Some poorly written notes that explain the size of conjugacy classes in the symmetric and alternating groups, works through the example of S_5 and A_5, and along the way proves that A_5 is simple.

The size of a conjugacy class in the symmetric group

Being able to find the number of conjugates of a permutation in S_n is an important skill that could prove useful someday in a life-threatening emergency situation. Let's find out how to do that, so that you too can be prepared.

The first important thing to know is that *conjugation preserves cycle type*, so that by specifying a cycle type, which is the same as specifying a partition of n, we specify a conjugacy class in S_n.

Let's look at an example:

$$(a_1a_2)(a_3a_4)(a_5a_6a_7)$$

We can arrange the numbers 1 to 7 in any of 7! ways and then just set them down into the cycles to get a permutation of this cycle type. But then we've overcounted. For instance, since the cycle $(a_5a_6a_7)$ is the same as $(a_6a_7a_5)$ and $(a_7a_5a_6)$, we've overcounted by a factor of 3. In general, each k-cycle will be overcounted by a factor of k. The two 2-cycles make us overcount by a factor of $2 \cdot 2$.

Since disjoint cycles commute, we can switch the 2-cycles above and get the same permutation: $(a_1a_2)(a_3a_4)$ is the same as $(a_3a_4)(a_1a_2)$. There's two ways to arrange two 2-cycles (!) so we've overcounted by a factor of 2.

Altogether, we have

$$\frac{7!}{3 \cdot 2 \cdot 2 \cdot 2} = 210$$

permutations of this cycle type in S_7.

In general, let's say the cycle type has c_1 1-cycles, c_2 2-cycles, and so on, up to c_k k-cycles, where $1c_1 + 2c_2 + \cdots + kc_k = n$. The above example has $c_1 = 0$, $c_2 = 2$, and $c_3 = 1$ with $k = 3$.

There are $n!$ ways to fill in the permutation and we need to correct our overcounting. Just copy the above reasoning:

- each of the c_j j-cycles can be rotated around j ways and be the same cycle, so divide by j^{c_j} for $j = 1, 2, \ldots, k$.
- there are c_j j-cycles which we can be permuted around in $c_j!$ ways, so we divide by $c_j!$ for $j = 1, 2, \ldots, k$.

Smushing all those things together into a big product, we see the number of permutations in the conjugacy class described by the c_i's is

$$n! \left(\prod_{i=1}^{k} i^{c_i} \prod_{i=1}^{k} c_i! \right)^{-1} .$$

That denominator is often called z_λ (for partitions of cycle type λ) when dealing with symmetric functions.

Conjugacy classes in the alternating group

That wasn't too bad. Now let's find the size of conjugacy classes in the alternating group A_n. This is slightly trickier than above. The basic idea is that a conjugacy class in S_n will either stay the same in S_n, or split into two, and so we need to figure out when a class splits.

To see why classes stay the same or split in two, we need to think about centralizers. The centralizer Z_g of some element g in a group G is the subgroup $\{x : xg = gx\}$, which can also be written $\{x : xgx^{-1} = g\}$. The cosets of the centralizer biject with the conjugates of g, which means

$$|Z_g||K_g| = |G|,$$

writing K_g for the conjugacy class of g. Back in our example, if $\pi \in A_n$, then if its conjugacy class stays the same in S_n, we need the following equations to hold. I've superscripted the Z's and K's with A or S to indicate which group it applies to:

$$|Z_\pi^A||K_\pi^A| = \frac{n!}{2}$$
$$|Z_\pi^S||K_\pi^S| = n!$$

The conjugacy classes are the same, so the centralizer must be twice as large. Conversely, we see that *a class of S_n splits in A_n if and only if the S_n-centralizer of one of its elements lies entirely in A_n*. (When that happens, the centralizer stays the same, so the class must be half as large.)

Write a permutation π as a product of disjoint cycles and observe that π is centralized by any power of a single cycle in its cycle decomposition. For example, if π contained the cycle (2537), then $(2537)^k\pi = \pi(2537)^k$ for any $k \geq 0$. This implies that π is centralized by any product of such cycles, and you can go through an argument exactly similar to the one in the previous section, and see that there are $z_{\lambda(\pi)}$ (where $\lambda(\pi)$ is the cycle type of π) such permutations, and since when we multiply the size of π's conjugacy class by that number we get $n!$, we've accounted for the entire centralizer.

If π contains an even-length (and hence odd sign) cycle, that cycle and its powers are in Z_π and so Z_π is not contained in A_n. Hence for any permutation whose cycle type contains an even-length cycle, its conjugacy class stays the same in A_n.

