NONVANISHING OF THE CENTRAL CRITICAL VALUE OF THE THIRD SYMMETRIC POWER L-FUNCTION

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ABSTRACT. We characterize in this paper the nonvanishing of the central critical value of the third symmetric power L-functions of irreducible cuspidal automorphic representations π of $GL_2(\mathbb{A})$ in terms of the occurrence of π in the spectral decomposition of the tensor product of two automorphic theta representations on the cubic cover of $GL_2(\mathbb{A})$, which was constructed by Kazhdan and Patterson in [KP].

1. Introduction

Let π be an irreducible cuspidal automorphic representation of $GL_2(\mathbb{A})$ and n be a positive integer. One can define, following Langlands, the n-th symmetric power L-function $L(s, \pi, Sym^n)$. For the significance of this family of automorphic L-functions, we refer to [Sh], for instance.

The symmetric cube L-functions, $L(s, \pi, Sym^3)$, was first studied by F. Shahidi in [Sh1] using the Langlands-Shahidi method to establishing the analytic properties as conjectured by Langlands (meromorphic continuation and functional equation, $\cdots$). In a recent work of D. Bump, D. Ginzburg and J. Hoffstein [BGH], an integral representation was found for the symmetric cube L-functions, by which they can prove that the partial symmetric cube L-function is holomorphic for $Re(s) > \frac{3}{4}$ except for $s = 1$. By using a very interesting local-global argument, Kim and Shahidi proved in a recent preprint [KS] that the symmetric cube L-function is entire unless the automorphic representation π is monomial.

The objective of this paper is to characterize in terms of co-period integral the nonvanishing of the complete symmetric cube L-function $L(s, \pi, Sym^3)$ at the central critical value (i.e. at $s = \frac{1}{2}$).

It is well known that if an automorphic representation is related by the Langlands reciprocity principle to a motive, then the Deligne's conjecture claims that the central critical value of the L-functions of π should be related to certain period associated to π. For more detail discussion on this aspect, we refer to [D], [Gr] and [H].

The recent study of the central critical value of automorphic L-functions was stimulated by the relative trace formula approach to Langlands functoriality lifting problem. In general, it is believed that the nonvanishing of the central critical value of

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automorphic L-functions should be related to the nonvanishing of certain period integral of the automorphic representation π. As discussed in [H], such period integrals may not be the right period in the sense of Deligne. However, it seems natural from the spectral theory of automorphic representations that the (generalized) period integrals should be basic objects which may have intrinsic relations with special values of automorphic L-functions in general. The recent work on this aspect are [HK], [JR], [GP], [Jng], [Jng1], [GJng], [GRS1], etc.

In this paper, we use the method developed in [JR] and [Jng] to characterize the nonvanishing of central critical value of the symmetric cube L-function $L(s,\pi,Sym^3)$ in terms of the occurrence of the irreducible cuspidal automorphic representation $\pi^\vee$ (the contragredient of π) in the spectral decomposition of $\theta_{\tilde{GL}_2(\mathbb{A}),\epsilon}\otimes\theta_{\tilde{GL}_2(\mathbb{A}),\epsilon^2}$ restricted to $GL_2(\mathbb{A})$. To be more precise for the notations, we consider the cubic cover of $GL_2(\mathbb{A})$,

$$1\to\mu_3\to\tilde{GL}_2(\mathbb{A})\to GL_2(\mathbb{A})\to 1,$$

where μ_3 is the group of the third roots of unity contained in the given number field F. The representations $\theta_{\tilde{GL}_2(\mathbb{A}),\epsilon}$ and $\theta_{\tilde{GL}_2(\mathbb{A}),\epsilon^2}$ are the automorphic theta representations of the cubic cover $\tilde{GL}_2(\mathbb{A})$, constructed by Kazhdan and Patterson in [KP], associated to the character ϵ, ϵ^2, respectively, of the group μ_3. Since the tensor product $\theta_{\tilde{GL}_2(\mathbb{A}),\epsilon}\otimes\theta_{\tilde{GL}_2(\mathbb{A}),\epsilon^2}$ is trivial on diagonal embedding of the group μ_3 into $\tilde{GL}_2(\mathbb{A})\times\tilde{GL}_2(\mathbb{A})$, it makes sense to consider the spectral decomposition of the tensor product $\theta_{\tilde{GL}_2(\mathbb{A}),\epsilon}\otimes\theta_{\tilde{GL}_2(\mathbb{A}),\epsilon^2}$ restricted to $GL_2(\mathbb{A})$. Our main result (part (2) of Theorem 3.2) is

Main Theorem. *If the co-period*

$$\mathcal{CP}_{GL_2}(\phi,\varphi'_\epsilon,\varphi'_{\epsilon^2})=\int_{GL_2(F)\backslash GL_2(\mathbb{A})^1}\phi(h)\varphi'_\epsilon(\tilde{h})\varphi'_{\epsilon^2}(\tilde{h})dh \tag{1.1}$$

does not vanish for a choice of data $\phi\in\pi$, $\varphi'_\epsilon\in\theta_{\tilde{GL}_2(\mathbb{A}),\epsilon}$, $\varphi'_{\epsilon^2}\in\theta_{\tilde{GL}_2(\mathbb{A}),\epsilon^2}$, *then the third symmetric power L-function* $L(s,\pi,Sym^3)$ *does not vanish at* $s=\frac{1}{2}$.

The idea to prove this result is based on the facts that (1) the nonvanishing of the symmetric cube L-function at $\frac{1}{2}$ is closely related to the nonvanishing of a residual representation of certain cuspidal Eisenstein series of $G_2(\mathbb{A})$, the exceptional group of type G_2 (following the Langlands-Shahidi theory [KS]) and (2) the part (1) of Theorem 3.2 in this paper, which says that the residual representation $E_{\frac{1}{2}}(g,\pi^\vee)$ of $G_2(\mathbb{A})$ occurs in the spectral decomposition of the tensor product $\theta_{\tilde{G}_2(\mathbb{A}),\epsilon}\otimes\theta_{\tilde{G}_2(\mathbb{A}),\epsilon^2}$ restricted to $G_2(\mathbb{A})$ if and only if the cuspidal data $\pi^\vee$ of $GL_2(\mathbb{A})$ (the Levi subgroup of G_2 generated by the long root) occurs in the spectral decomposition of the tensor product $\theta_{\tilde{GL}_2(\mathbb{A}),\epsilon}\otimes\theta_{\tilde{GL}_2(\mathbb{A}),\epsilon^2}$ restricted to $GL_2(\mathbb{A})$. The proof of this equivalence is based on the Arthur's truncation of the residual representations, the technique in the Rankin-Selberg method and the comparison principle of the 'outer' period and the 'inner' period ([JR] and [Jng]).

We remark that the method we are applying in this paper (as well as [JR] and [Jng]) proves the existence of distinguished residual representation of the Eisenstein series provided the nonvanishing of the co-period, without using the conjecture on the normalization of intertwining operators by means of an appropriate product of L-functions, which detects the possible poles of of the Eisenstein series. This key observation should be important to the formulation of the relative trace formula approach to study the distinguished residual spectrum ([JMR]).

In general, it is expected that the nonvanishing of the co-period $\mathcal{CP}_{GL_2}(\phi, \varphi'_\epsilon, \varphi'_{\epsilon^2})$ in (1.1) should be equivalent to the nonvanishing of the third symmetric power L-function $L(s, \pi, Sym^3)$ does not vanish at $s = \frac{1}{2}$, especially, in the case that the autmorphic L-function under consideration has a Rankin-Selberg type integral representation. In the case at hand, the integral representation for $L(s, \pi, Sym^3)$ was given in [BGH], which uses an Eisenstein series on the cubic cover of $GSp(6, \mathbb{A})$. We believe that by studying the first term identity related to the Eisenstein series at the center of symmetry of the functional equation for the Eisenstein series, the nonvanishing of $L(\frac{1}{2}, \pi, Sym^3)$ should imply the nonvanishing of the co-period $\mathcal{CP}_{GL_2}(\phi, \varphi'_\epsilon, \varphi'_{\epsilon^2})$ for a certain choice of data ([GJng]). Furthermore, we believe that an analogous conjecture of the Gross-Prasad type [GP] could be formulated provided the complete establishment of the regularized Siegel-Weil formula for the Eisenstein series under discussion [KR] and [HK].

In the case of the symmetric square L-functions for irreducible cuspidal automorphic representations π of $GL_2(\mathbb{A})$, following the early work of Shimura [Shm] and more generally of Gelbart and Jacquet [GJ], the simple pole at $s = 1$ of the symmetric square L-function $L(s, \pi, Sym^2)$ is characterized by the nonvanishing of the following co-period

$$\int_{Z(\mathbb{A})GL_2(F)\backslash GL_2(\mathbb{A})} \phi(g)\theta(g)\bar{\theta}(g)dg$$

for some choice of $\phi \in \pi$ and certain theta function θ on the double cover of $GL_2(\mathbb{A})$. This statement was later generalized by Patterson and Piatetski-Shapiro to GL_3 [PPS] and by Bump and Ginzburg to GL_n [BG]. It is expected that the nonvanishing of the co-period should imply that π is a functorial lift from $Sp(2m)$ if $n = 2m + 1$ and from $SO(2m)$ if $n = 2m$. By contrast, it is not sure in the symmetric cube case whether the nonvanishing of the co-period for certain choice of the data yields any lifting information on π.

It is also very mysterious whether there is the same type statement valid for the symmetric fourth power L-function $L(s, \pi, Sym^4)$, the integral representation for which was found in [G].

