Clay Mathematics Proceedings

Poles of L-functions and Theta Liftings for Orthogonal Groups, II.

David Ginzburg, Dihua Jiang, and David Soudry

Dedicated to Freydoon Shahidi

Abstract. We bound the first occurrence in the theta correspondence of irreducible cuspidal automorphic representations σ of orthogonal groups, in terms of their generalized Gelfand-Graev periods. We also obtain a local analog at a finite place. As a result, we determine a range of holomorphy of $L^S(s,\sigma)$ in the right half plane in terms of the local generalized Gelfand-Graev models of σ at one finite place.

1. Introduction

In [**GJS07**], we characterized the first occurrence of irreducible cuspidal automorphic representations of $\mathrm{O}_m(\mathbb{A})$ under the theta correspondence to $\mathrm{Mp}_{2n}(\mathbb{A})$, where $\mathrm{Mp}_{2n}(\mathbb{A})$ is either $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ (when m is odd) or $\mathrm{Sp}_{2n}(\mathbb{A})$ (when m is even) in terms of the existence of poles of certain Eisenstein series (Theorem 1.3, [**GJS07**]). Here, $\mathbb{A}$ is the ring of adeles of a number field k. As a consequence, we determined a range of holomorphy in the right half plane for the standard partial L-functions $L^S(s,\sigma)$ of irreducible cuspidal automorphic representations σ of $\mathrm{O}_m(\mathbb{A})$ (Theorem 1.1 in [**GJS07**]). These results can be viewed as a natural extension to orthogonal groups of the work of Kudla and Rallis on symplectic groups ([**KR94**]) and as a completion to Moeglin's work ([**M97a**] and [**M97b**]).

In this paper, we discuss the relations between the global or local theta correspondence and the generalized Gelfand-Graev periods or models. As a consequence, we determine a range of holomorphy in the right half plane of the standard partial L-functions $L^S(s,\sigma)$ in terms of local generalized Gelfand-Graev models supported by a local component σ_v at one finite place v. A preliminary version of such a result was given in [**GJS07**] (Theorem 1.7), and some related very interesting applications were discussed in §7 of [**GJS07**].

For an irreducible cuspidal automorphic representation (σ, V_σ) of $\mathrm{O}_m(\mathbb{A})$, we define in §2.2 $\psi_{t,\alpha}$-Fourier coefficients of $\phi_\sigma \in V_\sigma$ in (2.14). The characters $\psi_{t,\alpha}$ are parametrized by integers t and a square classes α in k. Let r be the Witt index

All three authors are partially supported by the USA-Israel Binational Science Foundation, and the second named author is also supported in part by NSF grant DMS-0653742.

of the quadratic space defining O_m. We assume that r is positive, $1 \le t \le r$, and $2t < m$. Similarly, we define in (2.17) the notion of a $\psi_{t,\alpha}$-functional in the local setting. The main result of this paper can be formulated as follows.

THEOREM 1.1 (Main). *Let σ be an irreducible cuspidal automorphic representation of $\mathrm{O}_m(\mathbb{A})$ and t -as above.*

1. *If there exists one finite local place v of k, such that the local component σ_v of σ has a nonzero $\psi_{t,\alpha}$-functional, then the partial L-function $L^S(s,\sigma)$ is holomorphic for $Re(s) > \frac{m}{2} - t$. In particular, if σ has a nonzero $\psi_{t,\alpha}$-Fourier coefficient, then the partial L-function $L^S(s,\sigma)$ is holomorphic for $Re(s) > \frac{m}{2} - t$.*
2. *Assume that σ has a nonzero $\psi_{t,\alpha}$-Fourier coefficient. If either $t < r-1$, or $t < r$ and α is represented by the quadratic form corresponding to the anisotropic kernel of the quadratic space defining O_m, then $L^S(s,\sigma)$ is holomorphic for $Re(s) > \frac{m}{2} - t - 1$.*

REMARK 1.2. *Write $m = m_0 + 2r$. When $m_0 = 1$, O_m is the split orthogonal group in $2r+1$ variables. When $t = r$, $\psi_{t,\alpha}$ is a Whittaker character, and the assertion of the first part of the main theorem is that $L^S(s,\sigma)$ is holomorphic when $Re(s) > \frac{1}{2}$. This is Theorem 1.5 in* [**GJS07**]. *If $t = r-1$, then σ has a nonzero $\psi_{r-1,\alpha}$-Fourier coefficient. This case was discussed in* § *7 of* [**GJS07**].

When $m_0 = 0$, O_m is the split orthogonal group in $2r$ variables. If $t = r-1$, then $\psi_{r-1,\alpha}$ is a Whittaker character, and the assertion of the first part of the main theorem is that $L^S(s,\sigma)$ is holomorphic when $Re(s) > 1$. This is Theorem 1.5 in [**GJS07**].

We first prove in §4 that the nonvanishing of $\psi_{t,\alpha}$-Fourier coefficients of σ determines a range of the lowest occurrence $\mathrm{LO}_\psi(\sigma)$ (defined in §3.1) of ψ-theta lifts of σ. Then we establish the corresponding local version of this global result. This is done by an explicit calculation of the $\psi_{t,\alpha}$-Fourier coefficient of theta lifts of cuspidal automorphic representations from $\mathrm{Mp}_{2n}(\mathbb{A})$ to $\mathrm{O}_m(\mathbb{A})$, and an analogous calculation in the local setting. At the first occurrence, we get a relation between these $\psi_{t,\alpha}$- Fourier coefficients (respectively, functionals in the local setting) and Whittaker coefficients (resp. models), corresponding to ψ and α, on the symplectic or metaplectic side. Finally, we use Theorem 1.1 in [**GJS07**]. Since we quote this theorem several times in this paper, we bring it here for convenience.

THEOREM 1.3 (Theorem 1.1 in [**GJS07**]). *Let σ be an irreducible cuspidal automorphic representation of $\mathrm{O}_m(\mathbb{A})$.*

1. *If $L^S(s,\sigma)$ has a pole at $s_0 = \frac{m}{2} - j > 0$, or if m is odd and $L^S(s,\sigma)$ does not vanish at $s = \frac{1}{2}$, and we let $j = 2[\frac{m}{2}]$, then there is an automorphic sign character ϵ of $\mathrm{O}_m(\mathbb{A})$, such that the ψ-theta lift of $\sigma \otimes \epsilon$ to $\mathrm{Mp}_{2j}(\mathbb{A})$ does not vanish, i.e. $\mathrm{LO}_\psi(\sigma) \le 2j$.*
2. *If $\mathrm{LO}_\psi(\sigma) = 2j_0 < m$, then $L^S(s,\sigma)$ is holomorphic for $Re(s) > \frac{m}{2} - j_0$.*
3. *If $\mathrm{LO}_\psi(\sigma) = 2j_0 \ge m$, then $L^S(s,\sigma)$ is holomorphic for $Re(s) \ge \frac{1}{2}$.*

The authors thank the referee for his comments and suggestions.

2. The Generalized Gelfand-Graev Periods

Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. Let $(X_m, (\cdot,\cdot))$ be a nondegenerate quadratic vector space over k of dimension m and Witt index r. We assume that $r \geq 1$. For any nonnegative integer a, we denote by

$$\mathcal{H}_a = \ell_a^+ \oplus \ell_a^- \tag{2.1}$$

the polarization of the $2a$-dimensional quadratic k-vector space $\mathcal{H}_a$, which is the direct sum of a-copies of the hyperbolic plane. Then X_m can be written as

$$X_m := X_{m_0} \perp (\ell_r^+ \oplus \ell_r^-) = X_{m_0} \perp \mathcal{H}_r, \tag{2.2}$$

where X_{m_0} ($m_0 = m-2r$) is the m_0-dimensional anisotropic quadratic vector space, which is called the anisotropic kernel of X_m.

We may choose a basis for X_m

$$\{e_1, \cdots, e_r; \epsilon_1, \cdots, \epsilon_{m_0}; e_{-r}, \cdots, e_{-1}\} \tag{2.3}$$

such that

$$(e_i, e_j) = \begin{cases} 1 & \text{if } j = -i; \\ 0 & \text{if } j \neq -i, \end{cases}$$

and $(e_i, \epsilon_j) = 0$ for all $i \in \{\pm 1, \cdots, \pm r\}$ and $j \in \{1, \cdots, m_0\}$, where $\{e_1, \cdots, e_r\}$ is a basis for ℓ_r^+, $\{e_{-r}, \cdots, e_{-1}\}$ is a basis for ℓ_r^- and $\{\epsilon_1, \cdots, \epsilon_{m_0}\}$ is a basis for X_{m_0}.

Denote the Gram matrix of $\{\epsilon_1, ..., \epsilon_{m_0}\}$ by T_{m_0}. Then the Gram matrix of the basis (2.3) is

$$T_m = \begin{pmatrix} & & \omega_r \\ & T_{m_0} & \\ \omega_r & & \end{pmatrix},$$

where ω_r is the $r \times r$ permutation matrix with 1 in its second main diagonal, i.e. $(\omega_r)_{i,j} = \delta_{i,r+1-j}$.

For each $t \in \{1, 2, \cdots, r\}$, we have the following partial polarization

$$X_m = \ell_t^+ \oplus X_{m-2t} \oplus \ell_t^- \tag{2.4}$$

where ℓ_t^+ (resp. ℓ_t^-) is the totally isotropic subspace of dimension t of ℓ_r^+ (resp. ℓ_r^-), generated by $\{e_1, \cdots, e_t\}$ (resp. by $\{e_{-t}, \cdots, e_{-1}\}$). We will write the elements of O_m as matrices according to (2.4) and (2.3). Denote by T_{m-2t} the Gram matrix of the basis $\{e_{t+1}, ..., e_r, \epsilon_1, ..., \epsilon_{m_0}, e_{-r}, ..., e_{-t-1}\}$ of X_{m-2t};

$$T_{m-2t} = \begin{pmatrix} & & \omega_{r-t} \\ & T_0 & \\ \omega_{r-t} & & \end{pmatrix}.$$

Assume that $m - 2t \geq 1$. Let $Q_t = L_t V_t$ be the standard parabolic subgroup of O_m such that

$$L_t = \mathrm{GL}_1^t \times \mathrm{O}_{m-2t} \subset \mathrm{O}_m \tag{2.5}$$

and

$$V_t = \{v = v'(u, x, z) = \begin{pmatrix} u & x^* & z \\ & \mathrm{I}_{m-2t} & x \\ & & u^* \end{pmatrix} \in \mathrm{O}_m\}, \tag{2.6}$$

where $u \in U_t$, the maximal standard (upper-triangular) unipotent subgroup of GL_t. Denote the first column of x by x_1. Then the vectors

$$x_1 = {}^t(x_{1,1}, \cdots, x_{m-2t,1}) \tag{2.7}$$

form a k-vector space which is isomorphic to X_{m-2t}. Consider the action of $\mathrm{GL}_1 \times \mathrm{O}_{m-2t}$ on X_{m-2t} by

$$(a,h)\circ x_1 := ahx_1. \tag{2.8}$$

By the Witt theorem, the space X_{m-2t} decomposes into the following disjoint union of k-rational $\mathrm{GL}_1 \times \mathrm{O}_{m-2t}$-orbits:

$$X_{m-2t} = \{0\} \cup \mathcal{O}_0 \cup (\cup_{\alpha\in k^\times/(k^\times)^2}\mathcal{O}_\alpha), \tag{2.9}$$

where $\mathcal{O}_0$ consists of all (nonzero) isotropic vectors in X_{m-2t} and $\mathcal{O}_\alpha$ consists of all vectors x_1 in X_{m-2t} with $(x_1,x_1)\equiv\alpha \mod (k^\times)^2$. It is clear that the disjoint union $\cup_{\alpha\in k^\times/(k^\times)^2}\mathcal{O}_\alpha$ is k-stable.

