Diffusion processes and parabolic differential equations with boundary conditions

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1 Parabolic Differential Equations without Boundary Conditions

There are close connections between diffusion processes and parabolic differential equations. For example, if a diffusion process admits a transition density, then the transition density will satisfy a parabolic differential equation. In this survey, we are going to summarize some of these facts. Let $(\Omega, \mathcal{F}, P)$ be the probability space and $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration on this space. We will assume that all the processes are built on this space. Before considering the case in some bounded smooth domain G, we look at diffusions in $\mathbb{R}^n$.

A diffusion process X_t in $\mathbb{R}^n$ can be described as a continuous Markovian process or the solution of a stochastic differential equation

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt \tag{1}$$

with the initial condition $X_0 = x_0 \in \mathbb{R}^n$, where B_t is a Brownian motion in $\mathbb{R}^n$, $\sigma(x) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $b(x) : \mathbb{R}^n \to \mathbb{R}^n$ are measurable. In order for (1) to have a unique strong solution, we also assume that

(i) $\sup_{x\in\mathbb{R}^n} \|\sigma(x)\| < A$ and $\sup_{x\in\mathbb{R}^n} \|b(x)\| < A$;

(ii) $\|\sigma(x) - \sigma(y)\| + \|b(x) - b(y)\| \leq A\|x - y\|$,

hold for some constant $A < \infty$ and any $x, y \in \mathbb{R}^n$. Here $\|\cdot\|$ denotes the Euclidean norm in either $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$. More details can be found in chapter 5 of Stroock and Varadhan [5]. Next, we state a version of the well-known Itô's formula, which can be found in almost any book in stochastic calculus, for example, again in [5].

Itô's Formula Let X_t satisfy (1) and $f(t,x) \in C^{1,2}([0,\infty) \times \mathbb{R}^n)$ such that f, $\partial_t f$,$\partial_{x_i} f$ and $\partial_{x_i x_j} f$ $(1 \leq i, j \leq n)$ are bounded pointwise by $Ae^{B\|x\|}$ for some A and B, then $f(t, X_t)$ satisfies the following stochastic differential equation

$$df(t,X_t) = \langle \sigma \nabla_x f, dB_t \rangle + (\partial_t f + \langle b, \nabla_x f \rangle + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma\sigma^T)_{ij} \partial_{x_i x_j} f) dt \ . \tag{2}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$.

From now on, denote

$$\mathcal{L}_x := \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \partial_{x_i x_j} + \sum_{i=1}^{n} b_i \partial_{x_i}, \tag{3}$$

where $a = \sigma\sigma^T$ is a symmetric and positive definite matrix. $\mathcal{L}_x$ is then the infinitesimal generator of the diffusion X_t, i.e., $\mathcal{L}_x \phi(x) = \lim_{t \downarrow 0} \frac{E[\phi(X_t)|X_0 = x] - \phi(x)}{t}$ for some test function $\phi(x) \in C_0^\infty(\mathbb{R}^n)$. For later use, we can also write $\mathcal{L}_x$ in its divergence form

$$\mathcal{L}_x = \frac{1}{2} \nabla_x \cdot (a \nabla_x) + \tilde{b} \cdot \nabla_x, \tag{4}$$

where $\tilde{b}_i = b_i - \frac{1}{2} \sum_{j=1}^{n} \partial_{x_j} a_{ij}$. This divergence form is easy to use when we are trying to apply integration by parts in some smooth domain. Now let us state a version of Feynman-Kac formula, which is adapted from Ikeda and Watanabe [3]. Before stating the formula, we assume that

(iii) $\sigma_{ij}(x)$ and $b_i(x)$, $1 \le i, j \le n$, are smooth functions.

(iv) $\lambda \|\theta\| \le \langle \theta a, \theta \rangle \le \lambda^{-1} \|\theta\|$, for any $\theta \in \mathbb{R}^n$ and some constant $0 < \lambda < 1$.

Feynman-Kac Formula Let $f(x), \Psi(x) \in C_0^\infty(\mathbb{R}^n)$ and $X_t, \mathcal{L}_x$ as above. Then the function

$$u(s,x) = E[\Psi(X_T) e^{-\int_s^T c(X_t) dt} | X_s = x] \tag{5}$$

is a bounded $C^\infty([0,T] \times \mathbb{R}^n)$ solution of the partial differential equation subject to a terminal condition

$$\begin{cases} u_s = -\mathcal{L}_x u + cu, & \text{for } s < T, \\ u(T,x) = \Psi(x). \end{cases} \tag{6}$$

Conversely, if $u(s,x)$ is a bounded $C^{1,2}([0,T] \times \mathbb{R}^n)$ solution of the (6), then $u(s,x)$ can be represented as (5).