The paper is organized as follows. We recall in §2 the construction of the automorphic theta representation of the cubic cover of $G_2(\mathbb{A})$ from [GRS] and that of the cubic cover of $GL_2(\mathbb{A})$ from [KP], and the basic relation between a cuspidal Eisenstein series of G_2 and the symmetric cube L-functions of GL_2 from [KS]. The third section is main part of this paper, in which we first apply the truncation technique to drive a formula (3.4) for the 'outer' co-period for residue $E_{s_0}(g,\phi)$, and then by using the technique from the Rankin-Selberg method we connect the 'outer' co-period to the 'inner' co-period, and finally by combining with the recent result form [KS] we prove our main theorem (Theorem 3.2).

2. Preliminaries

2.1. **The Split G_2.** We recall basic structure of G_2 from [GRS]. Let G be the split exceptional group of type G_2 over a field F of characteristics zero, with the long simple root α and the short simple root β. Then all the positive roots of G are

$$\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta.$$

Let N be the maximal unipotent subgroup of G generated by all the positive roots. Then the Borel subgroup $B = TN$ where T is the maximal split torus T of G. Let $h(t_1,t_2)$ be an element of T such that

$$(a\alpha+b\beta)(h(t_1,t_2)) = [t_1t_2^{-1}]^a \cdot t_2^b$$

for $a, b \in \mathbb{Z}$. Let w_γ be the Weyl group element associated to the root γ. Then one has

$$w_\alpha(h(t_1,t_2)) = h(t_2,t_1), \text{ and } w_\beta(h(t_1,t_2)) = h(t_1t_2, t_2^{-1}).$$

Let $h_\gamma(t)$ be the one-parameter subgroup in T associated to the root γ. One has

$$h_\alpha(t) = h(t,t^{-1}), \text{ and } h_\beta(t) = h(t^{-1},t^2).$$

Let $P = MU$ (resp. $Q = LV$) be the maximal parabolic subgroup of G associated to β (resp. α). Then both M and L are isomorphic to $GL(2)$ as algebraic groups. To indicate the difference, we denote

$$M = GL_\beta(2) \text{ and } L = GL_\alpha(2).$$

Specifically, one has

$$m\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = h(1,t) \text{ and } m\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = h(t^{-1},t^2),$$

and

$$l\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = h(t,1) \text{ and } l\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = h(t,t^{-1}).$$

2.2. **Automorphic Theta Representations of the Cubic Cover of G_2.** From now on, we assume that the ground field F is a number field containing the group μ_3 of the third roots of unit. Let $\mathbb{A}$ be the ring of adeles of F. According to [Ma], for any given cocycle $\sigma \in H^2(G_2, \mu_3)$, one may construct a cubic central extension of G_2, unique up to isomorphism, over a p-adic field or the ring of adeles of a number field (containing μ_3). The specific construction of a cubic covering group of $G = G_2$ was given in [GRS], for instance. In the following, we recall briefly from [GRS] the construction of the automorphic theta representations of a cubic cover of $G_2(\mathbb{A})$.

Let σ be a cocycle in $H^2(G(\mathbb{A}), \mu_3)$ which can be expressed as $\sigma = \prod_v \sigma_v$, i.e.

$$\sigma(g, h) = \prod_v \sigma_v(g_v, h_v),$$

for $g = (g_v), h = (h_v) \in G(\mathbb{A})$ and σ_v is the cocycle at the local place v. Corresponding to this cocycle, one can construct a cubic cover of $G(\mathbb{A})$,

$$1 \to \mu_3 \to \tilde{G}(\mathbb{A}) \to G(\mathbb{A}) \to 1,$$

and the local cocycle gives rise to the cubic cover of the local group $G(F_v)$,

$$1 \to \mu_3 \to \tilde{G}(F_v) \to G(F_v) \to 1.$$

There is a canonical section

$$\tilde{s} : G(\mathbb{A}) \to \tilde{G}(\mathbb{A})$$

which splits over subgroups $N(\mathbb{A})$, $SL_\gamma(2, \mathbb{A})$ when γ is a short root and $G(F)$.

Let $\mathbf{x}_\gamma(t)$ be the image of $x_\gamma(t)$ under $\tilde{s}$, where $x_\gamma(t)$ is the one-parameter additive subgroup associated to the root γ. For $t \in F^\times$,

$$\begin{aligned} \mathbf{w}_\gamma(t) &:= \mathbf{x}_\gamma(t)\mathbf{x}_{-\gamma}(-t^{-1})\mathbf{x}_\gamma(t), \\ \mathbf{h}_\gamma(t) &:= \mathbf{w}_\gamma(t)\mathbf{w}_\gamma(1)^{-1}. \end{aligned}$$

We denote by $\tilde{R}$ the preimage in $\tilde{G}(\mathbb{A})$ of a subgroup R of $G(\mathbb{A})$. Let $T_3(\mathbb{A})$ and $T^{(3)}(\mathbb{A})$ be subtori of $T(\mathbb{A})$ defined in p.265–266, [GRS]. Then the preimage $\tilde{T}_3(\mathbb{A})$ (resp. $\tilde{T^{(3)}}(\mathbb{A})$) is abelian (resp. maximal abelian) subgroup of $\tilde{T}(\mathbb{A})$.

Let χ_{s_1,s_2} be a character of $T(\mathbb{A})$ such that $\chi_{s_1,s_2}(h_\alpha(t)) := |t|^{\frac{1}{3}+s_1}$ and $\chi_{s_1,s_2}(h_\beta(t)) := |t|^{1+s_2}$. Then we define a character, which is still denoted by χ_{s_1,s_2}, of $\tilde{T}_3(\mathbb{A})$ by assuming its value on μ_3 to be ϵ ($\epsilon \in F$ with $\epsilon^3 = 1$). This character χ_{s_1,s_2} can be extended to be a character of the maximal abelian subgroup $\tilde{T}^{(3)}(\mathbb{A})$ of $\tilde{T}(\mathbb{A})$. Then one forms the induced representation

$$I_{\tilde{B}^{(3)}}(s_1, s_2) := Ind_{\tilde{B}^{(3)}(\mathbb{A})}^{\tilde{G}(\mathbb{A})}(\chi_{s_1,s_2}\delta_B^{\frac{1}{2}}).$$

From p. 266, [GRS], for $f_{s_1,s_2} \in I_{\tilde{B}^{(3)}}(s_1, s_2)$, the Eisenstein series

$$E(\tilde{g}, f_{s_1,s_2}) := \sum_{\gamma \in \mathbf{B}(F)\backslash \mathbf{G}(F)} \left(\sum_{t \in \mathbf{T}_3(F)\backslash \mathbf{T}(F)} f_{s_1,s_2}(t\gamma\tilde{g}) \right)$$

absolutely converges in the dominant Weyl chamber shifted by $(\frac{1}{3}, 1)$ and has meromorphic continuation when the section f_{s_1,s_2} is holomorphic. The residue at $(s_1, s_2) = (0,0)$ is the automorphic representation θ_ϵ, which can also be realized as the image of the (normalized) intertwining operator associated to the longest Weyl group element from $I_{\tilde{B}^{(3)}}((0,0))$ to $I_{\tilde{B}^{(3)}}((-\frac{2}{3}, -2))$.

Since the restriction of the cubic cover to the Levi subgroup $L(\mathbb{A}) = GL_\alpha(2,\mathbb{A})$ does not split, it gives rise to a cubic cover of $GL_\alpha(2,\mathbb{A})$. By induction on stages, one has

$$I_{\tilde{B}^{(3)}}(s_1, s_2) = Ind_{\tilde{Q}(\mathbb{A})}^{\tilde{G}(\mathbb{A})}[\rho_Q^{\frac{1}{2}+\frac{s_1+2s_2+3}{10}} Ind_{\tilde{B}_\alpha^{(3)}(\mathbb{A})}^{G\tilde{L}_\alpha(2,\mathbb{A})}(\delta_{B_\alpha}^{\frac{1}{6}+\frac{s_1}{2}+\frac{1}{2}})].$$

From [KP], the inner induced representation of $G\tilde{L}_\alpha(2,\mathbb{A})$, $Ind_{\tilde{B}_\alpha^{(3)}(\mathbb{A})}^{G\tilde{L}_\alpha(2,\mathbb{A})}(\delta_{B_\alpha}^{\frac{1}{6}+\frac{s_1}{2}+\frac{1}{2}})$ has a unique irreducible quotient at $s_1 = 0$, which is realized as the image of the (normalized) intertwining operator from $Ind_{\tilde{B}_\alpha^{(3)}(\mathbb{A})}^{G\tilde{L}_\alpha(2,\mathbb{A})}(\delta_{B_\alpha}^{\frac{1}{6}+\frac{1}{2}})$ to $Ind_{\tilde{B}_\alpha^{(3)}(\mathbb{A})}^{G\tilde{L}_\alpha(2,\mathbb{A})}(\delta_{B_\alpha}^{-\frac{1}{6}+\frac{1}{2}})$ and also as the residue at $s_1 = 0$ of the Eisenstein series

$$E(\tilde{h}, \phi_{s_1}) := \sum_{\gamma \in \mathbf{B}(F)\backslash \mathbf{GL}_\alpha(2,F)} \left(\sum_{t \in \mathbf{T}_3(F)\backslash \mathbf{T}(F)} \phi_{s_1}(t\gamma\tilde{g}) \right)$$

for holomorphic sections ϕ_{s_1} in $Ind_{\tilde{B}_\alpha^{(3)}(\mathbb{A})}^{G\tilde{L}_\alpha(2,\mathbb{A})}(\delta_{B_\alpha}^{\frac{1}{6}+\frac{s_1}{2}+\frac{1}{2}})$. This automorphic representation of $G\tilde{L}_\alpha(2,\mathbb{A})$ is denoted by $\theta_{\alpha,\epsilon}$, which is called the exceptional automorphic representation of $\tilde{GL}_\alpha(2,\mathbb{A})$.