2.1. Global periods. Let ψ be a nontrivial character of $\mathbb{A}/k$. Take μ_α in $\mathcal{O}_\alpha$ and define a character $\psi_{t,\alpha}$ of $V_t(\mathbb{A})$ as follows. For $v = v'(u,x,z)\in V_t(\mathbb{A})$, we define

$$\psi_{t,\alpha}(v) := \psi(u_{1,2}+\cdots+u_{t-1,1t})\psi^{-1}((\mu_\alpha,x_1)). \tag{2.10}$$

It is clear that the character $\psi_{t,\alpha}$ is trivial when restricted to $V_t(k)$. Since the Levi subgroup $L_t = \mathrm{GL}_1^t \times \mathrm{O}_{m-2t}$ normalizes V_t, the group of k-rational points $L_t(k)$ also acts on the characters $\psi_{t,\alpha}$, as α runs through a square class in $k^\times$. Consider the following decomposition

$$X_{m-2t} = (k\cdot\mu_\alpha)\perp(k\cdot\mu_\alpha)^\perp. \tag{2.11}$$

Since μ_α is anisotropic, the orthogonal complement $(k\cdot\mu_\alpha)^\perp$ is a nondegenerate quadratic k-vector space of dimension $m-2t-1$ with respect to the restriction of the bilinear form $(\cdot,\cdot)$ on X_m. The stabilizer of $\psi_{t,\alpha}$ in O_{m-2t} is

$$\mathrm{D}_{t,\alpha} := \mathrm{O}((k\cdot\mu_\alpha)^\perp) \tag{2.12}$$

We want to calculate the Witt index of $(k\cdot\mu_\alpha)^\perp$. Recall that $m-2t\geq 1$. If $t=r$, then $X_{m-2t}=X_{m_0}$ is anisotropic, and hence the Witt index of $(k\cdot\mu_\alpha)^\perp$ is zero. If $t<r$, then the Witt index of X_{m-2t} is $r-t$ and we have

$$X_{m-2t} = \ell^+_{r-t}\oplus X_{m_0}\oplus\ell^-_{r-t}. \tag{2.13}$$

If α is representable by X_{m_0}, then the Witt index of $(k\cdot\mu_\alpha)^\perp$ is $r-t$. If α is not representable by X_{m_0}, then the Witt index of $(k\cdot\mu_\alpha)^\perp$ is $r-t-1$.

For an automorphic form ϕ on $\mathrm{O}_m(\mathbb{A})$, we define the $\psi_{t,\alpha}$-Fourier coefficient of φ by the following integral

$$\mathcal{F}^{\psi_{t,\alpha}}(\phi)(g) := \int_{V_t(k)\backslash V_t(\mathbb{A})}\phi(vg)\psi^{-1}_{t,\alpha}(v)dv. \tag{2.14}$$

It is clear that the restriction of $\mathcal{F}^{\psi_{t,\alpha}}(\phi)$ to $\mathrm{D}_{t,\alpha}(\mathbb{A})$ is left $\mathrm{D}_{t,\alpha}(k)$-invariant. We note that when O_m is quasi-split, or split over k, and $t=[\frac{m-1}{2}]$, then the $\psi_{t,\alpha}$-Fourier coefficient is a Whittaker-Fourier coefficient.

Let ϕ' be an automorphic form on $\mathrm{D}_{t,\alpha}(\mathbb{A})$. Then we define the generalized Gelfand-Graev period (or Bessel period) of ϕ of type $(\mathrm{D}_{t,\alpha},\psi_{t,\alpha},\phi')$, or simply the $(\mathrm{D}_{t,\alpha},\psi_{t,\alpha},\phi)$-period of ϕ, by the following integral

$$\mathcal{P}_{\mathrm{D}_{t,\alpha};\psi_{t,\alpha}}(\phi,\phi') = \mathcal{P}_{\mathrm{D}_{t,\alpha}}(\phi,\phi') := \int_{\mathrm{D}_{t,\alpha}(k)\backslash\mathrm{D}_{t,\alpha}(\mathbb{A})}\mathcal{F}^{\psi_{t,\alpha}}(\phi)(h)\phi'(h)dh, \tag{2.15}$$

if the last integral converges. We refer to [**GPSR97**] for applications of such periods to the theory of automorphic L-functions.

2.2. Local models. Let v be a finite local place of k and k_v be the local field of k at v. Let ψ_v be a nontrivial character of k_v. We define the v-analogue of $\psi_{t,\alpha}$ for $V_t(k_v)$ by

$$\psi_{t,\alpha;v}(v'(u,x,z)) = \psi_v(u_{1,2} + \cdots + u_{t-1,1t})\psi^{-1}((\mu_{\alpha,v}, x_1)) \tag{2.16}$$

where $\alpha \in k_v^\times$ and $\mu_{\alpha,v} \in X_{m-2t}(k_v)$ is such that $(\mu_{\alpha,v}, \mu_{\alpha,v}) = \alpha$. Let (σ_v, V_{σ_v}) be an irreducible admissible representation of $\mathrm{O}_m(k_v)$. We say that σ_v has a nontrivial $\psi_{t,\alpha;v}$-functional if the following space

$$\mathrm{Hom}_{V_t(k_v)}(V_{\sigma_v}, \psi_{t,\alpha;v}) \tag{2.17}$$

is nonzero. It is clear that in case $\mathrm{O}_m(k_v)$ is quasi-split or split over k_v, and $t = [\frac{m-1}{2}]$, then a $\psi_{t,\alpha}$-functional is a Whittaker functional.

Let τ_v be an irreducible admissible representation of $\mathrm{D}_{t,\alpha}(k_v)$. Then $\tau_v \otimes \psi_{t,\alpha;v}$ is a representation of the semi-direct product

$$\mathcal{J}_{t,\alpha}(k_v) = \mathrm{D}_{t,\alpha}(k_v) \ltimes V_t(k_v). \tag{2.18}$$

We say that σ_v has a nontrivial generalized Gelfand-Graev model (or Bessel model) of type $(\mathcal{J}_{t,\alpha}, \tau_v \otimes \psi_{t,\alpha;v})$, or a nontrivial $(\mathcal{J}_{t,\alpha}, \tau_v \otimes \psi_{t,\alpha;v})$-model if the following space

$$\mathrm{Hom}_{\mathcal{J}_{t,\alpha}(k_v)}(V_{\sigma_v}, \tau_v \otimes \psi_{t,\alpha;v}) \tag{2.19}$$

is nonzero. In this case, take $0 \neq \ell_v \in \mathrm{Hom}_{\mathcal{J}_{t,\alpha}(k_v)}(V_{\sigma_v}, \tau_v \otimes \psi_{t,\alpha;v})$. Then the corresponding $(\mathcal{J}_{t,\alpha}, \tau_v \otimes \psi_{t,\alpha;v})$-model is the space consisting of all functions of the following type

$$\mathcal{B}_x^{\psi_{t,\alpha;v}}(g) := \ell_v(\sigma_v(g)(x)), \quad g \in \mathrm{O}_m(k_v) \tag{2.20}$$

when x runs through V_{σ_v}.

3. Global and Local Theta Correspondences

In this section we recall the global and local theta correspondences for O_m and then study the global and local first occurrences of theta correspondences in terms of the periods or models defined in the previous sections.

3.1. Global and local theta liftings. Let Sp_{2l} be the symplectic group of k-rank l. Then $(\mathrm{O}_m, \mathrm{Sp}_{2l})$ forms a reductive dual pair in Sp_{2lm} in the sense of R. Howe ([**H79**]). We denote by $\mathrm{Mp}_{2l}(\mathbb{A})$ the metaplectic double cover $\widetilde{\mathrm{Sp}}_{2l}(\mathbb{A})$ of $\mathrm{Sp}_{2l}(\mathbb{A})$ if $m = 2n+1$ or the $\mathbb{A}$-rational points $\mathrm{Sp}_{2l}(\mathbb{A})$ of Sp_{2l} if $m = 2n$. Similarly, we denote by $\mathrm{Mp}_{2l}(k_v)$ the metaplectic double cover $\widetilde{\mathrm{Sp}}_{2l}(k_v)$ of $\mathrm{Sp}_{2l}(k_v)$ if $m = 2n+1$ or the k_v-rational points $\mathrm{Sp}_{2l}(k_v)$ of Sp_{2l} if $m = 2n$. Details about $\mathrm{Mp}_{2l}(k_v)$ and $\mathrm{Mp}_{2l}(\mathbb{A})$ and their splitting properties can be found in many references. See, for instance, [**K94**] or [**JngS07b**].

For a non-trivial character ψ of $\mathbb{A}/k$, there exists the Weil representation ω_ψ of $\widetilde{\mathrm{Sp}}_{2lm}(\mathbb{A})$, which is realized in the Schrödinger model $\mathcal{S}(\mathbb{A}^{ml})$, where $\mathcal{S}(\mathbb{A}^{ml})$ is the space of $\mathbb{C}$-valued Schwartz-Bruhat functions on $\mathbb{A}^{ml}$.

