

HOMOLOGICAL METHODS FOR HYPERGEOMETRIC FAMILIES

LAURA FELICIA MATUSEVICH, EZRA MILLER, AND ULI WALTHER

UW dedicates this paper to the memory of his father, Hansjoachim Walther.

ABSTRACT. We analyze the behavior of the holonomic rank in families of holonomic systems over complex algebraic varieties by providing homological criteria for rank-jumps in this general setting. Then we investigate rank-jump behavior for hypergeometric systems $H_A(\beta)$ arising from a $d \times n$ integer matrix A and a parameter $\beta \in \mathbb{C}^d$. To do so we introduce an Euler–Koszul functor for hypergeometric families over \mathbb{C}^d , whose homology generalizes the notion of a hypergeometric system, and we prove a homology isomorphism with our general homological construction above. We show that a parameter $\beta \in \mathbb{C}^d$ is rank-jumping for $H_A(\beta)$ if and only if β lies in the Zariski closure of the set of \mathbb{Z}^d -graded degrees α where the local cohomology $\bigoplus_{i < d} H_{\mathfrak{m}}^i(\mathbb{C}[\mathbb{N}A])_{\alpha}$ of the semigroup ring $\mathbb{C}[\mathbb{N}A]$ supported at its maximal graded ideal \mathfrak{m} is nonzero. Consequently, $H_A(\beta)$ has no rank-jumps over \mathbb{C}^d if and only if $\mathbb{C}[\mathbb{N}A]$ is Cohen–Macaulay of dimension d .

CONTENTS

1. Introduction	1
2. Upper semi-continuity of rank in holonomic families	4
3. Rank-jumps as failures of flatness	6
4. Euler–Koszul homology of toric modules	8
5. Rigidity and holonomicity of Euler–Koszul homology	11
6. Euler–Koszul homology detects local cohomology	13
7. Global Euler–Koszul homology as a holonomic family	15
8. Isomorphism of the two homology theories	18
9. Combinatorics of hypergeometric ranks	19
References	21

Received by the editors June 22, 2004 and, in revised form, April 4, 2005.

1991 *Mathematics Subject Classification.* Primary 13N10, 13D45, 14D99, 13F99, 16E99; Secondary 32C38, 35A27, 14M25, 70F20, 33C70, 13C14, 13D07.

Key words and phrases. hypergeometric system, Cohen–Macaulay, toric, local cohomology, holonomic, D -module.

LFM was partially supported by a postdoctoral fellowship from MSRI and an NSF Postdoctoral Fellowship.

EM was partially supported by NSF Grant DMS-0304789.

UW was partially supported by the DfG, theHumboldt foundation, and NSF Grant DMS-0100509.

REFERENCES

- [Ado94] Alan Adolphson, *Hypergeometric functions and rings generated by monomials*, Duke Math. J. **73** (1994), no. 2, 269–290. MR **96c**:33020
- [Ado99] ———, *Higher solutions of hypergeometric systems and Dwork cohomology*, Rend. Sem. Mat. Univ. Padova **101** (1999), 179–190. MR **2001b**:14032
- [BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR **95h**:13020
- [Bjö79] J.-E. Björk, *Rings of differential operators*, North-Holland Mathematical Library, vol. 21, North-Holland Publishing Co., Amsterdam, 1979. MR **82g**:32013
- [CDD99] Eduardo Cattani, Carlos D’Andrea, and Alicia Dickenstein, *The A -hypergeometric system associated with a monomial curve*, Duke Math. J. **99** (1999), no. 2, 179–207. MR **2001f**:33018
- [CDS01] Eduardo Cattani, Alicia Dickenstein, and Bernd Sturmfels, *Rational hypergeometric functions*, Compositio Math. **128** (2001), no. 2, 217–239. MR **2003f**:33016
- [CK99] David Cox and Sheldon Katz, *Mirror symmetry and algebraic geometry*, Amer. Math. Soc., Providence, RI, 1999. MR **2000d**:14048
- [Eis95] David Eisenbud, *Commutative algebra, with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. MR **97a**:13001
- [GGZ87] I. M. Gel’fand, M. I. Graev, and A. V. Zelevinskii, *Holonomic systems of equations and series of hypergeometric type*, Dokl. Akad. Nauk SSSR **295** (1987), no. 1, 14–19. MR **88j**:58118
- [GKZ89] I. M. Gel’fand, A. V. Zelevinskii, and M. M. Kapranov, *Hypergeometric functions and toric varieties*, Funktsional. Anal. i Prilozhen. **23** (1989), no. 2, 12–26. Correction in *ibid.*, **27** (1993), no. 4, 91. MR **90m**:22025, MR **95a**:22010
- [Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York, 1977. MR **57** #3116
- [Hot98] Ryoshi Hotta, *Equivariant D -modules*, 1998. arXiv:math.RT/9805021
- [Mat01] Laura Felicia Matusevich, *Rank jumps in codimension 2 A -hypergeometric systems*, J. Symbolic Comput. **32** (2001), no. 6, 619–641. Effective methods in rings of differential operators. MR **2003f**:33017
- [Mat03] ———, *Exceptional parameters for generic A -hypergeometric systems*, Int. Math. Res. Not. (2003), no. 22, 1225–1248. MR **1** 967 406
- [MM05] Laura Felicia Matusevich and Ezra Miller, *Combinatorics of rank jumps in simplicial hypergeometric systems*, Proc. Amer. Math. Soc., to appear, 2005. arXiv:math.AC/0402071
- [MW04] Laura Felicia Matusevich and Uli Walther, *Arbitrary rank jumps for A -hypergeometric systems through Laurent polynomials*, 2004. arXiv:math.CO/0404183
- [Mil02] Ezra Miller, *Graded Greenlees–May duality and the Čech hull*, Local cohomology and its applications (Guanajuato, 1999), Lecture Notes in Pure and Appl. Math., vol. 226, Dekker, New York, 2002, pp. 233–253.
- [MS04] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics Vol. 227, Springer-Verlag, New York, 2004. MR **2** 110 098
- [Sai01] Mutsumi Saito, *Isomorphism classes of A -hypergeometric systems*, Compositio Math. **128** (2001), no. 3, 323–338. MR **2003f**:33019
- [Sai02] Mutsumi Saito, *Logarithm-free A -hypergeometric series*, Duke Math. J. **115** (2002), no. 1, 53–73. MR **1** 932 325
- [SST00] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama, *Gröbner deformations of hypergeometric differential equations*, Algorithms and Computation in Mathematics, vol. 6, Springer-Verlag, Berlin, 2000. MR **2001i**:13036
- [ST98] Bernd Sturmfels and Nobuki Takayama, *Gröbner bases and hypergeometric functions*, Gröbner bases and applications (Linz, 1998), London Math. Soc. Lecture Note Ser., vol. 251, Cambridge Univ. Press, Cambridge, 1998, pp. 246–258. MR **2001c**:33026