\tau,4m}^{L_{2(m-2r-1)},\psi}(\widetilde{y}(g_0, h)_1 z_0(g, 1))\widetilde{\psi}(y)dy$$

In (15), the function $\theta_{\phi,\psi}^{Sp_{8r(r+1)}}$ is the theta function defined on the double cover of $Sp_{8r(r+1)}(\mathbf{A})$. Also, ϕ and ϕ_1 are suitable Schwartz functions in $\mathcal{S}(\mathbf{A}^{2r\times 2(r+1)})$. We denote by l the projection from W_r onto the Heisenberg group with $8r(r+1)+1$ variables.

Denote the integral in (15) by $F(g_0, h; g)$. Here, g is a fixed element in $SO_{2m+1}(\mathbf{A})$. We have

Lemma 1. *For all $h \in Sp_{2r}(\mathbf{A})$, we have $F(g_0, h; g) = F(g_0, 1; g)$.*

The proof of this lemma will be given after the proof of the theorem. Assuming the lemma, we deduce that integral (14) is not zero for some choice of data if and only if the integral

$$(16) \qquad \int \varphi_\pi^{V,\psi}(g_0 g)\theta_{\phi,\psi}^{Sp_{8r(r+1)}}((g_0, h))F(g_0, 1; g)dg_0 dg dh$$

is not zero for some choice of data. Here all variables are integrated as before.

Next we calculate the inner integration of (16) along the variable h, which is

$$(17) \qquad \int\limits_{Sp_{2r}(F)\backslash Sp_{2r}(\mathbf{A})} \theta_{\phi,\psi}^{Sp_{8r(r+1)}}((g_0, h))dh.$$

This integral may not converge for general function $\phi \in \mathcal{S}(\mathbf{A}^{2r\times 2(r+1)})$. Thus we use regularization by an Hecke algebra element at one unramified local place to extend the integral to general Schwartz functions $\phi \in \mathcal{S}(\mathbf{A}^{2r\times 2(r+1)})$. This is done in [JS], Theorem 2.4. The theorem states that there exists an Hecke algebra element α_{ν_0} in the Hecke algebra of $SO_{4(r+1)}$ at a finite local place ν_0, such that the integral

$$(18) \qquad c_{\alpha_{\nu_0}}^{-1} \cdot \int\limits_{Sp_{2r}(F)\backslash Sp_{2r}(\mathbf{A})} \theta_{\omega_{\psi_{\nu_0}}(1,\alpha_{\nu_0})\phi,\psi}^{Sp_{8r(r+1)}}((g_0, h))dh$$

converges absolutely, and is the unique extension of integral (17) to general Schwartz functions ϕ in $\mathcal{S}(\mathbf{A}^{2r\times 2(r+1)})$. The second part of Theorem 2.4, [JS], states that the integral (17) or equivalently (19) is identified as the residue at $s=\frac{r+1}{2r+1}$ of the Eisenstein series $E(g,s)$ on $SO_{4(r+1)}(\mathbf{A})$, which is denoted by $\Theta_{[2^{2r}1^4]}$.

Hence, integral (16) is not zero for some choice of data, if and only if the integral

$$\int\limits_{V^m_{m-2r-2}(\mathbf{A})SO_{4(r+1)}(\mathbf{A})\backslash SO_{2m+1}(\mathbf{A})}\ \int\limits_{SO_{4(r+1)}(F)\backslash SO_{4(r+1)}(\mathbf{A})} \varphi_\pi^{V,\psi}(g_0g)F(g_0,1;g)\theta_{[2^{2r}1^4]}(g_0)dg_0dg$$

is not zero for some choice of data. By applying a standard argument, for example as in [GS] section 7, we deduce that the above integral is nonzero for some choice of data if and only if the integral

$$\int\limits_{SO_{4(r+1)}(F)\backslash SO_{4(r+1)}(\mathbf{A})} \varphi_\pi^{V,\psi}(g_0)F(g_0,1;1)\theta_{[2^{2r}1^4]}(g_0)dg_0$$

is not zero for some choice of data. Applying Proposition 2 from Appendix, it follows from the definitions of the group $L_{2(m-2r-1)}$ and the character defined on this group, that the above integral is not zero for some choice of data if and only if the integral

$$\int \varphi_\pi^{V,\psi}(g_0)\int\limits_{W_r(F)\backslash W_r(\mathbf{A})} \theta_{\tau,8r+4}(wg_0)\theta_{\phi_1,\psi}^{Sp_{8r(r+1)}}(l(w)(g_0,1))dw\theta_{[2^{2r}1^4]}(g_0)dg_0 \tag{19}$$