For $\varphi \in \mathcal{S}(\mathbb{A}^{ml})$, we form the theta function

$$\theta_{\psi,\varphi}(x) := \sum_{\xi \in k^{ml}} \omega_\psi(x)(\varphi)(\xi),$$

on $\widetilde{\mathrm{Sp}}_{2ml}(\mathbb{A})$. This series is absolutely convergent and defines a function of moderate growth on $\widetilde{\mathrm{Sp}}_{2ml}(\mathbb{A})$. There is a natural homomorphism

$$\mathrm{O}_m(\mathbb{A}) \times \mathrm{Mp}_{2l}(\mathbb{A}) \to \widetilde{\mathrm{Sp}}_{2ml}(\mathbb{A})$$

with kernel $C_2 = \{\pm 1\}$, and the center of $\mathrm{O}_m(\mathbb{A})$ diagonally embedded. We pull the Weil representation ω_ψ back to $\mathrm{O}_m(\mathbb{A}) \times \mathrm{Mp}_{2l}(\mathbb{A})$. This allows us to restrict $\theta_{\psi,\varphi}$ to $\mathrm{O}_m(\mathbb{A}) \times \mathrm{Mp}_{2l}(\mathbb{A})$. See [**JngS07b**], for instance.

For an irreducible cuspidal automorphic representation (σ, V_σ) of $\mathrm{O}_m(\mathbb{A})$, the following integral

$$\theta^{2l}_{\psi,m}(h; \phi_\sigma, \varphi) := \int_{\mathrm{O}_m(k)\backslash \mathrm{O}_m(\mathbb{A})} \phi_\sigma(g)\theta_{\psi^{-1},\varphi}(g,h)dg \tag{3.1}$$

with $\phi_\sigma \in V_\sigma$, defines an automorphic form on $\mathrm{Mp}_{2l}(\mathbb{A})$. We denote by $\theta^{2l}_{\psi,m}(\sigma)$ the space generated by all $\theta^{2l}_{\psi,m}(g; \phi_\sigma, \varphi)$ as φ and ϕ_σ vary. This defines a genuine automorphic representation of $\mathrm{Mp}_{2l}(\mathbb{A})$, which we denote by $\theta^{2l}_{\psi,m}(\sigma)$. We call this representation the ψ-theta lifting of σ to $\mathrm{Mp}_{2l}(\mathbb{A})$. Similarly, for a genuine irreducible cuspidal automorphic representation $(\tilde{\pi}, V_{\tilde{\pi}})$ of $\mathrm{Mp}_{2l}(\mathbb{A})$, we get the automorphic representation $\theta^m_{\psi,2l}(\tilde{\pi})$ of $\mathrm{O}_m(\mathbb{A})$. Its space is generated by the following automorphic forms

$$\theta^m_{\psi,2l}(g; \phi_{\tilde{\pi}}, \varphi) := \int_{\mathrm{Mp}_{2l}(k)\backslash \mathrm{Mp}_{2l}(\mathbb{A})} \phi_{\tilde{\pi}}(h)\theta_{\psi,\varphi}(g,h)dh \tag{3.2}$$

as φ and $\phi_{\tilde{\pi}}$ vary. We say that $\theta^m_{\psi,2l}(\tilde{\pi})$ is the ψ-theta lifting of $\tilde{\pi}$ to $\mathrm{O}_m(\mathbb{A})$. In this paper, all representations of the metaplectic group (global or local) are assumed to be genuine.

Recall that a basic problem in the theory of the theta correspondence is to determine when the ψ-theta lifting $\theta^{2l}_{\psi,m}(\sigma)$ is nonzero for a given irreducible cuspidal automorphic representation σ of $\mathrm{O}_m(\mathbb{A})$ (similarly for $\theta^m_{\psi,2l}(\tilde{\pi})$). In [**GJS07**], we introduced the notion of the lowest occurrence $\mathrm{LO}_\psi(\sigma)$ of σ, with respect to all twists by automorphic sign characters of $\mathrm{O}_m(\mathbb{A})$, in the tower $\mathrm{Mp}_{2l}(\mathbb{A})$, via the ψ-theta correspondence, namely

$$\mathrm{LO}_\psi(\sigma) := \min_{\epsilon}\{\mathrm{FO}_\psi(\sigma \otimes \epsilon)\}, \tag{3.3}$$

where ϵ runs through all automorphic sign characters of $\mathrm{O}_m(\mathbb{A})$. As the notation suggests, $\mathrm{FO}_\psi(\sigma \otimes \epsilon)$ denotes the first occurrence of $\sigma \otimes \epsilon$ in the tower $\mathrm{Mp}_{2l}(\mathbb{A})$ via the ψ-theta correspondence.

Next, we recall briefly from [**MVW87**] the local theta correspondence over the local field k_v, where v is a finite local place of k.

For a nontrivial character ψ_v of k_v, let ω_{ψ_v} be the Weil representation of the reductive dual pair $\mathrm{O}_m(k_v) \times \mathrm{Mp}_{2l}(k_v)$ acting on the local Schrödinger model $\mathcal{S}(k_v^{ml})$, where $\mathcal{S}(k_v^{ml})$ is the space of local k_v-valued Schwartz-Bruhat functions on k_v^{ml}. A detailed discussion of the splitting of the double cover and the related the cocycles can be found in [**JngS07b**], for example. See [**K94**] for general reductive dual pairs.

Let (σ_v, V_{σ_v}) ($(\tilde{\pi}_v, V_{\tilde{\pi}_v})$, resp.) be an irreducible admissible representation of $\mathrm{O}_m(k_v)$ ($\mathrm{Mp}_{2l}(k_v)$, resp.). If

$$\mathrm{Hom}_{\mathrm{O}_m(k_v)\times \mathrm{Mp}_{2l}(k_v)}(\mathcal{S}(k_v^{ml}), V_{\sigma_v} \otimes V_{\tilde{\pi}_v}) \neq 0, \tag{3.4}$$

then we say that $\tilde{\pi}_v$ is a local ψ_v-theta lift of σ_v, and σ_v is a local ψ_v-theta lift of $\tilde{\pi}_v$. We do not assume that the local Howe duality conjecture holds for the case we are discussing here. The local Howe duality conjecture was proved by J.-L. Waldspurger [**W90**], when the residual characteristic of k is odd. In such a circumstance, the local ψ_v-theta lift is the same as the local ψ_v-Howe lift. We refer to [**MVW87**] for more detailed discussions.

We define the first occurrence for the local ψ_v-theta liftings based on (3.4). More precisely, we say that the first occurrence of σ_v is $\mathrm{FO}_{\psi_v}(\sigma_v) = 2l_0$ if

$$\mathrm{Hom}_{\mathrm{O}_m(k_v)\times \mathrm{Mp}_{2l_1}(k_v)}(\mathcal{S}(k_v^{ml_1}), V_{\sigma_v} \otimes V_{\tilde{\pi}_{v,l_1}}) = 0,$$

for all $l_1 < l_0$ and for all irreducible admissible representations $\tilde{\pi}_{v,l_1}$ of $\mathrm{Mp}_{2l_1}(k_v)$, but there exists at least one irreducible admissible representation $\tilde{\pi}_{v,l_0}$ of $\mathrm{Mp}_{2l_0}(k_v)$ such that

$$\mathrm{Hom}_{\mathrm{O}_m(k_v)\times \mathrm{Mp}_{2l_0}(k_v)}(\mathcal{S}(k_v^{ml_0}), V_{\sigma_v} \otimes V_{\tilde{\pi}_{v,l_0}}) \neq 0.$$

By the local tower property of ([**K96**], for instance), if the first occurrence of σ_v is $\mathrm{FO}_{\psi_v}(\sigma_v) = 2l_0$, then for any $l > l_0$, there always exists an irreducible admissible representation $\tilde{\pi}_{v,l}$ of $\mathrm{Mp}_{2l}(k_v)$ such that the following space

$$\mathrm{Hom}_{\mathrm{O}_m(k_v)\times \mathrm{Mp}_{2l}(k_v)}(\mathcal{S}(k_v^{ml}), V_{\sigma_v} \otimes V_{\tilde{\pi}_{v,l}}) \neq 0.$$

We define the local lowest occurrence of σ_v by

$$\mathrm{LO}_{\psi_v}(\sigma_v) := \min\{\mathrm{FO}_{\psi_v}(\sigma_v), \mathrm{FO}_{\psi_v}(\sigma_v \otimes \det)\}.$$

We mention here the conservation relation conjectured by Kudla and Rallis, namely that $\mathrm{FO}_{\psi_v}(\sigma_v) + \mathrm{FO}_{\psi_v}(\sigma_v \otimes \det) = m$. See [**KR05**].
The local first occurrence for $\tilde{\pi}_v$ can be defined in the same way.

3.2. Vanishing of theta liftings. For an irreducible cuspidal automorphic representation σ of $\mathrm{O}_m(\mathbb{A})$, we are going to relate, by doing some explicit calculations, the nonvanishing of the $\psi_{t,\alpha}$-Fourier coefficient on σ to the first occurrence $\mathrm{FO}_\psi(\sigma)$ of σ.

Following [**MW87**], [**M96**] and [**GRS03**], we say that σ has $\psi_{t,\alpha}$ as a top Fourier coefficient, for given $t \in \{1, 2, \cdots, r\}$ and $\alpha \in k^\times \mod (k^\times)^2$, if there is some $\phi_\sigma \in V_\sigma$ such that the $\psi_{t,\alpha}$-Fourier coefficient $\mathcal{F}^{\psi_{t,\alpha}}(\phi_\sigma)$ is not identically zero, but the $\psi_{t',\alpha'}$-Fourier coefficients $\mathcal{F}^{\psi_{t',\alpha'}}(\phi_\sigma)$ are all identically zero, for all $\phi_\sigma \in V_\sigma$, $\alpha' \in k^\times \mod (k^\times)^2$, and $t' > t$. Recall again that we assume that $m - 2t \geq 1$. Note that if r, the k-rank of O_m, is zero, i.e. O_m is k-anisotropic, then σ has no such Fourier coefficients at all.

The first result in this paper is

THEOREM 3.1. *Let σ be an irreducible cuspidal automorphic representation of $\mathrm{O}_m(\mathbb{A})$. If σ has $\psi_{t,\alpha}$ as a top Fourier coefficient, for some $t \in \{1, 2, \cdots, r\}$, with $r = \frac{m-m_0}{2} \geq 1$, $m - 2t \geq 1$, and some $\alpha \in k^\times \mod (k^\times)^2$, then the lowest occurrence of σ, $\mathrm{LO}_\psi(\sigma)$ is greater than or equal to $2t$, i.e. for any automorphic sign character ϵ of $\mathrm{O}_m(\mathbb{A})$, the first occurrence $\mathrm{FO}_\psi(\sigma \otimes \epsilon)$ is greater than or equal to $2t$.*

Note that when $m_0 \leq 1$ and $t = [\frac{m-1}{2}]$, $\psi_{t,\alpha}$ is a Whittaker character. In these cases the theorem is well known (at least the version with special orthogonal groups). See [**W80**], for $m = 3$, [**PSS87**], for $m = 5$, [**F95**], for m odd, in general, and [**GRS97**], for m even.