Assume that X_t has a transition density $p(s,x,t,y)$, i.e., for $x, y \in \mathbb{R}^n$ and $s \le t$,

$$P(X_t = dy | X_s = x) = p(s,x,t,y) dy.$$

Let $c(x) \equiv 0$ in Feynman-Kac formula. Then $E[\Psi(X_T)|X_s = x] = \int \Psi(y)p(s,x,t,y)dy$ as a function of s and x solves (6). In particular, we have

$$\int \Psi(y)\partial_s p(s,x,t,y)dy = -\int \Psi(y)\mathcal{L}_x p(s,x,t,y)dy.$$

Choose $\Psi(y) \to \delta_y$, the Dirac function at y. It follows that $p(s,x,t,y)$ satisfies the parabolic equation

$$\partial_s p = -\mathcal{L}_x p. \tag{7}$$

This is the *backwards equation* for X_t.

Chapman-Kolmogorov equation tells us that, for $x, z \in \mathbb{R}^n$ and $s \le t \le r$,

$$\int_{\mathbb{R}^n} p(s,x,t,y)p(t,y,r,z)dy = p(s,x,r,z).$$

Taking derivative with respect to t, we get

$$\int_{\mathbb{R}^n} [\partial_t p(s,x,t,y)p(t,y,r,z) + p(s,x,t,y)\partial_t p(t,y,r,z)]dy = 0$$

Therefore, using (7), form (3) of $\mathcal{L}$ and integration by parts, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \partial_t p(s,x,t,y)p(t,y,r,z)dy \\
= &\int_{\mathbb{R}^n} p(s,x,t,y)(\frac{1}{2}\sum_{i,j=1}^n a_{ij}\partial_{y_iy_j} + \sum_{i=1}^n b_i\partial_{y_i})p(t,y,r,z)dy \\
= &\int_{\mathbb{R}^n} [\frac{1}{2}\sum_{i,j=1}^n \partial_{y_iy_j}(a_{ij}p(s,x,t,y)) - \sum_{i=1}^n \partial_{y_i}(b_ip(s,x,t,y))]p(t,y,r,z)dy,
\end{aligned}$$

which indicates that $p(s,x,t,y)$, as a function of t and y, satisfies the *forwards equation*

$$\partial_t p = \mathcal{L}_y^* p. \tag{8}$$

Here $\mathcal{L}^*$ is the dual of $\mathcal{L}$ with the explicit form as follows

$$\mathcal{L}_y^* = \frac{1}{2}\sum_{i,j=1}^n a_{ij}\partial_{y_iy_j} + \sum_{i=1}^n(-b_i + \sum_{j=1}^n \partial_{y_j}a_{ij})\partial_{y_i} + (\frac{1}{2}\sum_{i,j=1}^n \partial_{y_iy_j}a_{ij} - \sum_{i=1}^n \partial_{y_i}b_i). \tag{9}$$

2 Parabolic Differential Equations with Dirichlet Conditions

In this section, we study the relation between the Dirichlet problem and the stopped diffusion. Given a bounded smooth domain G in $\mathbb{R}^n$ and X_t a diffusion process as in the previous section starting at some point in G, a stopped diffusion process Y_t is defined as $Y_t = X_{t\wedge\tau}$ where $\tau = \inf\{t : X_t \notin G\}$. Intuitively, Y_t is X_t before X_t hits ∂G (the boundary of G) and Y_t does not change after X_t hits ∂G. Assume that there exists a function $p_1(s, x, t, y)$ such that

$$p_1(s, x, t, y)dy = P(Y_t = dy|Y_s = x) = P(X_t = dy, t < \tau_s|X_s = x),$$

where $\tau_s = \inf\{r > s : X_r \notin G\}$, $s < t$ and $x, y \in G$.

