Introduction

Practitioner Course: Interest Rate Models

John Dodson

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Symbols & Terms

- P risk-free discount factor
- r spot instantaneous risk-free rate
- f forward instantaneous risk-free rate
- R continuous zero-coupon bond yield
- τ conventional years
- L simple spot rate
- F simple forward rate

Note that

$$P(T) = e^{-R(T)T} = \frac{1}{1 + L(T)\,\tau(T)}$$

and

$$\frac{P(S)}{P(T)} = \frac{1}{1 + F(T,S)\,\tau(T,S)} \quad , \quad S > T$$

Symbols & Terms

Dynamics

$r(t)$ stochastic risk-free rate

$B(t)$ stochastic bank account balance

$D(t, T)$ stochastic discount factor

The bank account is defined by the initial condition $B(0) = 1$ and the SDE

$$dB(t) = B(t)r(t)\, dt$$

The discount factor is defined by

$$D(t, T) = \frac{B(t)}{B(T)} = e^{-\int_t^T r(T')\, dT'}$$

It represents the amount of money at time t that would grow on deposit to be worth exactly 1 at time $T > t$.

Fitting the Initial Term Structure

We will assume that the graph $T \mapsto P(T)$ is observable today, e.g.

- Bootstrap bond prices

$$\mathbf{CB}\left(c, \{T_i\}\right) = P\left(T_n\right) + c \sum_{i=1}^{n} \tau\left(T_{i-1}, T_i\right) P\left(T_i\right)$$

- Interpolate bond yields

$$y(T) = \frac{1 - P(T)}{\int_0^T P(T')\, d\tau(T')}$$

- Chain forward rates

$$P(T) = \frac{1}{1 + L(T_1)\,\tau(T_1)} \cdot \frac{1}{1 + F(T_1, T_2)\,\tau(T_1, T_2)} \cdots \frac{1}{1 + F(T_{n-1}, T_n)\,\tau(T_{n-1}, T)}$$

Term Structure Dynamics

Equivalent Martingale Measure

If rates are continuous in time and there are enough liquid securities, then interest rates can be completely hedged. In this case, the stochastic discount factor is integrable with respect to some measure, and we have

$$P(t, T) = \mathrm{E}_t\left[D(t, T)\right] = \mathrm{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(T')\, dT'} \middle| \mathcal{F}_t \right]$$

where the conditioning sigma-algebra $\mathcal{F}_t$ is in the filtration $\mathbb{F}$.

Black-Scholes PDE

Evaluating the expectation above may not be practical. The Feynman-Kac result suggests a connection between such expectations and linear second-order PDEs.

Term Structure Dynamics

In fact we can demonstrate that the simplest no-arbitrage setting[1] requires $P(T,T)=1$, $P_t(T,T)=r$, and

$$P_t(t,T)+P_r(t,T)b(t,r)+\frac{1}{2}P_{rr}(t,T)\sigma^2(t,r)=P(t,T)r$$

for all $t<T$, where $\sigma(r,t)$ is the diffusion rate of r and $b(r,t)$, which comes from the implicit function theorem, turns out to be the risk-neutral drift rate of r.

Short-rate Models

This PDE has several classes of analytical solutions, most notably in the affine setting[2]:

$$\begin{aligned}\sigma^2(r,t)&=\delta(t)+\gamma(t)\,r\\ b(r,t)&=\eta(t)+\lambda(t)\,r\end{aligned}$$

[1] all bonds being perfectly concordant

[2] including Vasicek, Hull-White, Ho-Lee, & Cox-Ingersoll-Ross

Term Structure Dynamics

Affine Short-rate Models

If all bond returns are all perfectly correlated, and σ^2 and b above are both affine in the instantaneous risk-free rate r, then

$$P(t,T)=\exp\left\{-\int_0^{\theta(t,T)}\left(r+\frac{\tilde{\eta}(u)u-\frac{1}{2}\tilde{\delta}(u)u^2}{1+\tilde{\lambda}(u)u-\frac{1}{2}\tilde{\gamma}(u)u^2}\right)du\right\}$$

where $\tilde{\eta}(u)=\eta\circ\{t:\theta(t,T)=u\}$ etc., and $\theta(t,T)$ solves the Riccati ODE for $t<T$,

$$\theta_t(t,T)=-1-\lambda(t)\theta(t,T)+\frac{1}{2}\gamma(t)\theta(t,T)^2$$

with the terminal condition $\theta(T,T)=0$.