Here is an outline of the proof of Theorem 3.1.
It suffices to show that for any $l < t$, the ψ-theta lifting $\theta^{2l}_{\psi,m}(\sigma \otimes \epsilon)$ is zero as an automorphic representation of $\mathrm{Mp}_{2l}(\mathbb{A})$, for all automorphic sign characters ϵ of $\mathrm{O}_m(\mathbb{A})$.

If this is not the case, then by the Rallis tower property of theta liftings ([**R84**]), there is an integer $l < t$ and an automorphic sign character ϵ of $\mathrm{O}_m(\mathbb{A})$ such that the ψ-theta lifting $\theta^{2l}_{\psi,m}(\sigma \otimes \epsilon)$ is nonzero and cuspidal. Clearly, $\sigma \otimes \epsilon$ has a nontrivial $\psi_{t,\alpha}$-Fourier coefficient (and this is its top Fourier coefficient). Thus, we may assume that ϵ is trivial, and hence that $\theta^{2l}_{\psi,m}(\sigma)$ is nonzero and cuspidal.

By the main theorem of [**M97b**] and Theorem 1.2 of [**JngS07b**], the ψ-theta lifting $\tilde{\pi}_{2l} := \theta^{2l}_{\psi,m}(\sigma)$ is a nonzero irreducible cuspidal automorphic representation of $\mathrm{Mp}_{2l}(\mathbb{A})$ and we have

$$\sigma = \theta^{m}_{\psi,2l}(\tilde{\pi}_{2l}) = \theta^{m}_{\psi,2l}(\theta^{2l}_{\psi,m}(\sigma)). \tag{3.5}$$

We consider the following polarizations for X_m and W_{2l}

$$X_m = \ell_t^+ \oplus X_{m-2t} \oplus \ell_t^-, \tag{3.6}$$

$$W_{2l} = Y_l^+ \oplus Y_l^-, \tag{3.7}$$

where W_{2l} is the nondegenerate symplectic k-vector space defining Sp_{2l}, and hence Mp_{2l}. We assume that O_m acts from the left on X_m and Sp_{2l} acts from the right on W_{2l}. We may take a canonical basis

$$\{f_1, \cdots, f_l; f_{-l}, \cdots, f_{-1}\} \tag{3.8}$$

for W_{2l}, such that Y_l^+ is generated by $\{f_1, \cdots, f_l\}$, Y_l^- is generated by $\{f_{-l}, \cdots, f_{-1}\}$, and $(f_i, f_{-j})_{W_{2l}} = \delta_{ij}$.

We consider the Weil representation ω_ψ on the mixed Schrödinger model

$$\mathcal{S}_{m\otimes 2l} := \mathcal{S}(\ell_t^-(\mathbb{A}) \otimes W_{2l}(\mathbb{A}) \oplus X_{m-2t}(\mathbb{A}) \otimes Y_l^+(\mathbb{A})). \tag{3.9}$$

The Schwartz-Bruhat function φ in $\mathcal{S}_{m\otimes 2l}$ is written as

$$\varphi(w_1, \cdots, w_t; y_1, \cdots, y_{m-2t}) \tag{3.10}$$

where $w_i \in W_{2l}(\mathbb{A})$ and $y_j \in Y_l^+(\mathbb{A})$ for $i = 1, \cdots, t$ and $j = 1, \cdots, m-2t$.

By assumption, σ has a nonzero $\psi_{t,\alpha}$-Fourier coefficient for $t \le r$ and for some $\alpha \in k^\times$ (see (2.14)), i.e.

$$\mathcal{F}^{\psi_{t,\alpha}}(\phi_\sigma)(g) := \int_{V_t(k)\backslash V_t(\mathbb{A})} \phi_\sigma(vg)\psi_{t,\alpha}^{-1}(v)dv \tag{3.11}$$

is nonzero, for some $\phi_\sigma \in V_\sigma$ and some $g \in \mathrm{O}_m(\mathbb{A})$. By (3.5), we may take ϕ_σ to be

$$\phi_\sigma(g) = \int_{\mathrm{Mp}_{2l}(k)\backslash \mathrm{Mp}_{2l}(\mathbb{A})} \phi_{\tilde{\pi}}(h)\theta_{\psi,\varphi}(g,h)dh \tag{3.12}$$

for some $\phi_{\tilde{\pi}} \in V_{\tilde{\pi}}$. Then the $\psi_{t,\alpha}$-Fourier coefficient $\mathcal{F}^{\psi_{t,\alpha}}(\phi_\sigma)$ can be written as

$$\begin{aligned} \mathcal{F}^{\psi_{t,\alpha}}(\phi_\sigma)(e) &= \int_{V_t(k)\backslash V_t(\mathbb{A})} \int_{\mathrm{Mp}_{2l}(k)\backslash \mathrm{Mp}_{2l}(\mathbb{A})} \phi_{\tilde{\pi}}(h)\theta_{\psi,\varphi}(v,h)dh\psi_{t,\alpha}^{-1}(v)dv \\ &= \int_{\mathrm{Mp}_{2l}(k)\backslash \mathrm{Mp}_{2l}(\mathbb{A})} \phi_{\tilde{\pi}}(h) \int_{V_t(k)\backslash V_t(\mathbb{A})} \theta_{\psi,\varphi}(v,h)\psi_{t,\alpha}^{-1}(v)dvdh. \end{aligned} \tag{3.13}$$

The switch of order of integrations is easily justified, since $V_t(k)\backslash V_t(\mathbb{A})$ is compact, $\phi_{\tilde{\pi}}$ is rapidly decreasing and $\theta_{\psi,\varphi}(v,h)$ is of moderate growth. The inner integral is the $\psi_{t,\alpha}$-Fourier coefficient of the theta function

$$\mathcal{F}^{\psi_{t,\alpha}}(\theta_{\psi,\varphi}(\cdot,h)) := \int_{V_t(k)\backslash V_t(\mathbb{A})} \theta_{\psi,\varphi}(v,h)\psi_{t,\alpha}^{-1}(v)dv \tag{3.14}$$

Then we have

Proposition 3.2. *The $\psi_{t,\alpha}$-Fourier coefficient of the theta function $\theta_{\psi,\varphi}(g,h)$, $\mathcal{F}^{\psi_{t,\alpha}}(\theta_{\psi,\varphi}(\cdot,h))$, is zero for all $\varphi \in \mathcal{S}(\mathbb{A}^{ml})$, if $l < t$.*

We postpone the proof of Proposition 3.2 to §4.1.

By Proposition 3.2, if $l < t$, the $\psi_{t,\alpha}$-Fourier coefficient of the theta function $\theta_{\psi,\varphi}(g,h)$, $\mathcal{F}^{\psi_{t,\alpha}}(\theta_{\psi,\varphi}(\cdot,h))$, is zero for all $\varphi \in \mathcal{S}(\mathbb{A}^{ml})$. It follows that the $\psi_{t,\alpha}$-Fourier coefficient $\mathcal{F}^{\psi_{t,\alpha}}(\phi_\sigma)$ as in (3.11) is zero for all $\phi_\sigma \in V_\sigma$. This contradicts our assumption. This will prove Theorem 3.1.

By applying Theorem 3.1 above to Theorem 1.1 in [**GJS07**], we obtain

Corollary 3.3. *Let σ be an irreducible cuspidal automorphic representation of $\mathrm{O}_m(\mathbb{A})$. If σ has $\psi_{t,\alpha}$ as a top Fourier coefficient, for some $t \in \{1, 2, \cdots, r\}$, with $r = \frac{m-m_0}{2} \geq 1$, $m - 2t \geq 1$, and some $\alpha \in k^\times \mod (k^\times)^2$, then the partial L-function $L^S(s,\sigma)$ is holomorphic for $Re(s) > \frac{m}{2} - t$.*

4. Fourier Coefficients of Theta Functions

We shall prove Proposition 3.2 first and then develop its local version afterwards.

4.1. Proof of Proposition 3.2. We shall use the notation in §3 for the calculation of the $\psi_{t,\alpha}$-Fourier coefficient of the theta function $\theta_{\psi,\varphi}(g,h)$ as in Proposition 3.2,

$$\mathcal{F}^{\psi_{t,\alpha}}(\theta_{\psi,\varphi}(\cdot,h)) := \int_{V_t(k)\backslash V_t(\mathbb{A})} \theta_{\psi,\varphi}(v,h)\psi_{t,\alpha}^{-1}(v)dv. \tag{4.1}$$

Let us rewrite the elements (2.6) in V_t in the form

$$v = v(u,x,z) = \begin{pmatrix} u & x & z \\ & \mathrm{I}_{m-2t} & x^* \\ & & u^* \end{pmatrix}$$

The subgroup $Z_t = \{v(z) = v(\mathrm{I}_t, 0, z) \in V_t\}$ is the center of $N_t = \{v(x,z) = v(\mathrm{I}_t, x, z) \in V_t\}$, and the subgroup $U_t = \{v(u) = v(u,0,0) \in V_t\}$ normalizes N_t. We may write the elements of $Z_t\backslash N_t$ as $v(x) = v(\mathrm{I}_t, x, z)Z_t$, for any z, such that $v(\mathrm{I}_t, x, z) \in V_t$. Note that

$$\psi_{t,\alpha}(v(\mathrm{I}_t, x, z)) = \psi(x_t \cdot \mu_\alpha),$$

where x_t is the last row of x. We have

$$\begin{aligned} \mathcal{F}^{\psi_{t,\alpha}}(\theta_{\psi,\varphi}(\cdot,h)) \quad &= \quad \int_{U_t(k)\backslash U_t(\mathbb{A})} \int_{M_{t\times(m-2t)}(k)\backslash M_{t\times(m-2t)}(\mathbb{A})} \\ &\quad \cdot \int_{Z_t(k)\backslash Z_t(\mathbb{A})} \theta_{\psi,\varphi}(v(z)v(x)v(u),h)\psi_{t,\alpha}^{-1}(v(x)v(u))dzdxdu. \end{aligned} \tag{4.2}$$

We first work out the dz-integration in (4.2). By the definition of the mixed Schrödinger model as in (3.9) and (3.10), $\theta_{\psi,\varphi}(v(z)g,h)$ can be written as

$$\sum_{w_i\in W_{2l}(k),y_j\in Y_l^+(k)} \omega_\psi(v(z)g,h)\varphi(w_1,\cdots,w_t;y_1,\cdots,y_{m-2t}). \tag{4.3}$$