Let us consider the following PDE

$$\begin{cases} \partial_s u = -\mathcal{L}_x u \quad \text{for } x \in G, s < T, \\ u(s, x) = 0 \quad \text{for } x \in \partial G, s < T, \\ u(T, x) = \Psi(x) \quad \text{for } x \in G, \end{cases} \tag{10}$$

where $T > 0$ is a constant, $\mathcal{L}$ is as in the previous section and $\Psi(x) \in C_0^\infty(G)$. Let $u(s, x)$ be a $C^{1,2}([0, T] \times \bar{G})$ solution of the above PDE (u can easily extended to $C_0^{1,2}([0, T] \times \mathbb{R}^n)$). Apply Itô's formula to $u(t, X_t)$, we obtain that

$$u(T \wedge \tau_s, X_{T\wedge\tau_s}) = u(s, x) + \int_s^{T\wedge\tau_s} (\partial_t + \mathcal{L})u(t, X_t)dt + \int_s^{T\wedge\tau_s} \sigma\nabla u dB_t.$$

Notice that $(\partial_t + \mathcal{L})u(t, X_t)$ is 0 for $s \le t \le (T \wedge \tau_s)$ by (10) and $E(\int_s^{T\wedge\tau_s} \sigma\nabla u dB_t) = 0$. Taking expectation, $u(s, x)$ can then be written as

$$\begin{aligned} u(s, x) &= E(u(T \wedge \tau_s, X_{T\wedge\tau_s})|X_s = x) = E(u(T \wedge \tau_s, X_{T\wedge\tau_s})|Y_s = x) \\ &= E(u(T \wedge \tau_s, Y_T)|Y_s = x) = E(u(T, Y_T)1_{\{T<\tau_s\}}|Y_s = x) \\ &= E(\Psi(Y_T)1_{\{T<\tau_s\}}|Y_s = x) \\ &= \int_G \Psi(y)p_1(s, x, T, y)dy. \end{aligned}$$

Substitute $u(s, x)$ back to the PDE (10) and choose a sequence $\Psi_n(y) \to \delta_y$ for $y \in G$, we get immediately the *backwards equation* for $p_1(s, x, t, y)$:

$$\begin{cases} \partial_s p_1 = -\mathcal{L}_x p_1 \quad \text{for } x \in G, s < t \\ p_1(s, x, t, y) = 0 \quad \text{for } x \in \partial G. \end{cases} \tag{11}$$

Toward the forwards equation, let us first state Dynkin's Formula, which can be found for example in Itô and Mckean [2].

Dynkin's Formula Let X_t be as in the first section and ζ a stopping time with $E^x[\zeta] < \infty$. f is $C^2(\mathbb{R}^n)$ with compact support. Dynkin's formula holds as follows:

$$E^x[f(X_\zeta)] = f(x) + E^x[\int_0^\zeta \mathcal{L}f(X_s)ds]. \tag{12}$$

If $x \in G$ and $X_0 = x$, then $Y_0 = x$. For any f compacted supported in $\mathbb{R}^n$, we replace ζ in Dynkin's formula by $t \wedge \tau$, so

$$\begin{aligned}
E^x[f(Y_t)] &= E^x[f(X_{t\wedge\tau})] = f(x) + E^x[\int_0^{t\wedge\tau} \mathcal{L}f(X_s)ds] \\
&= f(x) + E^x[\int_0^t \mathcal{L}f(Y_s)1_{\{Y_s\in G\}}ds].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
E^x[f(Y_t)] &= E^x[f(Y_t)1_{\{Y_t\in G\}}] + E^x[f(Y_\tau)1_{\{t\le\tau\}}] \\
&= \int_G f(y)p_1(0,x,t,y)dy + E^x[f(Y_\tau)1_{\{t\le\tau\}}].
\end{aligned}$$

Therefore, we have

$$\int_G f(y)p_1(0,x,t,y)dy + E^x[f(Y_\tau)1_{\{t\le\tau\}}] = f(x) + E^x[\int_0^t \mathcal{L}f(Y_s)1_{\{Y_s\in G\}}ds].$$

Take derivative with respect to t and use dS_y to denote the area measure on ∂G, and we obtain that

$$\begin{aligned}
& \int_G f(y)\partial_t p_1(0,x,t,y)dy + \partial_t(E^x[f(Y_\tau)1_{\{t\le\tau\}}]) \\
= \ & E^x[\mathcal{L}f(Y_t)1_{\{Y_t\in G\}}] = \int_G \mathcal{L}_y f(y)p_1(0,x,t,y)dy \\
= \ & \frac{1}{2}\int_G \nabla_y\cdot(a\nabla_y f(y))p_1(0,x,t,y)dy + \int_G \tilde{b}\cdot\nabla_y f(y)p_1(0,x,t,y)dy \\
= \ & \frac{1}{2}\int_{\partial G}[p_1(0,x,t,y)a\nabla_y f(y) - f(y)a\nabla_y p_1(0,x,t,y)]\cdot\overrightarrow{n}dS_y \\
& +\frac{1}{2}\int_G f(y)\nabla_y(a\nabla_y p_1(0,x,t,y))dy \\
& +\int_{\partial G} f(y)p_1(0,x,t,y)\tilde{b}\cdot\overrightarrow{n}dS_y - \int_G f(y)\nabla_y(p_1(0,x,t,y)\tilde{b})dy
\end{aligned}$$