is not zero for some choice of data. Here g_0 is integrated as before. Unfolding the theta function (for the action of the Weil representation, see [I]), and collapsing summation with integration, the inner integral is equal to

$$\int\limits_{W_r'(\mathbf{A})}\int\limits_{X(F)\backslash X(\mathbf{A})}\int\limits_{Y(F)\backslash Y(\mathbf{A})} \theta_{\tau,8r+4}((x,y)g_0w')\omega_\psi(l(x,y)g_0l(w'))\phi_1(0)dxdydw' = \tag{20}$$

$$\int\limits_{W_r'(\mathbf{A})}\int\limits_{X(F)\backslash X(\mathbf{A})}\int\limits_{Y(F)\backslash Y(\mathbf{A})} \theta_{\tau,8r+4}((x,y)g_0w')\psi_r(y)\phi_1(w')dxdydw'.$$

Here, the embedding of X and Y in SO_{8r+4} is given by

$$(x,y)g_0\mapsto \begin{pmatrix} I_r & & x & y_1 & y_2\\ & I_r & & y_3 & y_1^*\\ & & I_{4r+4} & & x^*\\ & & & I_r & \\ & & & & I_r\end{pmatrix}\begin{pmatrix} I_{2r} & & \\ & g_0 & \\ & & I_{2r}\end{pmatrix},\qquad y=\begin{pmatrix} y_1 & y_2\\ y_3 & y_1^*\end{pmatrix}.$$

The character ψ_r is defined by $\psi_r(y)=\psi(\mathrm{tr}y_1)$. Since ϕ_1 is an arbitrary Schwartz function, it follows that integral (19) is nonzero for some choice of data if and only if the integral

(21)
$$\int\limits_{SO_{4(r+1)}(F)\backslash SO_{4(r+1)}(\mathbf{A})}\varphi_\pi^{V,\psi}(g_0)\int\limits_{X(F)\backslash X(\mathbf{A})}\int\limits_{Y(F)\backslash Y(\mathbf{A})}\theta_{\tau,8r+4}((x,y)g_0)\psi_r(y)\theta_{[2^{2r}1^4]}(g_0)dxdydg_0$$

is not zero for some choice of data. Next we preform the following series of Fourier expansions. Recall that for the above matrix y to be in the orthogonal group, then we have

$$y=\begin{pmatrix} y_1 & y_2\\ y_3 & y_1^*\end{pmatrix}\in Mat_{2r}^0=\{A\in Mat_{2r}\ :\ J_{2r}A+A^tJ_{2r}=0\}$$

where J_{2r} is the matrix of size $2r$ which has ones on the other diagonal and zeros elsewhere. Let Y^0 denote the subgroup of Y which consists of all matrices y as above such that $y_{i,j}=0$ for all $i\le j$. Let Z_{2r}^e denote the group of all matrices of the form

$$Z_{2r}=\left\{z=\begin{pmatrix} z_1 & z_2\\ & I_r\end{pmatrix}\ :\ z_1\in Z_{2r}',\ z_2\in Z_{2r}''\right\}.$$

Here Z_{2r}' consists of all upper unipotent matrices in GL_r, and Z_{2r}'' consists of all matrices of size r such that J_rz_2 is lower triangular. Also, the matrix J_r is defined in a similar way as we defined J_{2r}. We shall denote by Z_{2r} the subgroup of Z_{2r}^e which consists of all matrices above such that the diagonal entries of J_rz_2 are also zero.

We embed these groups in SO_{8r+4} as all matrices of the form $z\mapsto \mathrm{diag}(z,I_{4r+4},z^*)$ and continue to denote it by Z_{2r}^e or Z_{2r}.

In integral (21) we expand the function $\theta_{\tau,8r+4}((x,y)g_0)$ along the group Z_{2r}. Collapsing summation and integration over Y^0, integral (21) is equal to

(22)
$$\int\int\limits_{Y^0(\mathbf{A})}\varphi_\pi^{V,\psi}(g_0)\int\int\limits_{Z_{2r}(F)\backslash Z_{2r}(\mathbf{A})}\theta_{\tau,8r+4}(z(x,yy_0)g_0)\psi_r(y)\theta_{[2^{2r}1^4]}(g_0)dzdxdydy_0dg_0.$$

Here g_0 and x are integrated as before, y is integrated over $Y(F)Y^0(\mathbf{A})\backslash Y(\mathbf{A})$, and y_0 is integrated over $Y^0(\mathbf{A})$.