All right then, consider permutations whose cycle type consists only of odd parts. If such a permutation has two cycles of the same length, that permutation is centralized by a product of transpositions of the same length that swap the two cycles. For example, if $\pi = (135)(246)$, then $(14)(25)(36) \cdot \pi \cdot (14)(25)(36) = \pi$, and since we're multiplying by disjoint transpositions, we have $(14)(25)(36) \cdot \pi = \pi \cdot (14)(25)(36)$. Thus π, despite being an even-sign permutation, has a centralizer that does not lie entirely in A_n. This argument works for any permutation with two cycles of the same length, so the only things we have left to check are permutations whose cycle type consists of odd and distinct parts. These are the only permutations whose conjugacy class splits in two in A_n. Note that for such cycle types, z_λ is odd—this is what one would expect, since if the class didn't split, its centralizer would be twice as large in S_n and hence have even size!

To summarize:

> Conjugacy classes of permutations in S_n stay the same size in A_n for all cycle types *except* those whose cycle type consists of parts that are all odd and distinct.

Worked example

Let's go from S_5 to A_5 and see what happens. The table below lists the conjugacy classes, their centralizer sizes, and the conjugacy class sizes.

cycle type	z_λ in S_5	size of S_5 conj class
11111	120	1
2111	12	10
221	8	15
311	6	20
32	6	20
41	4	30
5	5	24

Only the 5-cycle's class splits in A_n. The 24 cycles in that class can be written in the form "(1 [permutation of 2,3,4,5])", and it is easy to see why the conjugacy class splits in A_5: we can get from any permutation in the S_5-class to any other by a permutation in S_4 which permutes $2, 3, 4, 5$ around; when we restrict to A_5, we only have the permutations in A_4 (with size 12, half of 24!) to move around the 2,3,4, and 5.

In A_5, the above table turns into

cycle type	z_λ in A_5	size of A_5 conj class
11111	60	1
221	4	15
311	3	20
5	5	12
5	5	12

(Having done this, we might as well pause and see that we can easily prove that A_5 is simple: any proper normal subgroup would consist of a union of conjugacy classes, and its order must divide 60. But by inspection, there is no way to take the identity element and a union of other conjugacy classes and get a divisor of 60. Hence A_5 is simple.)

For further reference, here's a snippet of a GAP session that shows what these permutations actually are:

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gap> a5 := AlternatingGroup(5);
Alt( [ 1 .. 5 ] )
gap> ConjugacyClasses(a5);
[ ()^G, (1,2)(3,4)^G, (1,2,3)^G, (1,2,3,4,5)^G, (1,2,3,5,4)^G ]
gap> Elements(ConjugacyClass(a5, (1,2)(3,4)));
[ (2,3)(4,5), (2,4)(3,5), (2,5)(3,4), (1,2)(4,5), (1,2)(3,4), (1,2)(3,5),
  (1,3)(4,5), (1,3)(2,4), (1,3)(2,5), (1,4)(3,5), (1,4)(2,3), (1,4)(2,5),
  (1,5)(3,4), (1,5)(2,3), (1,5)(2,4) ]
gap> Elements(ConjugacyClass(a5, (1,2,3)));
[ (3,4,5), (3,5,4), (2,3,4), (2,3,5), (2,4,3), (2,4,5), (2,5,3), (2,5,4), (1,2,3),
  (1,2,4), (1,2,5), (1,3,2), (1,3,4), (1,3,5), (1,4,2), (1,4,3), (1,4,5), (1,5,2),
  (1,5,3), (1,5,4) ]
gap> Elements(ConjugacyClass(a5, (1,2,3,4,5)));
[ (1,2,3,4,5), (1,2,4,5,3), (1,2,5,3,4), (1,3,5,4,2), (1,3,2,5,4), (1,3,4,2,5),
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  (1,4,3,5,2), (1,4,5,2,3), (1,4,2,3,5), (1,5,4,3,2), (1,5,2,4,3), (1,5,3,2,4) ]
gap> Elements(ConjugacyClass(a5, (1,2,3,5,4)));
[ (1,2,3,5,4), (1,2,4,3,5), (1,2,5,4,3), (1,3,4,5,2), (1,3,2,4,5), (1,3,5,2,4),
  (1,4,5,3,2), (1,4,2,5,3), (1,4,3,2,5), (1,5,3,4,2), (1,5,4,2,3), (1,5,2,3,4) ]
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