In the above equation, if we choose f to be compactly supported in G, then $f(y) = 0$ for $y \in \partial_G$. Furthermore, if we choose a sequence of such functions $f_k \to \delta_y$ for $y \in G$, then it follows that $p_1(0, x, t, y)$ satisfies

$$\partial_t p_1(0, x, t, y) = \mathcal{L}_y^* p_1(0, x, t, y) \quad \text{for } y \in G,$$

where

$$\mathcal{L}_y^* = \frac{1}{2}\nabla_y(a\nabla_y) - \tilde{b} \cdot \nabla_y - \nabla_y \tilde{b} \tag{13}$$

is the same operator as (9). We can also choose a sequence of f_k's such that $\mathrm{supp} f_k \cap G \to \emptyset$ and $a\nabla f_k \cdot \overrightarrow{n} dS_y \to$ a point mass at $y \in \partial G$. Then it follows that $p_1(0, x, t, y)$ satisfies the Dirichlet boundary conditions

$$p_1(0, x, t, y) = 0 \quad \text{for } y \in \partial G.$$

To sum up, $p_1(0, x, t, y)$ (thus also $p_1(s, x, t, y)$ by the time homogeneous property) satisfies the *forwards equation*

$$\begin{cases} \partial_t p_1 = \mathcal{L}_y^* p_1 & \text{for } y \in G,\ t > s \\ p_1(s, x, t, y) = 0 & \text{for } y \in \partial G. \end{cases} \tag{14}$$

Before closing this section, we would like to show the consistency between the Chapman-Komolgorov equation and backwards/forwards equation (11)/(14).

The Chapman-Kolmogorov equation for Y_t tells us that, for any $s < t < T$ and $x, z \in G$,

$$p_1(s, x, T, z) = \int_G p_1(s, x, t, y) p_1(t, y, T, z) dy.$$

Take derivative with respect to t, we should get

$$0 = \int_G (\partial_t p_1(s, x, t, y)) p_1(t, y, T, z) + p_1(s, x, t, y)(\partial_t p_1(t, y, T, z)) dy.$$

Substitute the differential equations in backwards/forwards equations into the above identity, then we should get

$$\int_G \mathcal{L}_y^* p_1(s, x, t, y) p_1(t, y, T, z) = \int_G p_1(s, x, t, y) \mathcal{L}_y p_1(t, y, T, z).$$

Perform integration by parts to one side. The Dirichlet boundary conditions in backwards/forwards equations will kill some extra terms so that we end up with the other side of the equality. So the above identity is indeed the case.

3 Parabolic Differential Equations with Neumann Conditions

3.1 1-dimensional Brownian motion reflected at 0

In this subsection, we are going to discuss a simplified case, which is easy to check by elementary observations. There are some different but equivalent (in distribution) definitions of the 1-dim Brownian motion reflected at 0 (for simplicity, we call such a process RBM within this section).

Given a 1-dim standard Brownian motion B_t, the RBM Z_t can be defined as $|B_t|$, which is also the 1-dim Bessel process. The RBM Z_t can also be defined as $\max_{s=0}^t B_s - B_t$. Those two different processes are proved to have the same law in Itô and Mckean [2]. Z_t also has the same distribution as $B_t + L_t^0$, where

$$L_t^0 = \lim_{\epsilon \to 0} \frac{\int_0^t 1_{\{B_s \in (-\epsilon, \epsilon)\}} ds}{2\epsilon}, \qquad \text{a.s.}$$

is the local time of B_t at 0.

Let $p(s,x,t,y)$ be the transition density of the standard Brownian motion B_t and $p_2(s,x,t,y)$ be the transition density of RBM Z_t. The first definition above says that $Z_t = |B_t|$. It is then clear from the reflection principle that, for $s < t$ and $x, y \geq 0$,

$$p_s(s,x,t,y) = \frac{1}{2} p(s,x,t,y) + \frac{1}{2} p(s,x,t,-y).$$

Since it is well known that $p(s,x,t,y)$ satisfies $\partial_t p = \frac{1}{2}\Delta_y p$, p_2 also satisfies $\partial_t p_2 = \frac{1}{2}\Delta_y p_2$. And it is easy to check that

$$\partial_y p_2(s,x,t,0) = \frac{1}{2}\partial_y p(s,x,t,0) - \frac{1}{2}\partial_y p(s,x,t,0) = 0.$$

Therefore, $p_2(s,x,t,y)$, as a function of t and y, satisfies a parabolic equation with Neumann boundary conditions,

$$\begin{cases} \partial_t p_2(s,x,t,y) = \frac{1}{2}\Delta_y p_2(s,x,t,y) & \text{for } y > 0 \text{ and } t > s, \\ \partial_y p_2(s,x,t,0) = 0 \end{cases} \tag{15}$$

3.2 General Case

Now we want to extend the previous result to more general multi-dimensional diffusion processes. However, the main difficulty comes from the fact that, even in one dimension, a reflecting diffusion is not in general the absolute value of the diffusion process. We need to find an alternative way to describe the reflecting diffusion process.