N.B.: $\theta=-\frac{1}{P}\frac{\partial P}{\partial r}$ is the duration of a risk-free zero-coupon bond with respect to instantaneous risk-free rate.

Term Structure Dynamics

The integral in the solution for $P(t, T)$ is easy enough to evaluate numerically (or even analytically in some cases). So the only challenge that remains to valuation in this setting is the determination of $\theta(t, T)$.

Riccati equation

The Riccati equation comes up often in dynamic systems. Some special cases in this setting are:

- $\lambda = 0$ and $\gamma = 0 \implies \theta(t, T) = T - t$
- $\gamma = 0 \implies \theta(t, T) = \int_t^T e^{\int_t^{T'} \lambda(t')dt'} dT'$
- λ and γ constants $\implies$

$$\theta(t, T) = \frac{1}{\frac{1}{2}\left(\sqrt{\lambda^2 + 2\gamma} - \lambda\right) + \frac{\sqrt{\lambda^2+2\gamma}}{e^{\sqrt{\lambda^2+2\gamma}(T-t)}-1}}$$

where $\gamma < -\frac{1}{2}\lambda^2$ requires $T < t + \frac{\pi - \mathrm{sgn}(\lambda)\cos^{-1}\left(1+\frac{\lambda^2}{\gamma}\right)}{\sqrt{-\lambda^2-2\gamma}}$

Bonds

The net present value at t of a stream of certain cashflows with cumulative value given by $c(\cdot)$ is

$$\int_t^\infty P(t, T')\, dc(T')$$

For a prototypical bond, there is a fixed coupon rate c and maturity date T,

$$c(T') = c\tau(T' \wedge T) + H(T' - T)$$

so the net present value is

$$V(t, T, c) = c \int_t^T P(t, T')\, d\tau(T') + P(t, T)$$

Swaps

Interest rate swaps have two legs: a floating leg and a fixed leg

- The fixed leg has $c_{\text{fixed}}(T') = s(t,T)\,\tau(T' \wedge T)$ where $s(t,T)$ is the fair contractual swap rate at t
- The floating leg has $c_{\text{float}}(T') = -H(T'-t) + H(T'-T)$
  - Think of depositing \$1, collecting and distributing the periodic interest, then withdrawing it.

In order for the net present value at t to be zero, we must have

$$s(t,T) = \frac{1 - P(t,T)}{\int_t^T P(t,T')\, d\tau(T')}$$

N.B.: There is a connection between swaps and bonds. The swap rate is also the coupon rate (and the internal rate of return) on a bond whose value at t is par.

Swaps

A party to a swap is either a receiver or payer of the fixed leg cashflows. Since the owner of a bond receives fixed interest, we usually denote a receiver swap position with a positive notional. Conversely a payer swap position has a negative notional.

Valuation

Say we need to value a swap that was settled at some date $t = 0$ in the past. We can show that

$$V_{\text{swap}}(t, T, s(0, T)) = -\frac{1 - P(t, T)}{s(t, T)} (s(t, T) - s(0, T))$$

for continuous floating resets.

The dynamics of a swap value depends on the joint dynamics of the swap rate and the annuity factor $\frac{1-P(t,T)}{s(t,T)}$.

Bonds again

Duration

The annuity factor is the modern version of the classical concept of bond duration. Indeed, we can show that the value of a bond is

$$V(t, T, c) = 1 - A(t, T)\,(s(t, T) - c)$$

where

$$A(t, T) = \int_t^T P(t, T')\, d\tau(T')$$

so

$$-\left.\frac{1}{V}\frac{\partial V}{\partial s}\right|_{s=c} = A$$

The annuity factor is a proxy for the first-order exposure of a bond's value to changes in rates.

- notional $\times\, A(t, T) \times 10^{-4}$ is sometimes called the bond's present value of a basis point.