Proposition 2.1. *The automorphic theta representation θ_ϵ of $\tilde{G}(\mathbb{A})$ can be realized as the residue at $s_2 = 0$ of the Eisenstein series*

$$E(\tilde{g}, f_{s_2,\theta_{\alpha,\epsilon}}) := \sum_{\gamma \in \mathbf{Q}_F \backslash \mathbf{G}(F)} f_{s_2,\theta_{\alpha,\epsilon}}(\gamma\tilde{g}),$$

for holomorphic sections $f_{s_2,\theta_{\alpha,\epsilon}}$ in $Ind_{\tilde{Q}(\mathbb{A})}^{\tilde{G}(\mathbb{A})}(\theta_{\alpha,\epsilon} \otimes \delta_Q^{\frac{1}{2}+\frac{2s_2+3}{10}})$, which can also be realized as the image of the normalized intertwining operator from $Ind_{\tilde{Q}(\mathbb{A})}^{\tilde{G}(\mathbb{A})}(\theta_{\alpha,\epsilon} \otimes \delta_Q^{\frac{1}{2}+\frac{3}{10}})$ to $Ind_{\tilde{Q}(\mathbb{A})}^{\tilde{G}(\mathbb{A})}(\theta_{\alpha,\epsilon} \otimes \delta_Q^{\frac{1}{2}-\frac{3}{10}})$.

Proof. The automorphic theta representation θ_ϵ of $\tilde{G}(\mathbb{A})$ is equal to

$$res_{s_2=0}[res_{s_1=0}E(\tilde{g}, f_{s_1,s_2})] = res_{s_2=0}E(\tilde{g}, f_{s_2,\theta_{\alpha,\epsilon}}).$$

□

The following gives the structure of the constant term of the automorphic theta representation of $\theta_{\tilde{G}_2}$ along the maximal parabolic subgroup Q.

Theorem 2.1. *After restricted to the Levi subgroup $GL_\alpha(2)$, the constant term of the automorphic theta representation along the maximal parabolic subgroup Q, θ_ϵ^V, is equal to the exceptional automorphic representation*

$$\theta^1_{\alpha,\epsilon} := |\det| \otimes \theta_{\alpha,\epsilon}$$

of the cubic cover of the Levi subgroup $GL_\alpha(2,\mathbb{A})$.

Proof. By Proposition 2.1, the automorphic theta representation

$$\theta_\epsilon = res_{s_2=0} E(\tilde{g}, f_{s_2,\theta_{\alpha,\epsilon}})$$

for holomorphic sections $f_{s_2,\theta_{\alpha,\epsilon}}$ in $Ind_{\tilde{Q}(\mathbb{A})}^{\tilde{G}(\mathbb{A})}(\theta_{\alpha,\epsilon}\otimes\delta_Q^{\frac{1}{2}+\frac{2s_2+3}{10}})$.

To study the constant term θ_ϵ^V of θ_ϵ along the maximal parabolic subgroup Q, we collect the following data, which is needed to determine the structure of θ_ϵ^V as a representation of the Levi subgroup $GL_\alpha(2,\mathbb{A})$.

It is easy to see that

$$Ind_{\tilde{Q}(\mathbb{A})}^{\tilde{G}(\mathbb{A})}(\theta_{\alpha,\epsilon}\otimes\delta_Q^{\frac{1}{2}+\frac{2s_2+3}{10}}) \to Ind_{\tilde{B}^{(3)}(\mathbb{A})}^{\tilde{G}(\mathbb{A})}(\chi_{s_2}\cdot\delta_B^{\frac{1}{2}})$$

where $\chi_{s_2}(h(a,b)) := |a|^{s_2+\frac{4}{3}}\cdot|b|^{s_2+\frac{5}{3}}$. Then one has

$$\begin{array}{rclrcl} <\chi_{s_2},\alpha^\vee> &=& -\frac{1}{3}, & <\chi_{s_2},\beta^\vee> &=& s_2+2,\\ <\chi_{s_2},(\alpha+\beta)^\vee> &=& s_2+1, & <\chi_{s_2},(\alpha+2\beta)^\vee> &=& 2s_2+3,\\ <\chi_{s_2},(\alpha+3\beta)^\vee> &=& s_2+\frac{5}{3}, & <\chi_{s_2},(2\alpha+3\beta)^\vee> &=& s_2+\frac{4}{3}. \end{array} \quad (2.1)$$

The constant term of the Eisenstein series $E(\tilde{g}, f_{s_2,\theta_{\alpha,\epsilon}})$ along the maximal parabolic subgroup Q can be expressed as follows:

$$\begin{array}{rcl} E^V(\tilde{g}, f_{s_2,\theta_{\alpha,\epsilon}}) &=& f_{s_2,\theta_{\alpha,\epsilon}}(\tilde{g}) + M_{w_{\beta\alpha\beta\alpha\beta}}(s_2)(f_{s_2,\theta_{\alpha,\epsilon}})(\tilde{g})\\ && + \sum_{\gamma\in B_\alpha\backslash GL_\alpha(2)} M_{w_\beta}(s_2)(f_{s_2,\theta_{\alpha,\epsilon}})(\tilde{g})\\ && + \sum_{\gamma\in B_\alpha\backslash GL_\alpha(2)} M_{w_{\beta\alpha\beta}}(s_2)(f_{s_2,\theta_{\alpha,\epsilon}})(\tilde{g}). \end{array}$$

An easy calculation yields the c_w-functions:

$$\begin{aligned} c_{w_{\beta\alpha\beta\alpha\beta}}(s_2) &= \frac{\zeta(s_2+1)\zeta(2s_2+3)\zeta(3s_2+4)}{\zeta(s_2+3)\zeta(2s_2+4)\zeta(3s_2+6)} \\ c_{w_{\beta\alpha\beta}}(s_2) &= \frac{\zeta(s_2+2)\zeta(2s_2+3)\zeta(3s_2+5)}{\zeta(s_2+3)\zeta(2s_2+4)\zeta(3s_2+6)} \\ c_{w_{\beta}}(s_2) &= \frac{\zeta(s_2+2)}{\zeta(s_2+3)} \\ c_{w_{\alpha\beta}}(s_2) &= \frac{\zeta(s_2+2)\zeta(3s_2+5)}{\zeta(s_2+3)\zeta(3s_2+6)} \\ c_{w_{\alpha\beta\alpha\beta}}(s_2) &= \frac{\zeta(s_2+2)\zeta(2s_2+3)\zeta(3s_2+4)}{\zeta(s_2+3)\zeta(2s_2+4)\zeta(3s_2+6)}. \end{aligned}$$

It follows that $f_{s_2,\theta_{\alpha,\epsilon}}(\tilde{g})$, $M_{w_\beta}(s_2)(f_{s_2,\theta_{\alpha,\epsilon}})(\tilde{g})$, and $M_{w_{\beta\alpha\beta}}(s_2)(f_{s_2,\theta_{\alpha,\epsilon}})(\tilde{g})$ are holomorphic at $s_2 = 0$ and $M_{w_{\beta\alpha\beta\alpha\beta}}(s_2)(f_{s_2,\theta_{\alpha,\epsilon}})(\tilde{g})$ has a simple pole at $s_2 = 0$. By computing the constant term along the root α, we know that both

$$\sum_{\gamma\in B_\alpha\backslash GL_\alpha(2)} M_{w_\beta}(s_2)(f_{s_2,\theta_{\alpha,\epsilon}})(\tilde{g})$$

and

$$\sum_{\gamma\in B_\alpha\backslash GL_\alpha(2)} M_{w_{\beta\alpha\beta}}(s_2)(f_{s_2,\theta_{\alpha,\epsilon}})(\tilde{g})$$

are holomorphic at $s_2 = 0$. Hence the constant term θ_ϵ^V is included, as representation of $\tilde{GL}_\alpha(2,\mathbb{A})$, in the exceptional automorphic representation $\theta^1_{\alpha,\epsilon}$. They are actually equal to each other because $\theta_{\alpha,\epsilon}$ is irreducible and $[\theta_\epsilon^V]|_{\tilde{GL}_\alpha(2,\mathbb{A})}$ is nonzero. □

2.3. **Eisenstein Series and the Symmetric Cube L-functions.** In this subsection, we recall some basic facts about the Eisenstein series of G_2 associated to the cuspidal data (L,π).

More precisely, let $X(L)$ be the group of F-rational characters of L. Then one has

$$\mathfrak{a}^* = X(L)\otimes\mathbb{R} = \mathbb{R}(\alpha+2\beta).$$

As in [Sh1] and [KS], one takes $\tilde{\alpha} = \alpha+2\beta$ and identifies $\mathbb{C}$ with $\mathfrak{a}^*_{\mathbb{C}}$ by sending s to $s\cdot\tilde{\alpha}$. Let π be an irreducible cuspidal automorphic representation of $L(\mathbb{A})$ with the trivial central character, realizing in the space of square integrable cuspidal automorphic functions, $L^2(Z_L(\mathbb{A})L(F)\backslash L(\mathbb{A}))$. For a K-finite automorphic function ϕ in π, one follows [Sh1] to extend ϕ to a function $\tilde{\phi}$ on $G(\mathbb{A})$. Let H_Q be the Harish-Chandra map from L to $\mathfrak{a}$ ($\mathfrak{a} = Hom(X(L),\mathbb{R})$), which can be extended to a map from $G(\mathbb{A})$ by the Iwasawa decomposition. Then one defines

$$\Phi_s(g) := \tilde{\phi}(g)exp < s+\rho_Q; H_Q(g) >$$

for $g \in G(\mathbb{A})$. In this case, one notice that

$$exp < s+\rho_Q; H_Q(g) >= |\det l(g)|^{s+\frac{5}{2}}$$

where $g = vlk \in G(\mathbb{A}) = V(\mathbb{A})L(\mathbb{A})K$. The Eisenstein series associated to the cuspidal data (L,π) is defined by

$$E(g; s, \phi) := \sum_{\gamma \in Q\backslash G} \Phi_s(\gamma g). \tag{2.2}$$

The basic analytic properties (meromorphic continuation, $\cdots$) of this Eisenstein series is known from general theory of Eisenstein series, [La] and [MW]. The following important result is known from the Langlands-Shahidi theory, a recent work of Kim-Shahidi [KS] and a work of Ikeda [I].

