The p-adic Number System

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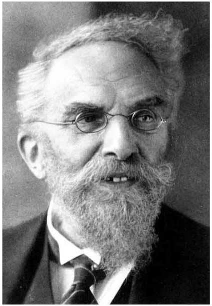
Background

The p-adic numbers were introduced by Kurt Hensel in 1897.



Since then, they have found their way into many other areas of mathematics and even physics.

Observation

Fix a prime p. Any rational number $\frac{s}{t}$ with $s, t \in \mathbb{Z}$ can be multiplied by some power of p to get a new rational number $\frac{s^*}{t^*}$ where $\mathrm{GCD}(s^*, p) = 1$ and $\mathrm{GCD}(t^*, p) = 1$.

For example, let $p = 3$.

$900 \cdot 3^{-2} = 100$ $\quad\quad$ $\frac{5}{21} \cdot 3 = \frac{5}{7}$

$-\frac{18}{23} \cdot 3^{-2} = -\frac{2}{23}$ $\quad\quad$ $\frac{1}{4} \cdot 3^0 = \frac{1}{4}$

Definition: The p-adic Valuation

Define $|\cdot|_p : \mathbb{Q} \to \mathbb{R}^+ \cup \{0\}$ by

$$|0|_p = 0$$

and for $r, s \neq 0$,

$$\left|\frac{r}{s}\right|_p = p^n$$

where $p^n \cdot \frac{r}{s} = \frac{r^*}{s^*}$ with $\mathrm{GCF}(r^*, p) = 1$ and $\mathrm{GCF}(s^*, p) = 1$.

3-adic Examples

Again, let $p = 3$.

$$900 \cdot 3^{-2} = 100 \qquad \frac{5}{21} \cdot 3 = \frac{5}{7}$$

$$|900|_3 = 3^{-2} \qquad \left|\frac{5}{21}\right|_3 = 3$$

$$-\frac{18}{23} \cdot 3^{-2} = -\frac{2}{23} \qquad \frac{1}{4} \cdot 3^{0} = \frac{1}{4}$$

$$\left|-\frac{18}{23}\right|_3 = 3^{-2} \qquad \left|\frac{1}{4}\right|_3 = 1$$

Properties of the p-adic Valuation

1. $|a|_p = 1$ if and only if $a = \frac{r}{s}$ with GCF$(p, r) = 1 =$GCF(p, s).

2. $|a|_p = p^n$ if and only if $|p^n a|_p = 1$.

3. For every integer n, $|p^n a|_p = p^{-n}|a|_p$.
   Let $a \neq 0$ and $|a|_p = p^m$.
   Then from (2), $|p^m a|_p = 1$.
   Rewrite this as $|p^{m-n} p^n a|_p = 1$.
   Again from (2), $|p^n a|_p = p^{m-n} = p^{-n}|a|_p$.

Back up: can we say that it's a valuation?

A valuation of $\mathbb{Q}$ is a map $|\cdot| : \mathbb{Q} \to \mathbb{R}^+ \cup \{0\}$ which satisfies 3 properties:

$$(i) \quad |0| = 0; \;\; |a| > 0 \text{ if } a \neq 0$$

We said that $|0|_p = 0$ and $\left|\frac{r}{s}\right|_p = 3^n > 0$.

$$(ii) \quad |ab| = |a| \cdot |b|$$

To eliminate all factors of p from ab we could multiply by a power of p to remove them from a, then multiply by a power of p to remove them from b.

$$(iii) \quad |a+b| \leq |a| + |b|$$

Third Property of Valuations

In fact, we'll prove something BETTER:

$$|a + b|_p \leq \max\{|a|_p, |b|_p\}$$

Proof: If $a = 0$ or $b = 0$, then this is clear.
Let $a, b \neq 0$ have p-adic values $|a|_p = p^n$ and $|b|_p = p^m$, where $n \leq m$.
The denominators of both $p^n a$ and $p^m b$ have no factors of p, and so neither does the denominator of $p^m(a + b)$.
That means $|p^m(a + b)|_p \leq 1$.
Then from property (3), $p^{-m}|a + b|_p \leq 1$.
$|a + b|_p \leq p^m = \max\{|a|_p, |b|_p\}$.

Series for Rational Numbers

Claim: Any rational number can be written in the form $\sum_{k=n}^{\infty} a_k p^k$, where $a_k \in \{0, \ldots, p-1\}$.