Caps & Floors

The valuation of caps and floors depend explicitly on the nature of interest rate volatility, and so are often the basis for calibrating volatility models.

Cap

A cap pays off whenever the floating rate is above the strike level.

$$V_{\text{cap}}(t, T, K) = \int_t^T \mathrm{E}_t \left[\left(r(T') - K \right)^+ D(t, T') \right] \, dT'$$

in the continuous reset version.

Floor

A floor pays out whenever the floating rate is below the strike level.

$$V_{\text{floor}}(t, T, K) = \int_t^T \mathrm{E}_t \left[\left(K - r(T') \right)^+ D(t, T') \right] \, dT'$$

Cap/Floor Parity

Since

$$P(t,T) = \mathrm{E}_t\left[D(t,T)\right] \quad \text{and}$$
$$P_T(t,T) = \mathrm{E}_t\left[-r(T)D(t,T)\right]$$

we can derive a parity arbitrage relationship similar to that for European puts and calls.

$$V_{\text{floor}}(t,T,K) - V_{\text{cap}}(t,T,K) = V_{\text{swap}}(t,T,K)$$

Moneyness

We can unambiguously define $K = s(t,T)$ as at-the-money, with higher strikes being in-the-money for floors and out-of-the-money for caps and vice versa for lower strikes.

Swaptions & Bond Options

The other class of vanilla interest rate derivatives are the swaptions.

- A swaption is the right, but not the obligation, to enter a swap.
- They are often denoted as right-to-receive or right-to-pay the fixed leg.

Pay/Receive Parity

In exact analogy to European put-call parity, if one is long the right to receive and short the right to pay fixed, then by arbitrage this is equivalent to one being long the underlying forward-start swap.

Embedded Bond Options

We can apply our learnings about swaptions to options on or embedded in (default-free) bonds.

- E.g., a callable bond is equivalent to a (non-callable) bond and a short receiver swaption.

Swaptions & Bond Options

For a (continuous reset) swaption with maturity T_0 and underlying swap maturity T_1, the value at $t < T_0 < T_1$ is

$$V_{\text{RTR}}(t, T_0, T_1, K) =$$
$$\mathrm{E}_t \left[D(t, T_0) \int_{T_0}^{T_1} \left(P(T_0, T')K + P_T(T_0, T') \right)^+ dT' \right]$$

and similarly for a right-to-pay swaption.

Decomposition

The difference between this and a cap/floor is that a swaption cannot be decomposed into a sum of options.

- A cap/floor is exercised continuously, while a swaption is exercised only once.

Jamshidian's Decomposition

The non-decomposition of swaptions can be an impediment to analysis.

Single-Factor Uncertainty

If we can write for $t < T_0 < T$

$$\mathsf{E}_t\left[D(T_0, T)\right] = \mathsf{E}_t\left[\Pi(T_0, T, r(T_0))\right]$$

where the expectation is over the terminal value $r(T_0)$ and not the whole path, and if $\Pi(T_0, T, r)$ is decreasing in r for $T > T_0$, then there is an r^* such that

$$K \int_{T_0}^{T_1} \Pi(T_0, T', r^*)\, d\tau(T') = 1 - \Pi(T_0, T_1, r^*)$$

N.B.: D, P, and Π are related. $P(t, T) = \Pi(t, T, r)$

Jamshidian's Decomposition

The holder of a right-to-pay swaption would exercise at T_0 iff $r > r^*$, because the swap at the strike rate would have a positive net present value.
Prior to exercise the swaption is worth

$$V_{\mathrm{RTR}}(t, T_0, T_1, K) = K \int_{T_0}^{T_1}$$

$$\mathrm{E}_t \left[D(t, T_0) \left(\Pi(T_0, T', r(T_0)) - \Pi(T_0, T', r^*) \right)^+ \right] d\tau(T')$$

$$+ \mathrm{E}_t \left[D(t, T_0) \left(\Pi(T_0, T_1, r(T_0)) - \Pi(T_0, T_1, r^*) \right)^+ \right]$$

and similarly for a right-to-pay fixed swaption.

- The advantage of this approach becomes apparent if we can determine analytical values for options on single cashflows.
- But the key assumption that all rates are perfectly correlated may not be reasonable.