Theorem 2.2. *Let $E(g; s, \phi)$ be the Eisenstein series associated to the cuspidal data (L,π) as defined in (2.2). Then $E(g; s, \phi)$ is analytic for $Re(s) \geq 0$ except for $s = \frac{1}{2}, 1$. $E(g; s, \phi)$ has a simple pole at $s = \frac{1}{2}$ if and only if the completed symmetric cube L-function $L(s, \pi, Sym^3)$ does not vanish at $s = \frac{1}{2}$; and $E(g; s, \phi)$ has a simple pole at $s = 1$ if and only if π is monomial.*

3. Co-Period over G_2

Let $E_{\frac{1}{2}}(\cdot, \phi)$ be the residue of the Eisenstein series $E(g; s, \phi)$ at $s = \frac{1}{2}$. For $\varphi_\epsilon \in \theta_{\tilde{G},\epsilon}$ and $\varphi_{\epsilon^2} \in \theta_{\tilde{G},\epsilon^2}$, we shall study an integral of following type

$$\mathcal{CP}(E_{\frac{1}{2}}(\cdot, \phi), \varphi_\epsilon, \varphi_{\epsilon^2}) := \int_{G(F)\backslash G(\mathbb{A})} E_{\frac{1}{2}}(g, \phi)\varphi_\epsilon(\tilde{g})\varphi_{\epsilon^2}(\tilde{g})dg. \tag{3.1}$$

The formulation of this type of integral makes sense by means of the truncation of the Eisenstein series, which we do in the next subsection.

3.1. Truncation of Eisenstein Series.

We shall use Arthur's truncation method to obtain a formula for the co-period $\mathcal{CP}(E_{\frac{1}{2}}(\cdot, \phi), \varphi_\epsilon, \varphi_{\epsilon^2})$. As we mentioned in the Introduction, we shall consider the residue $E_{s_0}(g, \phi)$ at $s = s_0 > 0$ of the Eisenstein series $E(g; s, \phi)$ without assuming $s_0 = \frac{1}{2}$. Then our method shows that the residue has nonzero co-period only if $s_0 = \frac{1}{2}$. In this way, we obtain the existence of the residue at $s = \frac{1}{2}$ without using the theory of L-functions.

We identity $\mathfrak{a}_Q$ with $\mathbb{R}$ and then a regular $T \in \mathfrak{a}_Q$ will correspond to a real number $c \in \mathbb{R}_{>1}$, the real numbers greater than one. We set $H(g) := \exp < 1; H_Q(g) >$. Then we have

$$H(g) = |\det l(g)| \tag{3.2}$$

for $g = vlk \in G(\mathbb{A})$. Let τ_c $(c \in \mathbb{R}_{>1})$ be the characteristic function over $\mathbb{R}_{>0}$ of the subset $\mathbb{R}_{>c}$.

Following [Ar], the truncation of the Eisenstein series is defined as follows:

$$\Lambda_c E(g;s,\phi) = E(g;s,\phi) - \sum_{\gamma\in Q\backslash G} E_Q(\gamma g;s,\phi)\tau_c(H(\gamma g)) \tag{3.3}$$

where $E_Q(g;s,\phi)$ is the constant term of the Eisenstein series $E(g;s,\phi)$ along the maximal parabolic subgroup Q, which can be written as

$$E_Q(g;s,\phi) = \int_{N(F)\backslash N(\mathbb{A})} E(ng;s,\phi)dn = \Phi_s(g) + M(s)(\Phi_s)(g)$$

where $M(s)$ is the relevant intertwining operator. It follows that the Eisenstein series $E(g;s,\phi)$ has a simple pole at $s = s_0$ if and only if the intertwining operator $M(s)$ has a simple pole at $s = s_0$. Since the Eisenstein series is concentrated on the maximal parabolic subgroup Q, the constant term has only two terms left. We remark that the summation in (3.3) has only finitely many terms and converges. Then we can rewrite the truncated Eisenstein series as follows

$$\begin{aligned}\Lambda_c E(g;s,\phi) &= \sum_{\gamma\in Q\backslash G} \Phi_s(\gamma g)(1-\tau_c(H(\gamma g))) - \sum_{\gamma\in Q\backslash G} M(s)(\Phi_s)(\gamma g)\tau_c(H(\gamma g)) \\ &= \mathcal{E}_1(g) - \mathcal{E}_2(g)\end{aligned}$$

where we denote

$$\begin{aligned}\mathcal{E}_1(g) &:= \sum_{\gamma\in Q\backslash G} \Phi_s(\gamma g)(1-\tau_c(H(\gamma g))) \\ \mathcal{E}_2(g) &:= \sum_{\gamma\in Q\backslash G} M(s)(\Phi_s)(\gamma g)\tau_c(H(\gamma g)).\end{aligned}$$

Hence the truncation of the residue is

$$\begin{aligned}\Lambda_c E_{s_0}(g,\phi) &= E_{s_0}(g,\phi) - \sum_{\gamma\in Q\backslash G} res_{s=s_0}[M(s)(\Phi_s)(\gamma g)]\tau_c(H(\gamma g)) \\ &:= E_{s_0}(g,\phi) - \mathcal{E}_3(g).\end{aligned}$$

We have the following formula for the co-period:

$$\begin{aligned}\mathcal{CP}(E_{s_0}(\cdot,\phi),\varphi_\epsilon,\varphi_{\epsilon^2}) &= \mathcal{CP}(\mathcal{E}_3,\varphi_\epsilon,\varphi_{\epsilon^2}) + \mathcal{CP}(\Lambda_c E_{s_0}(\cdot,\phi),\varphi_\epsilon,\varphi_{\epsilon^2}) \\ &= \mathcal{CP}(\mathcal{E}_3,\varphi_\epsilon,\varphi_{\epsilon^2}) + res_{s=s_0}[\mathcal{CP}(\mathcal{E}_1,\varphi_\epsilon,\varphi_{\epsilon^2}) - \mathcal{CP}(\mathcal{E}_2,\varphi_\epsilon,\varphi_{\epsilon^2})].\end{aligned}$$

Since both $\Lambda_c E(g;s,\phi)$ and $\Lambda_c E_{s_0}(g,\phi)$ rapidly decay in the usual sense, the co-periods $\mathcal{CP}(\Lambda_c E(g;s,\phi),\varphi_\epsilon,\varphi_{\epsilon^2})$ and $\mathcal{CP}(\Lambda_c E_{s_0}(g,\phi),\varphi_\epsilon,\varphi_{\epsilon^2})$ absolutely converge. Assuming that both $\mathcal{CP}(\mathcal{E}_1,\varphi_\epsilon,\varphi_{\epsilon^2})$ and $\mathcal{CP}(\mathcal{E}_2,\varphi_\epsilon,\varphi_{\epsilon^2})$ absolutely converge for $Re(s)$ large and have meromorphic continuation to the whole complex plane, one can easily see that

$$res_{s=s_0}\mathcal{CP}(\mathcal{E}_2,\varphi_\epsilon,\varphi_{\epsilon^2}) = \mathcal{CP}(\mathcal{E}_3,\varphi_\epsilon,\varphi_{\epsilon^2}),$$

since the summation in both $\mathcal{E}_2$ and $\mathcal{E}_3$ is finite. In other words, by the next Proposition, the formula above for the co-period $\mathcal{CP}(E_{s_0}(\cdot,\phi),\varphi_\epsilon,\varphi_{\epsilon^2})$ makes sense and finally we have

$$\mathcal{CP}(E_{s_0}(\cdot,\phi),\varphi_\epsilon,\varphi_{\epsilon^2}) = res_{s=s_0}[\mathcal{CP}(\mathcal{E}_1,\varphi_\epsilon,\varphi_{\epsilon^2})]. \tag{3.4}$$

Proposition 3.1. *The following co-periods*

$$\mathcal{CP}(\mathcal{E}_i,\varphi_\epsilon,\varphi_{\epsilon^2})$$

for $i=1,2$, absolutely converge for $Re(s)$ large and have meromorphic continuation to the whole complex plane.

Proof. We consider $\mathcal{CP}(\mathcal{E}_1,\varphi_\epsilon,\varphi_{\epsilon^2})$. By definition, one has

$$\begin{aligned}\mathcal{CP}(\mathcal{E}_1(g),\varphi_\epsilon,\varphi_{\epsilon^2}) &= \int_{G(F)\backslash G(\mathbb{A})}\mathcal{E}_1(g)\varphi_\epsilon(\tilde{g})\varphi_{\epsilon^2}(\tilde{g})dg\\ &= \int_{G(F)\backslash G(\mathbb{A})}\sum_{\gamma\in Q\backslash G}\Phi_s(\gamma g)(1-\tau_c(H(\gamma g)))\varphi_\epsilon(\tilde{g})\varphi_{\epsilon^2}(\tilde{g})dg\\ &= \int_{Q(F)\backslash G(\mathbb{A})}\Phi_s(g)(1-\tau_c(H(g)))\varphi_\epsilon(\tilde{g})\varphi_{\epsilon^2}(\tilde{g})dg.\end{aligned}$$

By Iwasawa decomposition $G(\mathbb{A})=V(\mathbb{A})L(\mathbb{A})K$, we get

$$\begin{aligned}\mathcal{CP}(\mathcal{E}_1(g),\varphi_\epsilon,\varphi_{\epsilon^2}) &= \int_K\int_{L(F)\backslash L(\mathbb{A})}\Phi_s(hk)(1-\tau_c(H(h)))\delta_Q(h)^{-1}\\ &\qquad \int_{V(F)\backslash V(\mathbb{A})}\varphi_\epsilon(\mathbf{v}\tilde{h}\mathbf{k})\varphi_{\epsilon^2}(\mathbf{v}\tilde{h}\mathbf{k})d\mathbf{v}dhdk.\end{aligned}$$

Since the inner integral

$$\int_{V(F)\backslash V(\mathbb{A})}\varphi_\epsilon(\mathbf{v}\tilde{h}\mathbf{k})\varphi_{\epsilon^2}(\mathbf{v}\tilde{h}\mathbf{k})d\mathbf{v}$$

gives rise to an automorphic function over $L(\mathbb{A})$ and $\Phi_s(hk)$ is cuspidal in h for a fixed k, the whole integral absolutely converges for $Re(s)$ large.