Examples:

$$\frac{1}{3} = 1 \cdot 3^{-1} + 0 \cdot 3^0 + 0 \cdot 3^1 + 0 \cdot 3^2 + \ldots$$

$$\frac{1}{3} = 10,000\ldots_3$$

$$25 = 1 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + 0 \cdot 3^3 + 0 \cdot 3^4 + \ldots$$

$$25 = 1,22000\ldots_3$$

More Interesting Example

$$\frac{2}{5} = 1 + 3 \cdot \frac{-1}{5} \qquad \frac{-4}{5} = 1 + 3 \cdot \frac{-3}{5}$$

$$\frac{-1}{5} = 1 + 3 \cdot \frac{-2}{5} \qquad \frac{-3}{5} = 0 + 3 \cdot \frac{-1}{5}$$

$$\frac{-2}{5} = 2 + 3 \cdot \frac{-4}{5} \qquad \frac{-1}{5} = 1 + 3 \cdot \frac{-2}{5} \dots$$

$$\frac{2}{5} = 3^0(1) + 3^1(1) + 3^2(2) + 3^3(1) + 3^4(0) + 3^5(1) + \dots$$

$$\frac{2}{5} = 1, 12101210\dots_3$$

Let's Build Spaces!

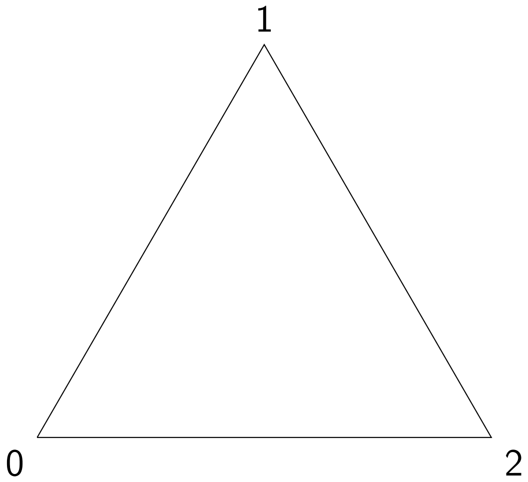
The *p-adic Numbers* $\mathbb{Q}_p$ are sequences

$$a = \sum_{k=n}^{\infty} a_k p^k,$$

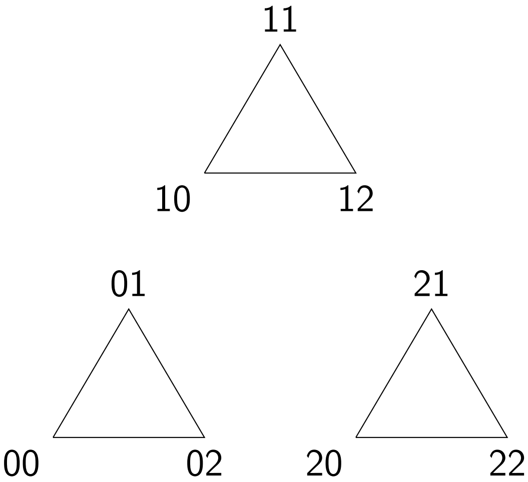
where $a_k \in \{0, \ldots, p-1\}$.

The *p-adic Integers* $\mathbb{Z}_p$ are elements $a \in \mathbb{Q}_p$ such that $|a|_p \leq 1$.

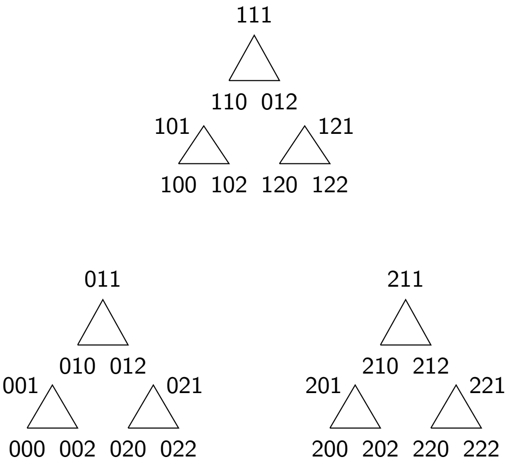
Visualization



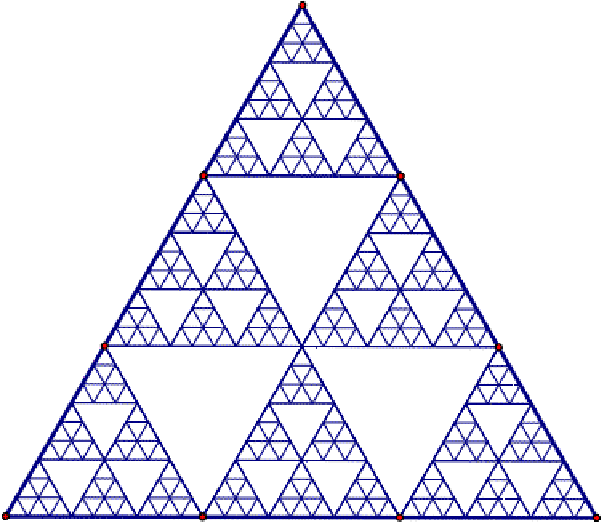
Visualization



Visualization



Sierpinski Triangle



References

- Mahler, K. *P-adic Numbers and Their Functions, 2nd ed.* Cambridge, England: Cambridge University Press, 1981.
- Albert A. Cuoco. Visualizing the p-adic integers. *Amer. Math. Monthly*, 98:355364, 1991.