We have the following formula for the action of $\omega_\psi(v(z),1)$ on the mixed Schrödinger model,

$$\begin{aligned}&\omega_\psi(v(z),1)\varphi(w_1,\cdots,w_t;y_1,\cdots,y_{m-2t})\\ =\ &\varphi(w_1,\cdots,w_t;y_1,\cdots,y_{m-2t})\psi(\frac{1}{2}\mathrm{tr}(\mathrm{Gr}(w_1,\cdots,w_t)\omega_t z)),\end{aligned}$$

where $\mathrm{Gr}(w_1,\cdots,w_t)$ is the Gram matrix of $(w_1,...,w_t)$ (see [**K96**], p. 37, and also [**JngS07b**], p. 727). Hence the dz-integration can be expressed as

$$\begin{aligned}\int_{Z_t(k)\backslash Z_t(\mathbb{A})}\theta_{\psi,\varphi}(v(z)g,h)dz &= \sum_{w_i,y_j}\omega_\psi(g,h)\varphi(w_1,\cdots,w_t;y_1,\cdots,y_{m-2t})\\ &\cdot\int_{Z_t(k)\backslash Z_t(\mathbb{A})}\psi^{-1}(\frac{1}{2}\mathrm{tr}(\mathrm{Gr}(w_1,\cdots,w_t)\omega_t z))dz,\end{aligned}$$

where the summation over w_i, y_j is the same as in (4.3). The order switch of integral and sum is easily justified, since $Z_t(k)\backslash Z_t(\mathbb{A})$ is compact and the summation over w_i, y_j is absolutely convergent. Note that

$$\int_{Z_t(k)\backslash Z_t(\mathbb{A})}\psi(\frac{1}{2}\mathrm{tr}(\mathrm{Gr}(w_1,\cdots,w_t)\omega_t z))dz$$

must be zero unless the Gram matrix $\mathrm{Gr}(w_1,\cdots,w_t)$ is zero, i.e. $(w_i,w_j)_{W_{2l}} = 0$ for all $i,j = 1,2,\cdots,t$. This means that the subspace of W_{2l} generated by $w_1,w_2,\cdots,w_t$ is totally isotropic. Since we assume that $l < t$, we deduce that $w_1,w_2,\cdots,w_t$ must be linearly dependent in W_{2l}. When $\mathrm{Gr}(w_1,\cdots,w_t)$ is zero,

$$\int_{Z_t(k)\backslash Z_t(\mathbb{A})}\psi(\frac{1}{2}\mathrm{tr}(\mathrm{Gr}(w_1,\cdots,w_t)\omega_t z))dz = 1$$

by the choice of the Haar measure on $Z_t(k)\backslash Z_t(\mathbb{A})$. Therefore we have

$$\int_{Z_t(k)\backslash Z_t(\mathbb{A})}\theta_{\psi,\varphi}(v(z)g,h)dz = \sum_{w_i,y_j}\omega_\psi(g,h)\varphi(w_1,\cdots,w_t;y_1,\cdots,y_{m-2t}) \tag{4.4}$$

where the summation is over all $y_1,\cdots,y_{m-2t}\in Y_l^+(k)$, and all $w_1,\cdots,w_t\in W_{2l}(k)$ with the property that $w_1,w_2,\cdots,w_t$ generate a totally isotropic subspace of $W_{2l}(k)$. Again, since $\dim_k \mathrm{Span}_k(w_1,\cdots,w_t)\le l < t$, $w_1,...,w_t$ are automatically linearly dependent. Hence we obtain

$$\begin{aligned}\mathcal{F}^{\psi_{t,\alpha}}(\theta_{\psi,\varphi}(\cdot,h)) &= \int_{U_t(k)\backslash U_t(\mathbb{A})}\int_{M_{t\times(m-2t)}(k)\backslash M_{t\times(m-2t)}(\mathbb{A})}\sum_{w_i,y_j}\omega_\psi(v(x)v(u),h)\\ &\cdot\varphi(w_1,\cdots,w_t;y_1,\cdots,y_{m-2t})\psi_{t,\alpha}^{-1}(v(x)v(u))dxdu,\end{aligned} \tag{4.5}$$

with summation as above. Denote

$$\begin{aligned}d &= \dim_k \mathrm{Span}_k(w_1,\cdots,w_t),\\ E_d &= \mathrm{Span}_k(w_1,\cdots,w_t).\end{aligned}$$

Then we can split the last integral as a sum over $0\le d\le l$, where in each summand we compute the last integral with $w_1,...,w_t\in E_d$ and E_d varies over all

d-dimensional totally isotropic subspaces of W_{2l}. Let P_d be the standard parabolic subgroup of Sp_{2l}, which preserves the totally isotropic subspace Y_d^- generated by $\{f_{-d},...,f_{-1}\}$. Then we may write $E_d = Y_d^- \gamma$, where $\gamma \in P_d(k)\backslash \mathrm{Sp}_{2l}(k)$. Thus,

$$\begin{aligned}\mathcal{F}^{\psi_{t,\alpha}}(\theta_{\psi,\varphi}(\cdot,h)) &= \sum_{d=0}^{l}\int_{U_t(k)\backslash U_t(\mathbb{A})}\int_{M_{t\times(m-2t)}(k)\backslash M_{t\times(m-2t)}(\mathbb{A})}\sum_{\gamma\in P_d(k)\backslash \mathrm{Sp}_{2l}(k)} \\ &\quad \sum_{w_i\in Y_d^-}\sum_{y_j\in Y_l^+}\omega_\psi(v(x)v(u),\gamma h)\cdot\varphi(w_1,\cdots,w_t;y_1,\cdots,y_{m-2t}) \\ &\quad \cdot\psi_{t,\alpha}^{-1}(v(x)v(u))dxdu. \end{aligned} \tag{4.6}$$

Here, we used the automorphy of theta series. More explicitly, if we write in (4.5), $w_i = v_i\gamma$, where $w_i \in E_d$ and $v_i \in Y_d^-$, then

$$\begin{aligned} &\sum_{y_j\in Y_l^+}\omega_\psi(g,h)\cdot\varphi(v_1\gamma,\cdots,v_t\gamma;y_1,\cdots,y_{m-2t}) \\ = &\sum_{y_j\in Y_l^+}\omega_\psi(g,\gamma h)\cdot\varphi(v_1,\cdots,v_t;y_1,\cdots,y_{m-2t}). \end{aligned}$$

The point is that the summation over $y_j \in Y_l^+$ defines the theta series on $\mathrm{O}_{m-2t}(\mathbb{A})\times \mathrm{Mp}_{2l}(\mathbb{A})$. To explain this, we may assume that $g = 1$, and that $\varphi = \varphi_1 \otimes \varphi_2$, where

$$\varphi(w_1,\cdots,w_t;y_1,\cdots,y_{m-2t}) = \varphi_1(w_1,...,w_t)\varphi_2(y_1,...,y_{m-2t}).$$

Then

$$\sum_{y_j\in Y_l^+}\omega_\psi(1,h)\cdot\varphi(w_1,\cdots,w_t;y_1,\cdots,y_{m-2t}) = \varphi_1(w_1h,...,w_th)\theta_{\psi,\varphi_2}(1,h),$$

where θ_{ψ,φ_2} is the corresponding theta series for $\mathrm{O}_{m-2t}(\mathbb{A}) \times \mathrm{Mp}_{2l}(\mathbb{A})$. Let us use now the action of $v(x)$, which follows from the formulae of the Weil representation on the mixed model. For $w_1,...,w_t \in Y_l^-$ and $y_1,...,y_{m-2t} \in Y_l^+$,

$$\begin{aligned} &\omega_\psi(v(x),1)\cdot\varphi(w_1,\cdots,w_t;y_1,\cdots,y_{m-2t}) \\ = &\psi(\sum_{i=1}^{t}\sum_{j=1}^{m-2t}x_{t+1-i,j}(w_i,y_j))\varphi(w_1,\cdots,w_t;y_1,\cdots,y_{m-2t}). \end{aligned}$$

As before, we may switch the order of summations and the dx-integration and get

$$\begin{aligned}\mathcal{F}^{\psi_{t,\alpha}}(\theta_{\psi,\varphi}(\cdot,h)) &= \sum_{d=0}^{l}\int_{U_t(k)\backslash U_t(\mathbb{A})}\sum_{\gamma\in P_d(k)\backslash \mathrm{Sp}_{2l}(k)}\sum_{W_{\alpha,d}(k)} \\ &\quad \omega_\psi(v(u),\gamma h)\cdot\varphi(w_1,\cdots,w_t;y_1,\cdots,y_{m-2t})\psi_{t,\alpha}^{-1}(v(u))du, \end{aligned} \tag{4.7}$$

where $W_{\alpha,d}$ is the variety of all $(w_1,...,w_t;y_1,...,y_{m-2t})$, such that the w_i lie in Y_d^-, the y_j lie in Y_l^+, $(w_i,y_j) = 0$, for all $2 \le i \le t$, $1 \le j \le m-2t$, and similarly,

$$(w_1,y_j) = (\mu_\alpha)_j.$$

Recall that μ_α is the (column) vector in $X_{m-2t}(k)$, such that $(\mu_\alpha,\mu_\alpha) = \alpha$, which enters into the definition of $\psi_{t,\alpha}$. The action of $\omega_\psi(v(u),1)$ is linear on $\ell_{-t}(\mathbb{A}) \otimes W_{2l}(\mathbb{A})$ and trivial on $X_{m-2t}(\mathbb{A}) \otimes Y_l^+(\mathbb{A})$. The precise form is

$$\omega_\psi(v(u),1)\cdot\varphi(w_1,\cdots,w_t;y_1,\cdots,y_{m-2t}) = \varphi((w_1,\cdots,w_t)\cdot\omega_tu\omega_t;y_1,\cdots,y_{m-2t}).$$

Note that

$$(w_1, ..., w_t; y_1, ..., y_{m-2t}) \mapsto ((w_1, \cdots, w_t) \cdot \omega_t u \omega_t; y_1, \cdots, y_{m-2t}) \tag{4.8}$$

defines a k-rational action of U_t on $W_{\alpha,d}$. The $U_t(k)$-orbits in $W_{\alpha,d}(k)$ are given by elements $(w_1, ..., w_t; y_1, ..., y_{m-2t}) \in W_{\alpha,d}(k)$, such that $(w_1, ..., w_t)$ is of the following form

$$(w_1, \cdots, 0, w_{i_2}, 0, \cdots, 0, w_{i_3}, 0, \cdots, 0, \cdots, 0, w_{i_d}, 0, \cdots, 0), \tag{4.9}$$

where $w_1, w_{i_1}, w_{i_2}, \cdots, w_{i_d}$ are linearly independent elements in Y_d^-. Note that by definition of $W_{\alpha,d}$, we must have $d \geq 1$ and $w_1 \neq 0$. Denote by $w'_{(t:d)}$ the element in (4.9), and let $w_{(t:d)} = (w'_{(t:d)}; y_1, ..., y_{m-2t})$. Denote its $U_t(k)$-orbit by $\mathcal{O}_{w_{(t:d)}}$ and its stabilizer in U_t, via the action (4.8), by $\mathcal{L}_{w_{(t:d)}}$, i.e.