The same argument applies to the other case.

The meromorphic continuation of both co-periods will follow from the explicit calculation in the next subsection. □

3.2. **Comparison of Co-Periods over two groups.** We shall compute here the co-periods explicitly and relate them to similar co-periods over the Levi subgroup $L=GL_\alpha(2)$.

We first consider the co-period $\mathcal{CP}(\mathcal{E}_1,\varphi_\epsilon,\varphi_{\epsilon^2})$. To simplify the notations, we denote by $x_{ij}=x_{i\alpha+j\beta}$ and $\mathbf{x}_{ij}=\mathbf{x}_{i\alpha+j\beta}$.

One notice that the two dimensional vector space generated by x_{13} and x_{23} (resp. by $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$) is stable under the adjoint action of the Levi subgroup $GL_{10}(2)=GL_\alpha(2)$

and decomposes into two orbits, which have representatives 0 and x_{13} (resp. $\mathbf{x}_{13}$), respectively. Then any φ_ϵ in θ_ϵ has the following Fourier expansion:

$$\begin{aligned}\varphi_\epsilon(\tilde{g}) &= \int_{[F\backslash\mathbb{A}]^2} \varphi_\epsilon(\mathbf{x}_{13}\mathbf{x}_{23}\tilde{g})d\mathbf{x}_{13}d\mathbf{x}_{23} \\ &+ \sum_{\delta\in B_{10}^\circ\backslash GL_{10}(2)} \int_{[F\backslash\mathbb{A}]^2} \varphi_\epsilon(\mathbf{x}_{13}\mathbf{x}_{23}\delta\tilde{g})\psi(\mathbf{x}_{13})d\mathbf{x}_{13}d\mathbf{x}_{23},\end{aligned} \tag{3.5}$$

where $B_{10}^\circ = \{\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix}\}$ in $GL_{10}(2)$.

By this formula, one has

$$\begin{aligned}\mathcal{CP}(\mathcal{E}_1, \varphi_\epsilon, \varphi_{\epsilon^2}) &= \int_{Q(F)\backslash G(\mathbb{A})} \Phi_s(g)(1-\tau_c(H(g)))\varphi_\epsilon(\tilde{g})\varphi_{\epsilon^2}(\tilde{g})dg \\ &= I_1 + I_2,\end{aligned} \tag{3.6}$$

where

$$I_1 = \int_{Q(F)\backslash G(\mathbb{A})} \Phi_s(g)(1-\tau_c(H(g)))\varphi_{\epsilon^2}(\tilde{g}) \int_{[F\backslash\mathbb{A}]^2} \varphi_\epsilon(\mathbf{x}_{13}\mathbf{x}_{23}\tilde{g})d\mathbf{x}_{13}d\mathbf{x}_{23}dg$$

and

$$I_2 = \int_{Q(F)\backslash G(\mathbb{A})} \Phi_s(g)(1-\tau_c(H(g)))\varphi_{\epsilon^2}(\tilde{g}) \sum_{\delta\in B_{10}^\circ\backslash GL_{10}(2)} \int_{[F\backslash\mathbb{A}]^2} \varphi_\epsilon(\mathbf{x}_{13}\mathbf{x}_{23}\delta\tilde{g})\psi(\mathbf{x}_{13})d\mathbf{x}_{13}d\mathbf{x}_{23}dg.$$

It will be proven in Proposition 3.2 that integral I_2 vanishes for all the choices of Φ_s, φ_ϵ and φ_{ϵ^2}.

To further study integral I_1, one observes that under the adjoint action of $GL_{10}(2)$, the one-dimensional vector space generated by the x_{12} is stable and has two orbits 0 and x_{12}. Hence one has

$$\begin{aligned}\varphi_\epsilon(\tilde{g}) &= \int_{F\backslash\mathbb{A}} \varphi_\epsilon(\mathbf{x}_{12}\tilde{g})d\mathbf{x}_{12} \\ &+ \sum_{\delta\in SL_{10}(2)\backslash GL_{10}(2)} \int_{F\backslash\mathbb{A}} \varphi_\epsilon(\mathbf{x}_{12}\delta\tilde{g})\psi(\mathbf{x}_{12})d\mathbf{x}_{12}.\end{aligned} \tag{3.7}$$

Applying this formula to integral I_1, one has

$$I_1 = I_{11} + I_{12}, \tag{3.8}$$

where

$$I_{11} = \int_{Q(F)\backslash G(\mathbb{A})} \Phi_s(g)(1-\tau_c(H(g)))\varphi_{\epsilon^2}(\tilde{g}) \int_{[F\backslash\mathbb{A}]^3} \varphi_\epsilon(\mathbf{x}_{12}\mathbf{x}_{13}\mathbf{x}_{23}\tilde{g})d\mathbf{x}_{12}d\mathbf{x}_{13}d\mathbf{x}_{23}dg$$

and

$$I_{12} = \int_{Q^\circ(F)\backslash G(\mathbb{A})} \Phi_s(g)(1-\tau_c(H(g)))\varphi_{\epsilon^2}(\tilde{g}) \int_{[F\backslash\mathbb{A}]^3} \varphi_\epsilon(\mathbf{x}_{12}\mathbf{x}_{13}\mathbf{x}_{23}\tilde{g})\psi(\mathbf{x}_{12})d\mathbf{x}_{12}d\mathbf{x}_{13}d\mathbf{x}_{23}dg,$$

where $Q^\circ = SL_{10}(2)V$.

We shall show in Proposition 3.2 that integral I_{12} vanishes for all the choices of Φ_s, φ_ϵ and φ_{ϵ^2}. It reduces to study integral I_{11}. Similarly, one has

$$\begin{aligned}\varphi_\epsilon(\tilde{g}) &= \int_{[F\backslash\mathbb{A}]^2} \varphi_\epsilon(\mathbf{x}_{01}\mathbf{x}_{11}\tilde{g})d\mathbf{x}_{01}d\mathbf{x}_{11} \\ &\quad + \sum_{\delta\in B_{10}^\circ\backslash GL_{10}(2)} \int_{[F\backslash\mathbb{A}]^2} \varphi_\epsilon(\mathbf{x}_{01}\mathbf{x}_{11}\delta\tilde{g})\psi(\mathbf{x}_{01})d\mathbf{x}_{01}d\mathbf{x}_{11},\end{aligned} \tag{3.9}$$

where $B_{10}^\circ = \{\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix}\}$ in $GL_{10}(2)$. Applying this formula to integral I_{11}, we get

$$I_{11} = I_{111} + I_{112}, \tag{3.10}$$

where

$$I_{111} = \int_{Q(F)\backslash G(\mathbb{A})} \Phi_s(g)(1-\tau_c(H(g)))\varphi_{\epsilon^2}(\tilde{g}) \int_{\mathbf{V}(F)\backslash\mathbf{V}(\mathbb{A})} \varphi_\epsilon(\mathbf{v}\tilde{g})d\mathbf{v}dg$$

and

$$\begin{aligned}I_{112} &= \int_{Q(F)\backslash G(\mathbb{A})} \Phi_s(g)(1-\tau_c(H(g)))\varphi_{\epsilon^2}(\tilde{g}) \int_{[F\backslash\mathbb{A}]^3} \\ &\quad \sum_{\delta\in B_{10}^\circ\backslash GL_{10}(2)} \int_{[F\backslash\mathbb{A}]^2} \varphi_\epsilon(\mathbf{x}_{01}\mathbf{x}_{11}\delta\mathbf{x}_{12}\mathbf{x}_{13}\mathbf{x}_{23}\tilde{g})\psi(\mathbf{x}_{01})d\mathbf{x}_{01}d\mathbf{x}_{11}d\mathbf{x}_{12}d\mathbf{x}_{13}d\mathbf{x}_{23}dg.\end{aligned}$$

It will follows from Proposition 3.2 that integral I_{112} vanishes for all the choices of Φ_s, φ_ϵ and φ_{ϵ^2}.

