1. Match the following using integral theorems we learned this semester. Be sure to check the hypotheses of the theorem, and if possible, compute the integral.

(a) $\displaystyle\iint_{x^2+y^2+z^2=1} \left(x^3/3, z(zy+x), y^2z\right)\, ds$

(b) $\displaystyle\int_0^1\int_0^{2\pi}\int_0^{\pi} \rho^4\sin(\varphi)\, d\varphi\, d\theta\, d\rho$

(c) $\displaystyle\int_{c(t)} \left(\frac{y}{2}, \frac{-x}{2}\right) ds$ With $c(t)=\begin{pmatrix} t^3-t \\ t^2\end{pmatrix}$, $-1\le t\le 1$.

(d) $\displaystyle\iint_{\frac{x^2}{4}+\frac{y^2}{16}\le 1} xy\, dA$

(e) $\displaystyle\int_{c(t)} \begin{pmatrix} y+ze^{xy} \\ x+2z\cos(yz)\sin(yz) \\ 2y\cos(yz)\sin(yz)+xe^{xy}\end{pmatrix} dS$

With $c(t)=\begin{pmatrix}\cos(t)\\ \sin(t)\\ 0\end{pmatrix}$,

$0\le t\le \pi$.

(f) $\displaystyle\int_{c(t)} \nabla\times F(x,y,z)\, dS$ With $\Phi(s,t)=\begin{pmatrix} s\cos(t)\\ s\sin(t)\\ 2s\end{pmatrix}$, $0\le t\le 2\pi$ and $F(x,y,z)= \left(x^4z, x^2+y^2-\frac{z}{2}, \tan(x^2y^3z)\right)$.

(α) $\displaystyle\int_{-1}^{0}\int_{-\sqrt{x^3-x}}^{\sqrt{x^3-x}} 1\, dA$

(β) $f(-1,0,0)-f(1,0,0)$ With $f=xy+\cos(yz)^2+e^{xz}$

(γ) $\displaystyle\int_{c(t)} F(x,y,z)\, dS$ With $c(t)=\begin{pmatrix}\cos(t)\\ \sin(t)\\ 2\end{pmatrix}$, $0\le t\le 2\pi$ and $F(x,y,z)= \left(x^4z, x^2+y^2-\frac{z}{2}, \tan(x^2y^3z)\right)$.

(δ) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{1-x^2-y^2}^{1-x^2-y^2} x^2+y^2+z^2\, dz\, dy\, dx$

(ϵ) $\displaystyle\int_0^{2\pi}\int_0^{16} 32r^3\sin(2\theta)\, dr\, d\theta$

(ζ) $\displaystyle\int_{c(t)} \begin{pmatrix} y+ze^{xy} \\ x+2z\cos(yz)\sin(yz) \\ 2y\cos(yz)\sin(yz)+xe^{xy}\end{pmatrix} dS$

With $c(t)=\begin{pmatrix}1-2t\\ 0\\ 0\end{pmatrix}$, $0\le t\le 1$.

Solution:

a=b=δ
c=α
d=ϵ
e=δ=ζ
f=γ

2. Your book defines the first and second order Taylor polynomials on page 196. Using more compact notation, we can write the Taylor Formula for the function f about the point x_0 as

$$p_2(x) = f(x_0) + Df(x_0)(\mathbf{x}-x_0) + (\mathbf{x}-x_0)^t Hf(x_0)(\mathbf{x}-x_0)$$

(a) Write out Taylor's Formula explicitly for scalar valued functions of one, two, and three variables.

(b) Using the function $f(x, y) = \sin(xy)$, compute the Hessian matrix of f at the point $\mathbf{a} = (0, 0)$.

(c) Compute the degree 2 Taylor Polynomial of $f(x, y)$ at the point $\mathbf{a} = (0, 0)$

Solution:

(a) In dimension 1 this is the same Taylor polynomial that we worked with in single variable calculus. We will let our point $\mathbf{a} = a$.

$$p_1(x) = f(a) + f'(a)(x - a)$$
$$p_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

Now for $n = 2$ the two dimensional case, rather than using x_1 and x_2 as variables we will use x and y, and we will write $\mathbf{a} = (a, b)$

$$\begin{aligned} p_1(x, y) &= f(\mathbf{a}) + f_x(\mathbf{a})(x - a) + f_y(\mathbf{a})(y - b) \\ p_2(x, y) &= f(\mathbf{a}) + f_x(\mathbf{a})(\mathbf{x} - \mathbf{a}) + f_y(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \\ &\quad + \frac{f_{xx}(\mathbf{a})}{2}(x - a)^2 + \frac{f_{yy}(\mathbf{a})}{2}(y - b)^2 + f_{xy}(\mathbf{a})(x - a)(y - b) \end{aligned}$$

Finally if $n = 3$, the 3 dimensional case, we will use x, y, z for our variables, and let the point $\mathbf{a} = (a, b, c)$ we have

$$\begin{aligned} p_1(x, y, z) &= f(\mathbf{a}) + f_x(\mathbf{a})(x - a) + f_y(\mathbf{a})(y - b) + f_z(\mathbf{a})(z - c) \\ p_2(x, y, z) &= f(\mathbf{a}) + f_x(\mathbf{a})(x - a) + f_y(\mathbf{a})(y - b) + f_z(\mathbf{a})(z - c) + \\ &\quad + \frac{f_{xx}(\mathbf{a})}{2}(x - a)^2 + \frac{f_{yy}(\mathbf{a})}{2}(y - b)^2 + \frac{f_{yy}(\mathbf{a})}{2}(z - c)^2 + \\ &\quad + f_{xy}(\mathbf{a})(x - a)(y - b) + f_{xz}(\mathbf{a})(x - a)(z - c) + f_{yz}(\mathbf{a})(y - b)(z - c) \end{aligned}$$