In fact we can extend trivially the integration from Z_{2r} to Z_{2r}^e. Indeed, we identify the quotient $Z_{2r}\backslash Z_{2r}^e$ with the group of unipotent matrices $I_{8r+4}+\sum_{i=1}^r r_i(e_{i,2r-i+1}-e_{6r+i+4,8r-i+5})$. Thus, expending integral (22) along this quotient, we get two terms. The first term corresponds to the constant term, and the second term corresponds to all other elements in the expansion. The second term contributes zero to the expansion. Indeed, a similar argument as follows after (8) shows that this term will contain as inner integration, a Fourier coefficient of $\theta_{\tau,8r+4}$ which corresponds to the unipotent orbit (31^{8r+1}). It follows from Proposition 1 from Appendix, applied to $8r+4$ instead of $4m$, that this Fourier coefficient is zero. Thus we can replace in integral (22), the integration over $Z_{2r}(F)\backslash Z_{2r}(\mathbf{A})$ by the integration over $Z^e_{2r}(F)\backslash Z^e_{2r}(\mathbf{A})$.

Next we define the following Weyl element $\widetilde{w}$ of SO_{8r+4}. For all $1 \le i \le r$ set

$$\widetilde{w}_{2i-1,i} = \widetilde{w}_{2i,6r+i+4} = \widetilde{w}_{8r-i+5,8r-2i+6} = \widetilde{w}_{2r-i+1,8r-2i+5} = 1.$$

Also, for $2r+1 \le i \le 6r+4$ set $\widetilde{w}_{i,i} = 1$. All other entries of $\widetilde{w}$ are zero. Matrix multiplication implies that $L'_{2r} = \widetilde{w}(Z^e_{2r} \cdot X \cdot Y^0\backslash Y)\widetilde{w}^{-1}$ is a subgroup of L_{2r}. The groups L_i were defined as subgroups of SO_{4m} right after (8). In a similar way we define them as subgroups of any even orthogonal group, in particular for SO_{8r+4}. Similarly, we define the character $\psi_{L_{2r}}(l)$. More precisely, for $l = (l_{i,j}) \in L_{2r}$ we define $\psi_{L_{2r}}(l) = \psi(l_{1,2} + l_{3,4} + \cdots + l_{2r-1,2r})$. It follows from the matrix multiplication, that after the conjugation by $\widetilde{w}$ the character ψ_r is conjugated to $\psi_{L_{2r}}$ restricted to L'_{2r}. Thus, integral (22) is not zero for some choice of data if and only if the integral

$$(23) \qquad \int\limits_{SO_{4(r+1)}(F)\backslash SO_{4(r+1)}(\mathbf{A})} \int\limits_{Y^0(\mathbf{A})} \varphi_\pi^{V,\psi}(g_0)\theta_{\tau,8r+4}^{L'_{2r},\psi}(\widetilde{w}y_0g_0)\theta_{[2^{2r}1^4]}(g_0)dy_0dg_0$$

is not zero for some choice of data. Here $\theta_{\tau,8r+4}^{L'_{2r},\psi}$ is defined in a similar way as in (10). We claim that $\theta_{\tau,8r+4}^{L'_{2r},\psi} = \theta_{\tau,8r+4}^{L_{2r},\psi}$. In other words, we claim that integral (23) is not zero for some choice of data if and only if the integral

$$(24) \qquad \int\limits_{SO_{4(r+1)}(F)\backslash SO_{4(r+1)}(\mathbf{A})} \int\limits_{Y^0(\mathbf{A})} \varphi_\pi^{V,\psi}(g_0)\theta_{\tau,8r+4}^{L_{2r},\psi}(\widetilde{w}y_0g_0)\theta_{[2^{2r}1^4]}(g_0)dy_0dg_0$$

is not zero for some choice of data. This argument follows in a similar way as the argument given right after (10) when we extended the integration from L_1 to L_2 as were defined there. In our case, we expand integral (23) along the quotient $L'_{2r}\backslash L_{2r}$. Each element in this expansion which is not the constant term contributes zero. This follows again by using Proposition 1 from Appendix. Indeed, all Fourier coefficients which are not the constant term corresponds to unipotent orbits which are larger or not related to $(2^{2(r+1)})$.