Hence we obtain by assuming Proposition 3.2 in the next subsection that

$$\begin{aligned}\mathcal{CP}(\mathcal{E}_1,\varphi_\epsilon,\varphi_{\epsilon^2}) &= I_{111} \\ &= \int_{Q(F)\backslash G(\mathbb{A})} \Phi_s(g)(1-\tau_c(H(g)))\varphi_{\epsilon^2}(\tilde{g})\varphi_\epsilon^V(\tilde{g})dg,\end{aligned}$$

where

$$\varphi_\epsilon^V(\tilde{g}) = \int_{\mathbf{V}(F)\backslash\mathbf{V}(\mathbb{A})} \varphi_\epsilon(\mathbf{v}\tilde{g})d\mathbf{v}.$$

It is easy to see that

$$\mathcal{CP}(\mathcal{E}_1,\varphi_\epsilon,\varphi_{\epsilon^2}) = \int_{GL_{10}(2,F)V(\mathbb{A})\backslash G(\mathbb{A})} \Phi_s(g)(1-\tau_c(H(g)))\varphi_{\epsilon^2}^V(\tilde{g})\varphi_\epsilon^V(\tilde{g})dg. \tag{3.11}$$

By the Iwasawa decomposition $G(\mathbb{A}) = V(\mathbb{A})L(\mathbb{A})K$, we obtain

$$\mathcal{CP}(\mathcal{E}_1,\varphi_\epsilon,\varphi_{\epsilon^2}) = \int_K\int_{GL_\alpha(2,F)\backslash GL_\alpha(\mathbb{A})} \tilde{\phi}(hk)(1-\tau_c(H(h)))\varphi_{\epsilon^2}^V(\tilde{h}\mathbf{k})\varphi_\epsilon^V(\tilde{h}\mathbf{k})|\det_\alpha(h)|^{s-\frac{5}{2}}dhdk.$$

Using the decomposition

$$GL_\alpha(\mathbb{A}) = GL_\alpha(\mathbb{A})^1\cdot A_\alpha^+$$

where $GL_\alpha(\mathbb{A})^1$ is the kernel of $|\det_\alpha|_\mathbb{A}$ and A_α^+ is the connected component of the $\mathbb{R}$-points in the $\mathbb{C}$-points of the center of $GL_\alpha(2)$ at one archimedean place.

Then the inner integral can be computed as follows:

$$\begin{aligned}
&\int_{GL_\alpha(2,F)\backslash GL_\alpha(\mathbb{A})} \tilde{\phi}(hk)(1-\tau_c(H(h)))\varphi^V_{\epsilon^2}(\tilde{h}\mathbf{k})\varphi^V_\epsilon(\tilde{h}\mathbf{k})|\det{}_\alpha(h)|^{s-\frac{5}{2}}dh \\
=\ & \int_{GL_\alpha(2,F)\backslash GL_\alpha(\mathbb{A})^1} \tilde{\phi}(hk)\varphi^V_{\epsilon^2}(\tilde{h}\mathbf{k})\varphi^V_\epsilon(\tilde{h}\mathbf{k})dh \int_{A^+_\alpha}(1-\tau_c(|\det{}_\alpha(a)|))|\det{}_\alpha(a)|^{s-\frac{5}{2}+2}da \\
=\ & \frac{c^{2s-1}}{2s-1}\cdot\int_{GL_\alpha(2,F)\backslash GL_\alpha(\mathbb{A})^1} \tilde{\phi}(hk)\varphi^V_{\epsilon^2}(\tilde{h}\mathbf{k})\varphi^V_\epsilon(\tilde{h}\mathbf{k})dh.
\end{aligned}$$

One notice that the elements in A^+_α are in $GL_\alpha(2,\mathbb{C})$ for one complex archimedean place and the cubic cover splits, so that for $a\in A^+_\alpha$, the element $\mathbf{a}=(a,\xi)$ comes out from $\theta_{G_2(\mathbb{C})\times\mu_3,\epsilon}$ as $|\det_\alpha(a)|\cdot\epsilon$ and from $\theta_{G_2(\mathbb{C})\times\mu_3,\epsilon^2}$ as $|\det_\alpha(a)|\cdot\epsilon^2$. We remark that this final formula shows that the co-period $\mathcal{CP}(\mathcal{E}_1,\varphi_\epsilon,\varphi_{\epsilon^2})$ has meromorphic continuation to the whole complex plane and has the only simple pole at $s=\frac{1}{2}$. Therefore, we obtain that if $s_0\neq\frac{1}{2}$, then the co-period $\mathcal{CP}(\mathcal{E}_1,\varphi_\epsilon,\varphi_{\epsilon^2})$ is holomorphic at $s=s_0$, and if $s_0=\frac{1}{2}$, then the co-period $\mathcal{CP}(\mathcal{E}_1,\varphi_\epsilon,\varphi_{\epsilon^2})$ has a simple pole at $s=\frac{1}{2}$ and its residue is

$$res_{s=\frac{1}{2}}\mathcal{CP}(\mathcal{E}_1,\varphi_\epsilon,\varphi_{\epsilon^2})=\int_K\int_{GL_\alpha(2,F)\backslash GL_\alpha(\mathbb{A})^1}\tilde{\phi}(hk)\varphi^V_{\epsilon^2}(\tilde{h}\mathbf{k})\varphi^V_\epsilon(\tilde{h}\mathbf{k})dhdk. \tag{3.12}$$

The same calculation can be applied to the case

$$\mathcal{CP}(\mathcal{E}_2,\varphi_\epsilon,\varphi_{\epsilon^2}).$$

It turns out that

$$\mathcal{CP}(\mathcal{E}_2,\varphi_\epsilon,\varphi_{\epsilon^2})=\frac{c^{-2s-1}}{2s+1}\cdot\int_K\int_{GL_\alpha(2,F)\backslash GL_\alpha(\mathbb{A})^1}M(s)\tilde{\phi}(hk)\varphi^V_{\epsilon^2}(\tilde{h}\mathbf{k})\varphi^V_\epsilon(\tilde{h}\mathbf{k})dhdk.$$

This formula also shows that the co-period $\mathcal{CP}(\mathcal{E}_2,\varphi_\epsilon,\varphi_{\epsilon^2})$ has meromorphic continuation to the whole complex plane and when $Re(s)>0$, the possible poles come only from those of the intertwining operator $M(s)$.

We conclude from formula (3.4) and the above calculation that

Theorem 3.1. *The nonvanishing of the co-period $\mathcal{CP}(E_{s_0}(\cdot,\phi),\varphi_\epsilon,\varphi_{\epsilon^2})$ for some choice of data $\phi,\varphi_\epsilon,\varphi_{\epsilon^2}$ implies that $s_0=\frac{1}{2}$. In particular, the Eisenstein series $E(g;s,\phi)$ has a possible simple pole at $s=\frac{1}{2}$. The co-period of the residue at $s=\frac{1}{2}$ has the following formula*

$$\mathcal{CP}(E_{\frac{1}{2}}(\cdot,\phi),\varphi_\epsilon,\varphi_{\epsilon^2})=\int_K\int_{GL_\alpha(2,F)\backslash GL_\alpha(2,\mathbb{A})^1}\tilde{\phi}(hk)\varphi^V_{\epsilon^2}(\tilde{h}\mathbf{k})\varphi^V_\epsilon(\tilde{h}\mathbf{k})dhdk. \tag{3.13}$$

The following theorem states the comparison principle of the 'outer' period with the 'inner' period.

Theorem 3.2. *The nonvanishing of the co-period* $\mathcal{CP}(E_{\frac{1}{2}}(\cdot,\phi),\varphi_\epsilon,\varphi_{\epsilon^2})$ *for some choice of data* $\phi,\varphi_\epsilon,\varphi_{\epsilon^2}$ *is equivalent to the nonvanishing of the following integral*

$$\int_{GL_\alpha(2,F)\backslash GL_\alpha(\mathbb{A})^1}\tilde{\phi}(h)\varphi_\epsilon^V(\tilde{h})\varphi_{\epsilon^2}^V(\tilde{h})dh$$

for the corresponding data.

Proof. It is easy to see from (3.13) that if $\mathcal{CP}(E_{\frac{1}{2}}(\cdot,\phi),\varphi_\epsilon,\varphi_{\epsilon^2})$ does not vanish for some choice of data $\phi,\varphi_\epsilon,\varphi_{\epsilon^2}$, then

$$\int_{GL_\alpha(2,F)\backslash GL_\alpha(\mathbb{A})^1}\tilde{\phi}(h)\varphi_\epsilon^V(\tilde{h})\varphi_{\epsilon^2}^V(\tilde{h})dh$$

does not vanish for the corresponding data.

We are going to prove the opposite as follows. Define

$$\mathcal{P}(\tilde{\phi}\otimes\varphi_\epsilon^V\otimes\varphi_{\epsilon^2}^V):=\int_{GL_\alpha(2,F)\backslash GL_\alpha(\mathbb{A})^1}\tilde{\phi}(h)\varphi_\epsilon^V(\tilde{h})\varphi_{\epsilon^2}^V(\tilde{h})dh.$$

It is easy to see that $\mathcal{P}$ is a continuous functional with respect to the topology of smooth vectors in the space of $\pi\otimes\theta_\epsilon^V\otimes\theta_{\epsilon^2}^V$ since π is cuspidal. Then define

$$I(\Phi):=\int_K\mathcal{P}(\Phi(k))dk$$

for smooth functions Φ on K with values in the space of $\pi\otimes\theta_\epsilon^V\otimes\theta_{\epsilon^2}^V$ satisfying

$$\Phi(pk)=(\pi\otimes\theta_\epsilon^V\otimes\theta_{\epsilon^2}^V)(p)\Phi(k)$$

for $p\in P(\mathbb{A})\cap K$. It follows that for a given $\tilde{\phi}\otimes\varphi_\epsilon^V\otimes\varphi_{\epsilon^2}^V$ in $\pi\otimes\theta_\epsilon^V\otimes\theta_{\epsilon^2}^V$, there is a smooth function Φ such that

$$\begin{aligned}I(\Phi)&=\int_K\mathcal{P}(\Phi(k))dk\\&=\int_K\int_{GL_\alpha(2,F)\backslash GL_\alpha(2,\mathbb{A})^1}\tilde{\phi}(hk)\varphi_{\epsilon^2}^V(\tilde{h}\mathbf{k})\varphi_\epsilon^V(\tilde{h}\mathbf{k})dhdk.\end{aligned}$$

It suffices to show that if the functional $\mathcal{P}$ is not identically zero, so is $I(\Phi)$.