(b) The Hessian matrix for $f(x, y)$ will be given as

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

To begin we will compute

$$\frac{\partial f}{\partial x} = f_x(x, y) = y\cos(xy)$$
$$\frac{\partial f}{\partial y} = f_y(x, y) = x\cos(xy)$$

Building off of this result, we can easily compute the whole Hessian

Which we can easily compute explicitly

$$H = \begin{pmatrix} -y^2\sin(xy) & \cos(xy) - xy\sin(xy) \\ \cos(xy) - xy\sin(xy) & -x^2\sin(xy) \end{pmatrix}$$

(c) Using the general form we compute above, and the Hessian matrix (which we also computed)

$$\begin{aligned} p_2(x, y) = {} & \sin(0\cdot 0) + 0\cdot\cos(0\cdot 0)(x - 0) + \\ & + 0\cdot\cos(0\cdot 0)(y - 0) - 0^2\sin(0\cdot 0) - 0^2\sin(0\cdot 0) + \\ & + \cos(0\cdot 0) - 0\cdot 0\sin(x\cdot y) + \cos(0\cdot 0) = 1 \end{aligned}$$

3. Find the second order Taylor expansion about the point $(0,0)$ of the function

$$f(x,y) = e^{xy}$$

Solution: We begin by computing the matrix of partial derivatives of f.

$$Df(x,y) = (e^{xy}y, e^{xy}x)$$

From this we compute the Hessian matrix

$$Hf(x,y) = \begin{pmatrix} e^{xy}y^2 & e^{xy} + e^{xy}xy \\ e^{xy} + e^{xy}xy & e^{xy}x^2 \end{pmatrix}$$

Then we evaluate at the point $(0,0)$ and find

$$Df(0,0) = (0,0)$$
$$Dg(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now we put these together to compute the degree 2 Taylor polynomial

$$T_2(x,y) = f(0,0) + Df(0,0) \cdot (x,y) + \frac{1}{2}(x,y)Hf(0,0)\begin{pmatrix} x \\ y \end{pmatrix}$$

Expanding out the linear algebra we obtain

$$T_2(x,y) = 1 + xy$$

Which is the degree 2 Taylor polynomial about $(0,0)$ of the function $f(x,y) = e^{xy}$

4. Find and classify all critical points of the function

$$f(x,y) = x^3 + x^2y + y^2 + xy + x + 1$$

Solution: As in the last problem we will begin by computing both the matrix of partial derivatives and the Hessian matrix.

$$Df(x,y) = (3x^2 + 2xy + y, x^2 + 2y + x)$$
$$Hf(x,y) = \begin{pmatrix} 6x + 2y & 2x + 1 \\ 2x + 1 & 2 \end{pmatrix}$$

Now to find the critical values we compute where $Df(x, y) = (0, 0)$

We easily find the following points $(0, 0)$, $(1, -1)$, and $(1/2, -3/8)$. Now using our criteria on each of these points we find that the points $(0, 0)$ and $(1, -1)$ are both saddle points, while $(1/2, -3/8)$ is a local minimum.

5. There has been an oil spill in a triangular ocean. The ocean has vertices at the points $(1,0),(0,1)$, and $(0,0)$. The plan to disperse the oil calls for installing an oil vacuum rig (which sucks up oil from the surface of the ocean) at the point (x,y). The amount of oil removed by the rig is proportional to the product of the distances from the three edges. Where should the oil vacuum be placed to most effectively remove the oil?

   Remember that the distance from (x,y) to the line $ax+by=c$ is given by $d = |ax+bc+c|/\sqrt{a^2+b^2}$. This ocean is populated entirely by baby seals, singing disneyfish, and the rare fluffy northern penguin, failure is not an option.

Solution: To begin we will want to compute the the distances from an arbitrary point (x,y) to each edge.

$$\begin{aligned} d_1 &= x \\ d_2 &= y \\ d_e &= \frac{1-x-y}{\sqrt{2}} \end{aligned}$$

We would like to maximize $d_1d_2d_3$, equivalently, we would like to maximize

$$f(x,y) = \frac{xy\,(1-x-y)}{\sqrt{2}}$$

Notice that f is zero along each edge and vertex of the triangle. On the interior we have

$$\nabla f = \begin{pmatrix} y-2xy-y^2 \\ x-x^2-2xy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To solve this system of equations we need to solve

$$\begin{aligned} y\,(1-2x-y) &= 0 \\ x\,(1-x-2y) &= 0 \end{aligned}$$

Since we are working on the interior of the ocean we have $x,y \neq 0$, and can solve the linear system

$$\begin{aligned} 1-2x-y &= 0 \\ 1-x-2y &= 0 \end{aligned}$$

Which gives us the solution of $x=\frac{1}{3}, y=\frac{1}{3}$.