Arguing as above, in a similar way as in [GS], we deduce that integral (24) is not zero for some choice of data if and only if the integral

$$\int\limits_{SO_{4(r+1)}(F)\backslash SO_{4(r+1)}(\mathbf{A})} \varphi_\pi^{V,\psi}(g_0)\theta_{\tau,8r+4}^{L_{2r},\psi}(\widetilde{w}g_0)\theta_{[2^{2r}1^4]}(g_0)dg_0$$

is not zero for some choice of data. From the definition of $\widetilde{w}$ we have $\widetilde{w}g_0 = g_0\widetilde{w}$. Applying Proposition 2 from Appendix we deduce that the above integral is not zero for some choice of data if and only if the integral

$$\int\limits_{SO_{4(r+1)}(F)\backslash SO_{4(r+1)}(\mathbf{A})} \varphi_\pi^{V,\psi}(g_0)\theta_{\tau,4r+4}(g_0)\theta_{[2^{2r}1^4]}(g_0)dg_0$$

is not zero for some choice of data. But this is integral $\mathcal{P}_r(\pi,\tau)$.

To complete the proof of this step we still need to prove Lemma 1, which is done in the next subsection.

Next we need to prove the converse of Conjecture 2. In other words, we need to prove that if $\mathcal{P}_r(\pi,\tau)$ is not zero for some choice of data, then π is an endoscopic representation with respect to $SO_3\times SO_{2r+1}\times SO_{2(m-r)-1}$. This is done by reversing the arguments in the first part. Indeed, starting with the fact that $\mathcal{P}_r(\pi,\tau)$ is not zero for some choice of data, we obtain by applying the converse of the above computations, that the integral (6) is not zero for some choice of data. But this implies that if we denote by σ the space of representation of $SO_{2m-1}(\mathbf{A})$, generated by the functions given by integral (26), then σ is not zero and by (5), we obtain that $\mathcal{Q}_r(\sigma)$ is not zero. Also, from Theorem 4 from Appendix we deduce that σ is a cuspidal representation. Moreover, since σ is not zero, then integral (27) is not zero for some choice of data, which implies that π is endoscopic with respect to τ and σ. Applying Conjecture 1 we obtain that σ is endoscopic with respect to $SO_{2r+1}\times SO_{2(m-r)-1}$, and we are done.

This completes the proof.

3.2. **Proof of Lemma 1.** Let $p(k)=I_{2r}+ke_{1,2r}$. Then the group generated by these matrices is a unipotent subgroup of Sp_{2r} which corresponds to the highest root of Sp_{2r}. Thus, to prove the lemma it is enough to prove that $F(g_0,p(k)h;g)=F(g_0,h;g)$ for all $k\in\mathbf{A}$. To prove that it is enough to prove that the integral

$$\int\limits_{F\backslash\mathbf{A}} F(g_0,p(k)h;g)\psi(\alpha k)dk \tag{25}$$

is zero for all choice of data, and all $\alpha\in F^*$. To prove that, we may assume that g_0,h and g are the identity elements. Thus, from the definition of the function $F(g_0,h;g)$ we need to prove that the integral

$$\int\limits_{W_r(F)\backslash W_r(\mathbf{A})}\int\limits_{F\backslash\mathbf{A}} \theta_{\tau,8r+4}(wp(k))\theta_{\phi_1,\psi}^{Sp_{8r(r+1)}}(l(w)(1,p(k)))\psi(\alpha k)dkdw$$

is zero for all choice of data. We unfold the theta function, and as in (20) and (21), we deduce that it is enough to prove that the integral

$$\int\limits_{X(F)\backslash X(\mathbf{A})}\int\limits_{Y(F)\backslash Y(\mathbf{A})}\int\limits_{F\backslash \mathbf{A}}\theta_{\tau,8r+4}((x,y)p(k))\psi_r(y)\psi(\alpha k)dkdxdy$$

is zero for all choice of data.

Defining the group Z_{2r} and Z_{2r}^e as defined right before (22), it is enough to prove that the integral

$$\int\limits_{Y^0(\mathbf{A})}\int\limits_{X(F)\backslash X(\mathbf{A})}\int\limits_{Y(F)\backslash Y(\mathbf{A})}\int\limits_{F\backslash \mathbf{A}}\int\limits_{Z_{2r}^e(F)\backslash Z_{2r}^e(\mathbf{A})}\theta_{\tau,8r+4}(z(x,yy_0)p(k))\psi_r(y)\psi(\alpha k)dzdkdxdydy_0$$

is zero for all choice of data. The subgroup generated by all $p(k)$ is contained in $Z_{2r}\backslash Z_{2r}^e$, and as explained right after (22), the character ψ_r is trivial on this quotient. Hence it is trivial on $p(k)$, and conjugating this matrix to the left, and changing variables, we obtain the integral $\int\limits_{F\backslash \mathbf{A}}\psi(\alpha k)dk$ as inner integration. Since $\alpha\in F^*$, this integral is zero. Hence integral (25) is zero for all choice of data.