Without loss of generality, we may assume that there is a factorizable function $\tilde{\phi}\otimes\varphi_\epsilon^V\otimes\varphi_{\epsilon^2}^V$ in $\pi\otimes\theta_\epsilon^V\otimes\theta_{\epsilon^2}^V$, such that

$$\mathcal{P}(\tilde{\phi}\otimes\varphi_\epsilon^V\otimes\varphi_{\epsilon^2}^V)\neq 0.$$

Of course, we can write

$$\tilde{\phi}\otimes\varphi_\epsilon^V\otimes\varphi_{\epsilon^2}^V=\Psi_S\otimes\Psi^S$$

where Ψ_S is a finite tensor product of the local components of $\tilde{\phi}\otimes\varphi_\epsilon^V\otimes\varphi_{\epsilon^2}^V$ over the infinite places and the finite places where one of the local components of $\tilde{\phi}$, φ_ϵ^V, $\varphi_{\epsilon^2}^V$ is ramified, and Ψ^S is the infinite tensor product of all unramified local components of $\tilde{\phi}\otimes\varphi_\epsilon^V\otimes\varphi_{\epsilon^2}^V$.

If we write, for the set S,

$$\pi \otimes \theta_\epsilon^V \otimes \theta_{\epsilon^2}^V = (\pi \otimes \theta_\epsilon^V \otimes \theta_{\epsilon^2}^V)_S \otimes (\pi \otimes \theta_\epsilon^V \otimes \theta_{\epsilon^2}^V)^S,$$

then there is a continuous functional $\mathcal{P}_S$, $\mathcal{P}^S$ of $(\pi \otimes \theta_\epsilon^V \otimes \theta_{\epsilon^2}^V)_S$, $(\pi \otimes \theta_\epsilon^V \otimes \theta_{\epsilon^2}^V)^S$, respectively, such that

$$\mathcal{P}(\tilde{\phi} \otimes \varphi_\epsilon^V \otimes \varphi_{\epsilon^2}^V) = \mathcal{P}_S(\Psi_S) \cdot \mathcal{P}^S(\Psi^S).$$

Since Ψ^S is unramified, we take

$$\Phi^S(k^S) := \Psi^S$$

for $k^S \in K^S$, where $K^S := \prod_{v \notin S} K_v$. Since S is finite, one may assume that

$$\mathcal{P}_S(\Psi_S) = \prod_{v \in S} \mathcal{P}_v(\Psi_v).$$

By the admissibility of the local component representations of $(\pi \otimes \theta_\epsilon^V \otimes \theta_{\epsilon^2}^V)_S$, it follows from the standard argument used in Proposition 2 of [JR] and in §5 in [Jng] that there is a function Φ_S such that

$$I_S(\Phi_S) = \int_{K_S} \mathcal{P}_S(\Phi_S(k_S))dk_S \neq 0$$

where $K_S := \prod_{v \in S} K_v$.

Finally, we take $\Phi := \Phi_S \otimes \Phi^S$ and have

$$I(\Phi) = I_S(\Phi_S) \cdot I^S(\Phi^S) \neq 0.$$

This proves the Theorem. □

We remark that the argument used in the proof of Theorem 3.1 is much simpler than that used in the proof of the corresponding theorems in [JR] and [Jng]. Combining with Theorem 2.1, we have the main result of this paper.

Theorem 3.3 (Main)**.** *Let $E(g, s, \phi)$ be the Eisenstein series of $G_2(\mathbb{A})$ defined in (2.1) and $\theta_{\tilde{G}_2(\mathbb{A}),\epsilon}$, $\theta_{\tilde{G}_2(\mathbb{A}),\epsilon^2}$ (which are denoted by θ_ϵ and θ_{ϵ^2}, respectively, in the previous sections) be the cubic automorphic theta representations of the cubic cover $\tilde{G}_2(\mathbb{A})$ of $G_2(\mathbb{A})$ defined in §2.2. Let $\theta_{\tilde{GL}_2(\mathbb{A}),\epsilon}$, $\theta_{\tilde{GL}_2(\mathbb{A}),\epsilon^2}$ (which are denoted by $\theta_{\alpha,\epsilon}^1$ and $\theta_{\alpha,\epsilon^2}^1$, respectively, in the previous sections) be the exceptional automorphic representations of the cubic cover $\tilde{GL}_\alpha(2, \mathbb{A})$ of $G_2(\mathbb{A})$ defined in §2.2.*

(1) *The nonvanishing of the co-period*

$$\mathcal{CP}_{G_2}(E_{\frac{1}{2}}(\cdot, \phi), \varphi_\epsilon, \varphi_{\epsilon^2}) = \int_{G(F)\backslash G(\mathbb{A})} E_{\frac{1}{2}}(g, \phi)\varphi_\epsilon(\tilde{g})\varphi_{\epsilon^2}(\tilde{g})dg$$

for some choice of data $\phi \in \pi$, $\varphi_\epsilon \in \theta_{\tilde{G}_2(\mathbb{A}),\epsilon}$, $\varphi_{\epsilon^2} \in \theta_{\tilde{G}_2(\mathbb{A}),\epsilon^2}$, is equivalent to the nonvanishing of the co-period

$$\mathcal{CP}_{GL_2}(\phi, \varphi'_\epsilon, \varphi'_{\epsilon^2}) = \int_{GL_2(F)\backslash GL_2(\mathbb{A})^1} \phi(h)\varphi'_\epsilon(\tilde{h})\varphi'_{\epsilon^2}(\tilde{h})dh$$

for some choice of data $\phi \in \pi$, $\varphi'_\epsilon \in \theta_{\tilde{GL_2}(\mathbb{A}),\epsilon}$, $\varphi'_{\epsilon^2} \in \theta_{\tilde{GL_2}(\mathbb{A}),\epsilon^2}$. The relation between those two pieces of data is given by Theorem 2.1.

(2) *If the co-period*

$$\mathcal{CP}_{GL_2}(\phi, \varphi'_\epsilon, \varphi'_{\epsilon^2}) = \int_{GL_2(F)\backslash GL_2(\mathbb{A})^1} \phi(h)\varphi'_\epsilon(\tilde{h})\varphi'_{\epsilon^2}(\tilde{h})dh$$

does not vanish for a choice of data $\phi \in \pi$, $\varphi'_\epsilon \in \theta_{\tilde{GL_2}(\mathbb{A}),\epsilon}$, $\varphi'_{\epsilon^2} \in \theta_{\tilde{GL_2}(\mathbb{A}),\epsilon^2}$, then the third symmetric power L-function $L(s, \pi, Sym^3)$ does not vanish at $s = \frac{1}{2}$.

Proof. Part (1) follows from Theorem 2.1 and Theorem 3.1 and part (2) follows from part (1) and Theorem 2.2. □

Remark 3.1. *It is easy to check that for any given local place v, the co-period $\mathcal{CP}_{GL_2}(\phi, \varphi'_\epsilon, \varphi'_{\epsilon^2})$ produces a nonzero $GL_2(F_v)$-invariant linear functional on the space of the tensor product $\pi_v \otimes \theta_{\alpha,\epsilon,v} \otimes \theta_{\alpha,\epsilon^2,v}$ of three local representations π_v, $\theta_{\alpha,\epsilon,v}$, and $\theta_{\alpha,\epsilon^2,v}$, where $\theta_{\alpha,\epsilon,v}$ and $\theta_{\alpha,\epsilon^2,v}$ are the exceptional representations of the cubic cover of $GL_\alpha(2, F_v)$ as defined in* [KP] *(§2 before Proposition 2.1).*

3.3. **Vanishing of Certain Integrals.** We shall prove the vanishing of integrals I_2, I_{12} and I_{112} occurring in explicit calculation of co-period $\mathcal{CP}_{G_2}(E_{\frac{1}{2}}(\cdot, \phi), \varphi_\epsilon, \varphi_{\epsilon^2})$ in the previous subsection.

Proposition 3.2. *Integral I_2, I_{12}, and I_{112} vanish for all the choices of Φ_s, φ_ϵ and φ_{ϵ^2}.*

Proof. We first consider integral

$$I_2 = \int_{B^\circ_{10}(F)V(F)\backslash G(\mathbb{A})} \Phi_s(g)(1 - \tau_c(H(g)))\varphi_{\epsilon^2}(\tilde{g}) \int_{[F\backslash\mathbb{A}]^2} \varphi_\epsilon(\mathbf{x}_{13}\mathbf{x}_{23}\delta\tilde{g})\psi(\mathbf{x}_{13})d\mathbf{x}_{13}d\mathbf{x}_{23}dg.$$

Let $Z := \mathbf{x}_{23}$ be the one-parameter additive subgroup of $\tilde{G}_2(\mathbb{A})$ generated by the root $2\alpha + 3\beta$. Then the inner integral can be written as

$$\int_{[F\backslash\mathbb{A}]^2} \varphi_\epsilon(\mathbf{x}_{13}\mathbf{x}_{23}\tilde{g})\psi(\mathbf{x}_{13})d\mathbf{x}_{13}d\mathbf{x}_{23} = \int_{F\backslash\mathbb{A}} \varphi^Z_\epsilon(\mathbf{x}_{13}\tilde{g})\psi(\mathbf{x}_{13})d\mathbf{x}_{13}.$$

