Math 5285, Honors abstract algebra
Midterm #2, Fall 2008

Instructor: Ezra Miller

Due in class, Wednesday 12 November 2008

Instructions: This is an open book, open library, open notes, take-home exam, but you are not allowed to collaborate; the instructor is the only human source you are allowed to consult. You must cite each source in every solution where you use it. Be specific: "the primality lemma" is not precise; neither is "by a lemma in our textbook"; in contrast, "[Artin, Lemma 3.2.8]" is. If you rely on a named theorem from a source other than our class textbook (by M. Artin), then you must state the theorem—or at least give some idea of what it says—and say where you found it; theorems are often known by many names, so I'm likely not to recognize some theorems by names you might attach.

Your solutions must be typed; please print double-sided.

1. [15 points] Fix a vector space V and a linear operator $T : V \to V$ satisfying $T^2 = T$.
   (a) Prove that $V = \operatorname{im} T \oplus \ker T$ is the direct sum of the image and kernel of T.
   (b) If V is finite-dimensional, conclude that there is a basis of V with respect to which the matrix of T is diagonal and all entries equal 0 or 1.

2. [15 points] How many solutions does $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} X = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ have for $X \in (\mathbb{F}_7)^3$?

3. [15 points] Fix a linear operator T having two linearly independent eigenvectors with the same eigenvalue λ. Must λ be a multiple root of the characteristic polynomial of T? Justify your response with a proof or counterexample.

4. [15 points] Is the matrix $A = \begin{bmatrix} 5 & 4 \\ -3 & -3 \end{bmatrix}$ diagonalizable? If not, prove it. If so, find the eigenvectors and eigenvalues, and find a matrix P such that PAP^{-1} is diagonal.

5. [15 points] For $A \in \mathbb{C}^{n \times n}$, prove that $e^{\operatorname{tr} A} = \det(e^A)$, where $\operatorname{tr} A$ is the trace of A. (Hint: try it first when A is upper-triangular.)

6. [15 points] Fix a field F and $f \in F$. Let K be the vector space with basis $\{1, \mu\}$. The multiplication on K defined by the rule $\mu^2 = f$ is associative and commutative: $(a + b\mu)(a' + b'\mu) = (aa' + bb'f) + (ab' + a'b)\mu$. Prove that K is a field whenever f has no square root in F. Conclude that for any odd prime p, there is a field of order p^2.

7. [15 points] Determine the change-of-basis matrix P when the old and new (ordered) bases are $e_1 + e_2, e_1 - e_2$ and $2e_1 - e_2, -3e_1 + e_2$, where e_1, e_2 is the standard basis.