$$\mathcal{L}_{w_{(t:d)}} := \{u \in U_t(k) \mid w'_{(t:d)} \cdot \omega_t u \omega_t = w'_{(t:d)}\}.$$

Again, in (4.7), we may switch order of summations and the du-integration. Then, the contribution of the $U_t(k)$-orbit $\mathcal{O}_{w_{(t:d)}}$ is

$$\begin{aligned}
& \int_{U_t(k)\backslash U_t(\mathbb{A})} \sum_{\mathcal{O}_{w_{(t:d)}}} \omega_\psi(v(u), \gamma h)\varphi(w_1, \cdots, w_t; y_1, ..., y_{m-2t})\psi_{t,\alpha}^{-1}(v(u))du \\
= & \int_{U_t(k)\backslash U_t(\mathbb{A})} \sum_{\eta \in \mathcal{L}_{w_{(t:d)}}(k)\backslash U_t(k)} \omega_\psi(v(\eta u), \gamma h)\varphi(w_{(t:d)})\psi_{t,\alpha}^{-1}(v(u))du \\
= & \int_{\mathcal{L}_{w_{(t:d)}}(k)\backslash U_t(\mathbb{A})} \omega_\psi(v(u), \gamma h)\varphi(w_{(t:d)})\psi_{t,\alpha}^{-1}(v(u))du \\
= & \int_{\mathcal{L}_{w_{(t:d)}}(\mathbb{A})\backslash U_t(\mathbb{A})} \omega_\psi(v(u), \gamma h)\varphi(w_{(t:d)})\psi_{t,\alpha}^{-1}(v(u))du \\
& \cdot \int_{\mathcal{L}_{w_{(t:d)}}(k)\backslash \mathcal{L}_{w_{(t:d)}}(\mathbb{A})} \psi_{t,\alpha}^{-1}(v(a))da.
\end{aligned} \tag{4.10}$$

Hence, if the restriction of the character $\psi_{t,\alpha}$ to the stabilizer $\mathcal{L}_{w_{(t:d)}}(\mathbb{A})$ is nontrivial, then we must have

$$\int_{\mathcal{L}_{w_{(t:d)}}(k)\backslash \mathcal{L}_{w_{(t:d)}}(\mathbb{A})} \psi_{t,\alpha}^{-1}(v(a))da = 0.$$

This implies that for such a $U_t(k)$-orbit $\mathcal{O}_{w_{(t:d)}}$, we have

$$\int_{U_t(k)\backslash U_t(\mathbb{A})} \sum_{\mathcal{O}_{w_{(t:d)}}} \omega_\psi(v(u), \gamma h)\varphi(w_1, \cdots, w_t; y_1, ..., y_{m-2t})\psi_{t,\alpha}^{-1}(v(u))du = 0. \tag{4.11}$$

Note again that in the orbit $\mathcal{O}_{w_{(t:d)}}$, $y_1, ..., y_{m-2t}$ are fixed. Now for a $U_t(k)$-orbit $\mathcal{O}_{w_{(t:d)}}$ with representative of the form $(w'_{(t:d)}; y_1, ..., y_{m-2t})$, where $w'_{(t:d)}$ is as in (4.9), since $r \leq l < t$, it is easy to check that the stabilizer $\mathcal{L}_{w_{(t:d)}}$ contains at least one simple root of GL_t in U_t. Recall again, that in $W_{\alpha,d}$, we must have $w_1 \neq 0$. Hence we must have that the restriction of the character $\psi_{t,\alpha}$ to the stabilizer $\mathcal{L}_{w_{(t:d)}}(\mathbb{A})$ is nontrivial. This proves Proposition 3.2.

4.2. Genericity of theta liftings. We are going to calculate the $\psi_{t,\alpha}$-Fourier coefficient $\mathcal{F}^{\psi_{t,\alpha}}(\phi_\sigma)$ of $\phi_\sigma \in V_\sigma$, as defined in (3.12) when $l = t$. In other words, we will calculate explicitly the following integral

$$\mathcal{F}^{\psi_{t,\alpha}}(\phi_\sigma) = \int_{\mathrm{Mp}_{2t}(k)\backslash \mathrm{Mp}_{2t}(\mathbb{A})} \phi_{\tilde{\pi}}(h) \int_{V_t(k)\backslash V_t(\mathbb{A})} \theta_{\psi,\varphi}(v,h)\psi_{t,\alpha}^{-1}(v)dvdh \tag{4.12}$$

for $\phi_\sigma \in V_\sigma$ and $\varphi \in \mathcal{S}(\mathbb{A}^{mt})$. For this, we simply continue our calculation in §4.1, with $l = t$. What we did shows that in (4.7) only $d = t$ may contribute a nonzero summand. Note also that for $(w_1, ..., w_t; y_1, ..., y_{m-2t}) \in W_{\alpha,t}(k)$, $w_1, ..., w_t$ form a basis of Y_t^-. Denote by S the unipotent radical of the Siegel parabolic subgroup P_t. Then we may replace in (4.7) ($l = t$) the summation over $\gamma \in P_t(k)\backslash \mathrm{Sp}_{2t}(k)$ by the summation over $\gamma \in S(k)\backslash \mathrm{Sp}_{2t}(k)$, but now $(w_1, ..., w_t) = (f_{-t}, ..., f_{-1})$ are fixed to be the standard basis of Y_t^-. We get

$$\begin{aligned}\mathcal{F}^{\psi_{t,\alpha}}(\theta_{\psi,\varphi}(\cdot,h)) &= \int_{U_t(k)\backslash U_t(\mathbb{A})} \sum_{\gamma\in S(k)\backslash \mathrm{Sp}_{2t}(k)} \sum_{Y_\alpha(k)} \\ &\qquad \omega_\psi(v(u),\gamma h)\varphi(f_{-t},\cdots,f_{-1};y_1,\cdots,y_{m-2t})\psi_{t,\alpha}^{-1}(v(u))du,\end{aligned} \tag{4.13}$$

where Y_α is the set of all $(y_1, ..., y_{m-2t})$, such that the y_j lie in Y_t^+, satisfy $(f_{-i}, y_j) = 0$, for all $1 \le i < t$ and $1 \le j \le m-2t$, and similarly

$$(f_{-t}, y_j) = (\mu_\alpha)_j.$$

This implies that Y_α is a single point. Indeed, we must have $y_j = a_j f_t$, where $a_j \in k$, and

$${}^t(a_1, ..., a_{m-2t}) = \mu_\alpha.$$

In terms of our notation (3.9), (3.10), it is now more convenient to redenote the vector $(a_1 f_t, ..., a_{m-2t} f_t)$ by $\mu_\alpha \otimes f_t \in X_{m-2t}(k) \otimes Y_t^+(k)$. We conclude that

$$\begin{aligned}\mathcal{F}^{\psi_{t,\alpha}}(\theta_{\psi,\varphi}(\cdot,h)) &= \sum_{\gamma\in S(k)\backslash \mathrm{Sp}_{2t}(k)} \int_{U_t(k)\backslash U_t(\mathbb{A})} \\ &\qquad \omega_\psi(v(u),\gamma h)\cdot\varphi(f_{-t},\cdots,f_{-1};\mu_\alpha\otimes f_t)\psi_{t,\alpha}^{-1}(v(u))du,\end{aligned} \tag{4.14}$$

Substitute this in (3.13). We get

$$\begin{aligned}\mathcal{F}^{\psi_{t,\alpha}}(\phi_\sigma)(e) &= \int_{S(k)\backslash \mathrm{Mp}_{2t}(\mathbb{A})} \phi_{\tilde{\pi}}(h) \int_{U_t(k)\backslash U_t(\mathbb{A})} \\ &\qquad \omega_\psi(v(u),h)\cdot\varphi(f_{-t},\cdots,f_{-1};\mu_\alpha\otimes f_t)\psi_{t,\alpha}^{-1}(v(u))dudh,\end{aligned} \tag{4.15}$$

In the mixed Schrödinger model (3.9), we have, for $s = \begin{pmatrix} I_t & b \\ & I_t \end{pmatrix} \in S(\mathbb{A})$,

$$\omega_\psi(1,s)\cdot\varphi(f_{-t},\cdots,f_{-1};\mu_\alpha\otimes f_t) = \psi(\frac{1}{2}\alpha b_{t,1})\varphi(f_{-t},\cdots,f_{-1};\mu_\alpha\otimes f_t).$$

Factoring integration in the last integral through $S(\mathbb{A})$, we get

$$\begin{aligned}\mathcal{F}^{\psi_{t,\alpha}}(\phi_\sigma)(e) &= \int_{S(\mathbb{A})\backslash \mathrm{Mp}_{2t}(\mathbb{A})} \int_{S(k)\backslash S(\mathbb{A})} \phi_{\tilde{\pi}}(sh)\psi(\frac{1}{2}\alpha s_{t,t+1})ds \\ &\qquad \cdot\int_{U_t(k)\backslash U_t(\mathbb{A})} \omega_\psi(v(u),h)\cdot\varphi(f_{-t},\cdots,f_{-1};\mu_\alpha\otimes f_t)\psi_{t,\alpha}^{-1}(v(u))dudh,\end{aligned} \tag{4.16}$$

Next, we have the following formula, for $u \in U_t(\mathbb{A})$,

$$\omega_\psi(u,1)\cdot\varphi(f_{-t},\cdots,f_{-1};\mu_\alpha\otimes f_t)=\omega_\psi(1,u^{-1}h)\varphi(f_{-t},\cdots,f_{-1};\mu_\alpha\otimes f_t),$$

Then by changing variable $h\mapsto uh$ in the last integral, we get that

$$\begin{aligned}\mathcal{F}^{\psi_{t,\alpha}}(\phi_\sigma)(e) &= \int_{S(\mathbb{A})\backslash \mathrm{Mp}_{2t}(\mathbb{A})}\omega_\psi(1,h)\cdot\varphi(f_{-t},\cdots,f_{-1};\mu_\alpha\otimes f_t)\\ &\quad\cdot\int_{U_t(k)\backslash U_t(\mathbb{A})}\int_{S(k)\backslash S(\mathbb{A})}\phi_{\tilde\pi}(suh)\psi(\frac{1}{2}\alpha s_{t,t+1})\psi^{-1}_{t,\alpha}(u)dudsdh. \end{aligned}\tag{4.17}$$

It is clear that the semi-direct product $U_t \ltimes S$ is the unipotent radical R_t of the standard Borel subgroup of Sp_{2t} and the product of the two characters $\psi_{t,\alpha}(u)$ and $\psi(-\frac{1}{2}\alpha\cdot s_{t,t+1})$ is a generic character $\psi_{R_t,\alpha}$ of R_t. Hence the inner integrations ds and du give a Whittaker-Fourier coefficient of $\phi_{\tilde\pi}$, which is denoted by $\mathcal{W}^{\psi_{R_t,\alpha}}_{\phi_{\tilde\pi}}(h)$. Hence $\mathcal{F}^{\psi_{t,\alpha}}(\phi_\sigma)(e)$ is equal to

$$\int_{S(\mathbb{A})\backslash \mathrm{Mp}_{2t}(\mathbb{A})}\omega_\psi(1,h)\varphi(f_{-t},\cdots,f_{-1};\mu_\alpha\otimes f_t)\mathcal{W}^{\psi_{R_t,\alpha}}_{\phi_{\tilde\pi}}(h)dh.\tag{4.18}$$

We record the calculation above in the following proposition.