Hence this completes the proof.

4. Appendix

In this appendix we review the basic results regarding the construction of the endoscopic lifting as appears in [G]. Some more details can be found also in [G1]. Since we will need the details only for a special case, we will concentrate only on the relevant details.

Let τ denote an irreducible cuspidal representation of GL_2 with a trivial central character. Let $\Theta_{\tau,4m}$ denote the residue representation defined on $SO_{4m}(\mathbf{A})$ as constructed in [G] page 461 case 5. (In the notations of that reference take $m=1$ and $n=r$). The basic property of this representation is given by

Proposition 1. *We have* $\mathcal{O}_{SO_{4m}}(\Theta_{\tau,4m})=(2^{2m})$.

The definition of this notation and the proof of a similar proposition, for the symplectic group, can be found in [G] Theorem 1. See also [G] page 465 case (d).

Let π denote a generic irreducible cuspidal representation of $SO_{2m+1}(\mathbf{A})$. Following [G], we consider the space of functions defined on $SO_{2m-1}(\mathbf{A})$ given by

$$f(h)=\int\limits_{SO_{2m+1}(F)\backslash SO_{2m+1}(\mathbf{A})}\varphi_\pi(g)\theta_{\tau,4m}((g,h))dg. \tag{26}$$

Here $\theta_{\tau,4m}$ is a vector in the space of the representation $\Theta_{\tau,4m}$. The function $f(h)$ defines an automorphic function of $SO_{2m-1}(\mathbf{A})$, and we denote by σ the representation of $SO_{2m-1}(\mathbf{A})$ generated by the above functions. We have

Theorem 4. *The representation σ is a cuspidal representation of $SO_{2m-1}(\mathbf{A})$.*

This is Theorem 2 in [G]. The details in that case are given for the symplectic group, but are the same for the orthogonal group.

Next, we consider the nonvanishing of the representation σ. We have

Theorem 5. *The representation σ is a generic representation if and only if the representation π is an endoscopic representation with respect to τ and σ.*

This is Theorem 6 in [G]. The sketch of the proof is as follows. A computation of the Whittaker coefficient of σ implies that it is not zero for some choice of data if and only if integral (2) is not zero for some choice of data. But as argued in [G], the nonvanishing of this period integral is equivalent to the fact that π is an endoscopic with respect to τ and a certain automorphic representation of $SO_{2m-1}(\mathbf{A})$. If σ is not zero, then the period integral

$$(27) \qquad \int\limits_{SO_{2m+1}(F)\backslash SO_{2m+1}(\mathbf{A})} \int\limits_{SO_{2m-1}(F)\backslash SO_{2m-1}(\mathbf{A})} \varphi_\pi(g)\varphi_\sigma(h)\theta_{\tau,4m}((g,h))dhdg$$

is not zero for some choice of data. But then, a similar argument given in Sections 4 and 5 in [G], or in the last section of [G1] proves that π is endoscopic with respect to τ and σ.

Finally, we need to study a certain Fourier coefficient of $\Theta_{\tau,4m}$. Let L_2 denote the unipotent radical of the standard parabolic subgroup of SO_{4m} whose Levi part is $GL_1^2 \times SO_{4m-4}$. We define a character ψ_{L_2} on this unipotent group as follows. For $l = (l_{i,j}) \in L_2$ define $\psi_{L_2}(l) = \psi(l_{1,2})$. For $g \in SO_{4m-4}(\mathbf{A})$, the following functions

$$g \mapsto \int\limits_{L_2(F)\backslash L_2(\mathbf{A})} \theta_{\tau,4m}(lg)\psi_{L_2}(l)dl$$

are automorphic functions of $SO_{4m-4}(\mathbf{A})$. We have

Proposition 2. *The representation of $SO_{4m-4}(\mathbf{A})$ generated by all the above functions is $\Theta_{\tau,4m-4}$.*

The proof of this proposition follows from the definition of $\Theta_{\tau,4m}$ as a multi residue of a certain Eisenstein series. See proposition 1 in [G] for some details in a similar case done for the symplectic group.

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