Since Z is the center of the Heisenberg unipotent radical U of the maximal parabolic subgroup P, the Fourier expansion for φ^Z_ϵ ((3.14) in [GRS]) is

$$\varphi^Z_\epsilon(\tilde{g}) = \varphi^U_\epsilon(\tilde{g}) + \sum_{\gamma\in B^\circ_{01}\backslash GL_{01}(2)} \int_{U(F)\backslash U(\mathbb{A})} \varphi_\epsilon(\mathbf{u}\gamma\tilde{g})\psi_{-\alpha}(u)d\mathbf{u}, \quad (3.14)$$

where $B_{01}^{\circ}=\{\chi_{01}\cdot h(t,t)\}$. By this formula, one has

$$\begin{aligned}
&\int_{F\backslash\mathbb{A}}\varphi_{\epsilon}^{Z}(\mathbf{x}_{13}\tilde{g})\psi(\mathbf{x}_{13})d\mathbf{x}_{13}\\
=&\int_{F\backslash\mathbb{A}}\varphi_{\epsilon}^{U}(\mathbf{x}_{13}\tilde{g})\psi(\mathbf{x}_{13})d\mathbf{x}_{13}\\
&+\int_{F\backslash\mathbb{A}}\sum_{\gamma\in B_{01}^{\circ}\backslash GL_{01}(2)}\int_{U(F)\backslash U(\mathbb{A})}\varphi_{\epsilon}(\mathbf{u}\gamma\mathbf{x}_{13}\tilde{g})\psi_{-\alpha}(u)d\mathbf{u}\psi(\mathbf{x}_{13})d\mathbf{x}_{13}.
\end{aligned}$$

It is easy to see that

$$\int_{F\backslash\mathbb{A}}\varphi_{\epsilon}^{U}(\mathbf{x}_{13}\tilde{g})\psi(\mathbf{x}_{13})d\mathbf{x}_{13}=0.$$

To simplify the second term, one uses the Bruhat decomposition for $B_{01}^{\circ}\backslash GL_{01}(2)$, from which an element in $B_{01}^{\circ}\backslash GL_{01}(2)$ is equal to $h(a,1)$ or $h(a,1)w_{01}\chi_{01}(x)$. If $\gamma=h(a,1)$, one has

$$\int_{F\backslash\mathbb{A}}\int_{U(F)\backslash U(\mathbb{A})}\varphi_{\epsilon}(\mathbf{u}\gamma\mathbf{x}_{13}\tilde{g})\psi_{-\alpha}(u)d\mathbf{u}\psi(\mathbf{x}_{13})d\mathbf{x}_{13}=0.$$

If $\gamma=h(a,1)w_{01}\chi_{01}(x)$, then

$$\begin{aligned}
&\int_{F\backslash\mathbb{A}}\int_{U(F)\backslash U(\mathbb{A})}\varphi_{\epsilon}(\mathbf{u}h(a,1)w_{01}\chi_{01}(x)\mathbf{x}_{13}\tilde{g})\psi_{-\alpha}(u)d\mathbf{u}\psi(\mathbf{x}_{13})d\mathbf{x}_{13}\\
=&\int_{U(F)\backslash U(\mathbb{A})}\varphi_{\epsilon}(\mathbf{u}h(a,1)w_{01}\chi_{01}(x)\tilde{g})\psi_{-\alpha}(u)d\mathbf{u}\int_{F\backslash\mathbb{A}}\psi((1-a)\mathbf{x}_{13})d\mathbf{x}_{13}\\
=&\ 0,
\end{aligned}$$

if $a\neq 1$. This implies that

$$\int_{F\backslash\mathbb{A}}\varphi_{\epsilon}^{Z}(\mathbf{x}_{13}\tilde{g})\psi(\mathbf{x}_{13})d\mathbf{x}_{13}=\sum_{F}\varphi_{\epsilon}^{U,\psi_{-10}}(w_{01}\chi_{01}(x)\tilde{g}). \tag{3.15}$$

Hence integral I_2 can be written as follows:

$$\begin{aligned}
I_2=&\int_{B_{10}^{\circ}(F)V(F)\backslash G(\mathbb{A})}\Phi_s(g)(1-\tau_c(H(g)))\varphi_{\epsilon^2}(\tilde{g})\sum_{F}\varphi_{\epsilon}^{U,\psi_{-10}}(w_{01}\chi_{01}(x)\tilde{g})dg\\
=&\int_{T_{10}^{\circ}(F)U(F)\backslash G(\mathbb{A})}\Phi_s(g)(1-\tau_c(H(g)))\varphi_{\epsilon^2}(\tilde{g})\varphi_{\epsilon}^{U,\psi_{-10}}(w_{01}\tilde{g})dg\\
=&\int_{T_{10}^{\circ}(F)U(\mathbb{A})\backslash G(\mathbb{A})}\varphi_{\epsilon}^{U,\psi_{-10}}(w_{01}\tilde{g})\\
&\cdot\int_{U(F)\backslash U(\mathbb{A})}\Phi_s(x_{10}g)(1-\tau_c(H(g)))\varphi_{\epsilon^2}(\mathbf{u}\tilde{g})\psi(-\mathbf{x}_{13})d\mathbf{u}dg.
\end{aligned}$$

Now applying (3.14) to $\varphi^Z_{\epsilon^2}$, we have

$$\begin{aligned}
&\int_{U(F)\backslash U(\mathbb{A})} \Phi_s(x_{10}g)(1-\tau_c(H(g)))\varphi_{\epsilon^2}(\mathbf{u}\tilde{g})\psi(-\mathbf{x}_{13})d\mathbf{u} \\
= &\int_{[U/Z](F)\backslash[U/Z](\mathbb{A})} \Phi_s(x_{10}g)(1-\tau_c(H(g)))\varphi^Z_{\epsilon^2}(\bar{\mathbf{u}}\tilde{g})\psi(-\mathbf{x}_{13})d\bar{\mathbf{u}} \\
= &\int_{[U/Z](F)\backslash[U/Z](\mathbb{A})} \Phi_s(x_{10}g)(1-\tau_c(H(g)))\sum_{\gamma\in B^\circ_{01}\backslash GL_{01}(2)} \varphi^{U,\psi_{-\alpha}}_{\epsilon^2}(\gamma\bar{\mathbf{u}}\tilde{g})d\bar{\mathbf{u}}.
\end{aligned}$$

The same argument shows that if $\gamma = h(a,1)$ or $h(a,1)w_{01}\chi_{01}(x)$ with $a \neq 1$, the integral vanishes as in the case of φ_ϵ. We thus get

$$\begin{aligned}
I_2 = &\int_{T^\circ_{10}(F)U(\mathbb{A})\backslash G(\mathbb{A})} \varphi^{U,\psi_{-10}}_{\epsilon}(w_{01}\tilde{g}) \\
&\cdot\int_{[F\backslash\mathbb{A}]^3} \Phi_s(x_{10}g)(1-\tau_c(H(g)))\sum_{x\in F}\varphi^{U,\psi_{-\alpha}}_{\epsilon^2}(w_{01}\chi_{01}(x)\mathbf{x}_{10}\mathbf{x}_{11}\mathbf{x}_{12}\tilde{g})d\mathbf{x}_{10}d\mathbf{x}_{11}d\mathbf{x}_{12}dg.
\end{aligned}$$

Since $[\chi_{01}(t),\chi_{12}(s)] = \chi_{13}(3ts)$, one has

$$\varphi^{U,\psi_{-\alpha}}_{\epsilon^2}(w_{01}\chi_{01}(x)\mathbf{x}_{10}\mathbf{x}_{11}\mathbf{x}_{12}\tilde{g}) = \psi(-3x\mathbf{x}_{12})\cdot\varphi^{U,\psi_{-\alpha}}_{\epsilon^2}(w_{01}\chi_{01}(x)\mathbf{x}_{10}\mathbf{x}_{11}\tilde{g}),$$

and for $x \in F$,

$$\int_{[F\backslash\mathbb{A}]^3} \Phi_s(x_{10}g)(1-\tau_c(H(g)))\varphi^{U,\psi_{-\alpha}}_{\epsilon^2}(w_{01}\chi_{01}(x)\mathbf{x}_{10}\mathbf{x}_{11}\mathbf{x}_{12}\tilde{g})d\mathbf{x}_{10}d\mathbf{x}_{11}d\mathbf{x}_{12}$$

vanishes unless $x = 0$. This implies that

$$\begin{aligned}
I_2 = &\int_{T^\circ_{10}(F)U(\mathbb{A})\backslash G(\mathbb{A})} \varphi^{U,\psi_{-10}}_{\epsilon}(w_{01}\tilde{g}) \\
&\cdot\int_{[F\backslash\mathbb{A}]^3} \Phi_s(x_{10}g)(1-\tau_c(H(g)))\varphi^{U,\psi_{-\alpha}}_{\epsilon^2}(w_{01}\mathbf{x}_{10}\mathbf{x}_{11}\tilde{g})d\mathbf{x}_{10}d\mathbf{x}_{11}dg.
\end{aligned}$$

It is easy to see that

$$\varphi^{U,\psi_{-\alpha}}_{\epsilon^2}(w_{01}\mathbf{x}_{10}\mathbf{x}_{11}\tilde{g}) = \varphi^{U,\psi_{-\alpha}}_{\epsilon^2}(w_{01}\tilde{g}).$$

From this, we conclude that

$$\begin{aligned}
I_2 = &\int_{T^\circ_{10}(F)U(\mathbb{A})\backslash G(\mathbb{A})} \varphi^{U,\psi_{-10}}_{\epsilon}(w_{01}\tilde{g})\varphi^{U,\psi_{-\alpha}}_{\epsilon^2}(w_{01}\tilde{g})\int_{F\backslash\mathbb{A}} \Phi_s(x_{10}g)(1-\tau_c(H(g)))d\mathbf{x}_{10}dg \\
= &\ 0.
\end{aligned}$$

The vanishingness comes from the cuspidality of $\Phi_s(x_{10}g)$ in $\mathbf{x}_{10}$. This proves the Proposition for integral I_2.

We remark that the above proof is based on certain Fourier expansion of the automorphic theta functions and the cuspidality of the $\Phi_s(x_{10}g)$ on the Levi component $GL_{10}(2)$. The same argument applies to the cases of integrals I_{12} and I_{112} for the vanishing property.

□

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