Proposition 4.1. *Let $\tilde\pi$ be an irreducible cuspidal automorphic representation of $\mathrm{Mp}_{2t}(\mathbb{A})$. Let $\sigma=\theta^m_{\psi,2t}(\tilde\pi)$ be the theta lift of $\tilde\pi$ to $\mathrm{O}_m(\mathbb{A})$. We assume that $r\geq t$ and $m>2t$. Let ϕ_σ be the element of V_σ given by (3.12), namely*

$$\phi_\sigma(g)=\int_{\mathrm{Mp}_{2l}(k)\backslash \mathrm{Mp}_{2l}(\mathbb{A})}\phi_{\tilde\pi}(h)\theta_{\psi,\varphi}(g,h)dh.$$

Then the $\psi_{t,\alpha}$- Fourier coefficient of σ is related to the $\psi_{R_t,\alpha}$-Whittaker -Fourier coefficient of $\tilde\pi$ by

$$\mathcal{F}^{\psi_{t,\alpha}}(\phi_\sigma)(e)=\int_{S(\mathbb{A})\backslash \mathrm{Mp}_{2t}(\mathbb{A})}\omega_\psi(1,h)\varphi(f_{-t},\cdots,f_{-1};\mu_\alpha\otimes f_t)\mathcal{W}^{\psi_{R_t,\alpha}}_{\phi_{\tilde\pi}}(h)dh.\tag{4.19}$$

In particular, if $\tilde\pi$ is not generic, or if α is not represented by X_{m-2t}, then σ has zero $\psi_{t,\alpha}$-Fourier coefficients.

As a corollary, we get

Proposition 4.2. *Let σ be an irreducible cuspidal automorphic representation of $\mathrm{O}_m(\mathbb{A})$. Assume that σ has a nonzero $\psi_{t,\alpha}$-Fourier coefficient ($1\leq t\leq r$, $m-2t\geq 1$). Assume that the first occurrence of σ, $\mathrm{FO}_\psi(\sigma)$ is $2t$. Let $\tilde\pi=\theta^{2t}_{\psi,m}(\sigma)$ be the ψ-theta lift of σ to $\mathrm{Mp}_{2t}(\mathbb{A})$. Then $\tilde\pi$ is globally generic, with respect to the Whittaker character $\psi_{R_t,\alpha}$, as above. Moreover, the formula relating the $\psi_{t,\alpha}$-Fourier coefficient of σ and the $\psi_{R_t,\alpha}$-Whittaker-Fourier coefficient of $\tilde\pi$ is given (in the above notation) by* (4.19).

Conversely, start with an irreducible, cuspidal automorphic representation $\tilde\pi$ of $\mathrm{Mp}_{2t}(\mathbb{A})$, which is globally generic with respect to a character of the form $\psi_{R_t,\alpha}$, where $\alpha\in k^\times$. Assume that $t\leq r$ and $2t<m$. We use the same notation pertaining to X_m (X_{m-2t}, the symmetric nondegenerate matrices T_{m-2t} etc.). Since the quadratic form defined by T_{m_0+2} is not anisotropic, α is represented by T_{m_0+2}. Let $\mu_\alpha\in X_{m_0+2}(k)$ be such that $(\mu_\alpha,\mu_\alpha)=\alpha$, and consider the r.h.s. of (4.19), where we take ω_ψ to be the Weil representation of the dual pair $\mathrm{O}_{m_0+2+2t}(\mathbb{A})\times\mathrm{Mp}_{2t}(\mathbb{A})$.

It is easy to see that the r.h.s is not identically zero. By Proposition 4.1, we conclude that the $\psi_{t,\alpha}$-Fourier coefficient of the ψ theta lift of $\tilde{\pi}$ to $\mathrm{O}_{m_0+2+2t}(\mathbb{A})$ is nontrivial. In particular, the ψ-theta lift of $\tilde{\pi}$ to $\mathrm{O}_{m_0+2+2t}(\mathbb{A})$ is nonzero. Now, let σ be an irreducible cuspidal automorphic representation of $\mathrm{O}_m(\mathbb{A})$, which has a nontrivial $\psi_{t,\alpha}$-Fourier coefficient. We already proved that $\mathrm{FO}_\psi(\sigma) \geq 2t$. If $\mathrm{FO}_\psi(\sigma) = 2t$, then, by what we just explained, we must have that $r \leq t+1$. Thus, if we assume that $t < r-1$, we get that $\mathrm{FO}_\psi(\sigma) \geq 2t+2$, and hence, by Theorem 1.1 in [**GJS07**], $L^S(s,\sigma)$ is holomorphic at $Re(s) > \frac{m}{2} - t - 1$.

We can repeat the same considerations if α is represented by T_{m_0}. Let $\mu_\alpha \in X_{m_0}(k)$ represent α. Now we repeat the same argument with X_{m_0} replacing X_{m_0+2} and obtain that if $\mathrm{FO}_\psi(\sigma) = 2t$ and $\tilde{\pi}$ is the ψ-theta lift of σ to $\mathrm{Mp}_{2t}(\mathbb{A})$, then since $\tilde{\pi}$ is globally $\psi_{R_t,\alpha}$-generic, it has a nontrivial ψ-theta lift to $\mathrm{O}_{m_0+2t}(\mathbb{A})$, and we conclude that $t \geq r$. Thus, if we assume, in this case, that $t < r$, then we get, as before, that $\mathrm{FO}_\psi(\sigma) \geq 2t+2$, and that $L^S(s,\sigma)$ is holomorphic at $Re(s) > \frac{m}{2} - t - 1$. Let us summarize this.

Theorem 4.3. *Let σ be an irreducible cuspidal automorphic representation of $\mathrm{O}_m(\mathbb{A})$. Assume that σ has a nonzero $\psi_{t,\alpha}$-Fourier coefficient ($1 \leq t \leq r$, $m-2t \geq 1$).*

1. *Assume that $t < r-1$. Then the partial L-function $L^S(s,\sigma)$ is holomorphic for $Re(s) > \frac{m}{2} - t - 1$.*

2. *Assume that α is represented by the quadratic form corresponding to T_{m_0}, and that $t < r$. Then the partial L-function $L^S(s,\sigma)$ is holomorphic for $Re(s) > \frac{m}{2} - t - 1$.*

5. Completion of the Proof of Theorem 1.1

The proof of Part (1) of Theorem 1.1 is completely analogous to the one in §4.1. We will use the same notation as before, adapted to the local setting.

5.1. Local models and theta lifts. We determine the vanishing of local theta lifts in terms of the local $\psi_{t,\alpha}$-functional. Here is the result.

Theorem 5.1. *Let F be a finite extension of the p-adic field $\mathbb{Q}_p$. Let σ be an irreducible admissible representation of $\mathrm{O}_m(F)$. Assume that σ has a nonzero $\psi_{t,\alpha}$-functional as defined in (2.17) for some $t \leq r$, the Witt index of the quadratic space X_m defining $\mathrm{O}_m(F)$. Then $\mathrm{LO}_\psi(\sigma) \geq 2t$.*

This is the local analogue of Theorem 3.1. The proof uses the local version of the global arguments used in the proof of Theorem 3.1 and is modeled after the proof of Proposition 2.1, [**JngS03**] and of Theorem 4.2, [**JngS07a**]. Here is a sketch.

It is enough to show that the local ψ-theta lift $\theta^{2l}_{\psi,m}(\sigma)$ of σ to $\mathrm{Mp}_{2l}(F)$ is zero for all $l < t$. Assume that this is not the case. Then there is a integer $l < t$ such that the ψ-theta lift $\theta^{2l}_{\psi,m}(\sigma)$ of σ to $\mathrm{Mp}_{2l}(F)$ is nonzero. This means by (3.4) that there is an irreducible admissible representation $\tilde{\pi}$ of $\mathrm{Mp}_{2l}(F)$ such that

$$\mathrm{Hom}_{\mathrm{O}_m(F)\times\mathrm{Mp}_{2l}(F)}(\omega_\psi, \sigma \otimes \tilde{\pi}) \neq 0 \tag{5.1}$$

or equivalently,

$$\mathrm{Hom}_{\mathrm{O}_m(F)}(\omega_\psi \otimes \tilde{\pi}^\vee, \sigma) \neq 0 \tag{5.2}$$

where ω_ψ is the local Weil representation of $\mathrm{Mp}_{2lm}(F)$, restricted to the dual pair $(\mathrm{O}_m(F), \mathrm{Mp}_{2l}(F))$.

Following §3, we consider the analogous polarizations for X_m and W_{2l}

$$X_m = \ell_t^+ \oplus X_{m-2t} \oplus \ell_t^-, \tag{5.3}$$

$$W_{2l} = Y_l^+ \oplus Y_l^-. \tag{5.4}$$

We consider the local Weil representation ω_ψ on the mixed Schrödinger model

$$\mathcal{S}_{m\otimes 2l} := \mathcal{S}(\ell_t^- \otimes W_{2l} \oplus X_{m-2t} \otimes Y_l^+). \tag{5.5}$$

Using similar bases as in §3, we write a local Schwartz-Bruhat function φ in $\mathcal{S}_{m\otimes 2l}$ as

$$\varphi(w_1, \cdots, w_t; y_1, \cdots, y_{m-2t}) \tag{5.6}$$

where $w_i \in W_{2l}$ and $y_j \in Y_l^+$ for $i = 1, \cdots, t$ and $j = 1, \cdots, m-2t$.

By hypothesis, σ has a nonzero $\psi_{t,\alpha}$-functional ℓ, i.e. a nonzero element in

$$\mathrm{Hom}_{V_t(F)}(V_\sigma, \psi_{t,\alpha}).$$

By (5.2), the functional ℓ induces a nonzero functional β over $\mathcal{S}_{m\otimes 2l} \otimes V_{\tilde{\pi}^\vee}$, such that

$$\beta(\omega_\psi(v,h)\varphi, \xi) = \psi_{t,\alpha}(v)\beta(\varphi, \xi) \tag{5.7}$$

for $v \in V_t(F)$, $h \in \mathrm{Mp}_{2l}(F)$, $\xi \in V_{\tilde{\pi}^\vee}$, and φ is a function in the mixed model.

We consider the local version of the dz-integration as in (4.4), and obtain as in [**JngS03**], p. 755, that for each fixed ξ, β is supported on

$$C_0 = \{(w_1, \cdots, w_t; y_1, \cdots, y_{m-2t}))|(w_i, w_j) = 0, \forall\ 1 \leq i, j \leq t\}.$$

Indeed, let i be the restriction map from $\mathcal{S}(W_{2l}^t \oplus (Y_l^+)^{m-2t})$ to $\mathcal{S}(C_0)$. It is surjective. Let i^* be the corresponding map on Jacquet modules with respect to Z_t and the trivial character. Then i^* is an isomorphism, i.e.

$$J_{Z_t}(\mathcal{S}(W_{2l}^t \oplus (Y_l^+)^{m-2t})) \cong J_{Z_t}(\mathcal{S}(C_0)).$$

Let C be the complement of C_0 in $W_{2l}^t \oplus (Y_l^+)^{m-2t}$. Then it is easy to see that $J_{Z_t}(\mathcal{S}(C)) \cong 0$ and $J_{Z_t}(\mathcal{S}(C_0)) \cong \mathcal{S}(C_0)$. We regard $\mathcal{S}(C_0)$ as a module over $(Z_t(F)\backslash V_t(F)) \times \mathrm{Mp}_{2l}(F)$. Denote $U'_t(F) = Z_t(F)\backslash V_t(F)$. We identify $U'_t(F)$ with $U_t(F)M_{t\times(m-2t)}(F)$ and regard $\psi_{t,\alpha}$ as a character of $U'_t(F)$. Thus, we have to prove that $J_{U'_t(F),\psi_{t,\alpha}}(\mathcal{S}(C_0)) = 0$, when $l < t$. Write C_0 as the disjoint union, over $0 \leq d \leq l$, of the varieties

$$C_0^d = \{(w_1, \cdots, w_t; y_1, \cdots, y_{m-2t}) \in C_0 \mid \dim_F \mathrm{Span}\{w_1, ..., w_t\} = d\}.$$

Then it is enough to prove that $J_{U'_t(F),\psi_{t,\alpha}}(\mathcal{S}(C_0^d)) = 0$, for all $0 \leq d \leq l$. We can embed $\mathcal{S}(C_0^d)$ inside $ind_{U'_t(F)\times\bar{P}_d(F)}^{U'_t(F)\times\mathrm{Mp}_{2l}(F)}\mathcal{S}(C_0^{d,-})$, where

$$C_0^{d,-} = \{(w_1, \cdots, w_t; y_1, \cdots, y_{m-2t}) \in C_0^d \mid w_1, ..., w_d \in Y_d^-\},$$

and $\bar{P}_d(F)$ is the inverse image of $P_d(F)$ inside $\mathrm{Mp}_{2l}(F)$. Thus, we have an embedding

$$J_{U'_t(F),\psi_{t,\alpha}}(\mathcal{S}(C_0^d)) \hookrightarrow ind_{\bar{P}_d(F)}^{\mathrm{Mp}_{2l}(F)} J_{U'_t(F),\psi_{t,\alpha}}(\mathcal{S}(C_0^{d,-})),$$

and so it is enough to show that $J_{U'_t(F),\psi_{t,\alpha}}(\mathcal{S}(C_0^{d,-}))=0$, when $l<t$. Using the action of $M_{t\times(m-2t)}(F)$ on $\mathcal{S}(C_0^{d,-})$ through the formulae of the Weil representation, we conclude, as above, and as in §4.1 that

$$J_{U'_t(F),\psi_{t,\alpha}}(\mathcal{S}(C_0^{d,-}))\cong J_{U_t(F),\psi_{t,\alpha}}(\mathcal{S}(W_{\alpha,d})),$$

where $W_{\alpha,d}$ is defined exactly by the same relations as in the global case. Finally, it remains to show that $J_{U_t(F),\psi_{t,\alpha}}(\mathcal{O}_{w_{(t:d)}})=0$, for every $U_t(F)$-orbit $\mathcal{O}_{w_{(t:d)}}$ of $W_{\alpha,d}$ (same definition as in the global case). This follows, as in the global case, from the fact that since $d\le l<t$, there is a simple root subgroup in $U_t(F)$, which lies in the stabilizer of the representative $w_{t:d}$. This proves Theorem 5.1.

5.2. Proof of part (1) of Theorem 1.1. Let σ be an irreducible cuspidal automorphic representation of $\mathrm{O}_m(\mathbb{A})$. Assume that there is a finite local place v of the number field k such that the local v-component σ_v of σ has a nonzero local $\psi_{t,\alpha}$-functional. Then by Theorem 5.1, the local ψ-theta lift of σ_v to $\mathrm{Mp}_{2l}(k_v)$ is zero for all $l<t$. Hence the global ψ-theta lift of σ to $\mathrm{Mp}_{2l}(\mathbb{A})$ must be zero for all $l<t$. This property holds also for all twists of σ by automorphic sign characters, since the twist of σ_v by any sign character also has a nonzero local $\psi_{t,\alpha}$-functional. Hence the lowest occurrence of σ in the global ψ-theta liftings, $\mathrm{LO}_\psi(\sigma)$, must be greater than or equal to $2t$. By Theorem 1.1 of [**GJS07**], the partial L-function $L^S(s,\sigma)$ must be holomorphic for $Re(s)>\frac{m}{2}-t$. This completes the proof of Theorem 1.1.

References

[F95] Furusawa, Masaaki *On the theta lift from SO_{2n+1} to $\widetilde{\mathrm{Sp}}_n$.* J. Reine Angew. Math. 466 (1995), 87–110.

[GJS07] Ginzburg, David; Jiang, Dihua; Soudry, David *Poles of L-functions and theta liftings for orthogonal groups.* (2007), to appear in JIMJ.

[GPSR97] Ginzburg, D.; Piatetski-Shapiro, I.; Rallis, S. *L functions for the orthogonal group.* Mem. Amer. Math. Soc. 128 (1997), no. 611.

[GRS97] Ginzburg, David; Rallis, Stephen; Soudry, David *Periods, poles of L-functions and symplectic-orthogonal theta lifts.* J. Reine Angew. Math. 487 (1997), 85–114.

[GRS03] Ginzburg, D.; Rallis, S.; Soudry, D. *On Fourier coefficients of automorphic forms of symplectic groups.* Manuscripta Math. 111 (2003), no. 1, 1–16.

[H79] Howe, Roger *θ-series and invariant theory.* Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, pp. 275–285, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.

[JngS03] Jiang, Dihua; Soudry, David *The local converse theorem for $\mathrm{SO}(2n+1)$ and applications.* Ann. of Math. (2) 157 (2003), no. 3, 743–806.

[JngS07a] Jiang, Dihua; Soudry, David *On the genericity of cuspidal automorphic forms of SO_{2n+1}.* J. Reine Angew. Math. 604 (2007), 187–209.

[JngS07b] Jiang, Dihua; Soudry, David *On the genericity of cuspidal automorphic forms of SO_{2n+1}, II.* Compositio Math. 143 (2007), 721–748.

[K94] Kudla, Stephen S. *Splitting metaplectic covers of dual reductive pairs.* Israel J. Math.,Vol. 87 (1994), 361-401.

[K96] Kudla, Stephen S. *Notes on the local theta correspondence.* Lecture series in the European School in Group Theory, at Schloss Hirschberg, September 1996 (http://www.math.umd.edu/ ssk/ssk.research.html).

[KR94] Kudla, Stephen; Rallis, Stephen. *A regularized Siegel-Weil formula: the first term identity.* Ann. of Math.(2) 140 (1994), no. 1, 1-80.

[KR05] Kudla, Stephen; Rallis, Stephen. *On first occurrence in the local theta correspondence.* Automorphic Representations, L-functions and Applications: Progress and Prospects. Ohio-State University MRI Publications 11, de Gruyter (2005), 273–308.

[M96] Moeglin, Colette *Front d'onde des représentations des groupes classiques p-adiques.* (French) [Wave front of representations of p-adic classical groups] Amer. J. Math. 118 (1996), no. 6, 1313–1346.

[M97a] Moeglin, Colette *Non nullité de certaines relêvements par séries théta.* Journal of Lie Theory, 7(1997) 201–229.

[M97b] Moeglin, Colette *Quelques propriétés de base des séries théta.* Journal of Lie Theory, 7(1997) 231–238.

[MVW87] Moeglin, Colette; Vignéras, Marie-France; Waldspurger, Jean-Loup *Correspondances de Howe sur un corps p-adique.* (French) [Howe correspondences over a p-adic field] Lecture Notes in Mathematics, 1291. Springer-Verlag, Berlin, 1987

[MW87] Moeglin, Colette; Waldspurger, Jean-Loup *Modéles de Whittaker dégénérés pour des groupes p-adiques.* (French) [Degenerate Whittaker models for p-adic groups] Math. Z. 196 (1987), no. 3, 427–452.

[PSS87] Piatetski-Shapiro, Ilya; Soudry David *On a correspondence of automorphic forms of orthogonal groups of order five.* J. Math. Pures Appl. 66 (1987), 407–436.

[R84] Rallis, Stephen *On the Howe duality conjecture.* Compositio Math. 51 (1984), no. 3, 333–399.

[W80] Waldspurger, J.-L. *Correspondance de Shimura.* (French) [Shimura Correspondence] J. Math Pures Appl. 59 (1980), 1–132.

[W90] Waldspurger, J.-L. *Démonstration d'une conjecture de dualité de Howe dans le cas p-adique,* $p \neq 2$. (French) [Proof of a duality conjecture of Howe in the p-adic case, $p \neq 2$] Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989), 267–324, Israel Math. Conf. Proc., 2, Weizmann, Jerusalem, 1990.

David Ginzburg: School of Mathematical Sciences, The Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat-Aviv, 69978 Israel
E-mail address: ginzburg@post.tau.ac.il

Dihua Jiang: School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
E-mail address: dhjiang@math.umn.edu

David Soudry: School of Mathematical Sciences, The Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat-Aviv, 69978 Israel
E-mail address: soudry@post.tau.ac.il