Besides the definitions in the previous section, an RBM Z_t in G can also be defined as a diffusion process such that

$$f(t, Z_t) - \int_0^t (\partial_u f + \frac{1}{2}\Delta f)(u, Z_u)du$$

is a submartingale whenever f is a smooth function whose inner normal derivative on ∂G is non-negative.

This gives another equivalent description of the process in section 3.1, which can be easily seen from Itô's formula and the fact that L_t^0 only increases when B_t (and also Z_t) touches the boundary.

We are now ready to define general reflecting diffusion processes. Let G be a bounded smooth domain in $\mathbb{R}^n$ and $\{\overrightarrow{n}(x) : x \in \partial G\}$ be the inner normal vectors on ∂G. Z_t is a reflecting diffusion process in G if, for any smooth function $f(t, x)$ with compact support satisfying $\nabla_x f \cdot \overrightarrow{n} \geq 0$ on ∂G, we have

$$f(t, Z_t) - \int_0^t (\partial_u f + \mathcal{L} f)(u, Z_u)\chi_G(Z_u)du \tag{16}$$

is a submartingale. Here $\mathcal{L}$ is as in (3) or (4).

Then it can be proved (see Stroock and Varahdan [4]) that there is a continuous non-decreasing process $\xi(t)$ which increases only when $Z_t \in \partial G$ and has the property that

$$f(t, Z_t) - \int_0^t \chi_G(\partial_u f + \mathcal{L} f)(u, Z_u)du - \int_0^t \nabla_x f \cdot \overrightarrow{n}(u, Z_u)d\xi(u) \tag{17}$$

is a martingale for all smooth function f.

Intuitively, if X_t is a diffusion process with the infinitesimal generator $\mathcal{L}$, Z_t acts like X_t in G and gets a 'push' towards the interior of G according to $\int_0^t \overrightarrow{n}(Z_u)d\xi(u)$ when Z_t hits ∂G. With (17) in hand, we will follow the same pattern as in Section 2. First let us work on the backwards equation.

For fixed $T > 0$ and $\Psi \in C^\infty(\overline{G})$, consider the following PDE problem

$$\begin{cases} \partial_s u = -\mathcal{L}_x u & \text{for } x \in G, t < T \\ \nabla_x u \cdot \overrightarrow{n} = 0 & \text{for } x \in \partial G, t < T \\ u(T, x) = \Psi(x) & \text{for } x \in \overline{G}. \end{cases} \tag{18}$$

Let $u(s, x)$ be the classical solution of (18) and Z_t the reflecting diffusion process described above, then $u(t, Z_t)$ is a martingale by (17) and (18). Therefore,

$$u(s,x) = E[u(T,Z_T)|Z_s = x] = E[\Psi(Z_T)|Z_s = x].$$

If Z_t admits a transition density $p_2(s,x,t,y)$ where $s < t$ and $x, y \in G$, then

$$E[\Psi(Z_T)|Z_s = x] = \int_{\overline{G}} \Psi(y) p_2(s,x,T,y)dy.$$

Substitute this in (18), then we have

$$\begin{cases} \int_{\overline{G}} \Psi(y)\partial_s p_2(s,x,T,y)dy = \int_{\overline{G}} \Psi(y)(-\mathcal{L}_x)p_2(s,x,T,y)dy, & \text{for } x \in G, t < T, \\ \int_{\overline{G}} \Psi(y)(\nabla_x p_2(s,x,T,y) \cdot \overrightarrow{n})dy = 0, & \text{for } x \in \partial G, t < T. \end{cases}$$

Choosing a sequence of such $\Psi_k(y)'s$ which approach δ_y, we find the *backwards equation* for Z_t, i.e., $p_2(s,x,t,y)$, as a function of s and x, satisfies the Neumann problem

$$\begin{cases} \partial_s p_2 = -\mathcal{L}_x p_2 & \text{for } x \in G, \\ \nabla_x p_2 \cdot \overrightarrow{n} = 0 & \text{for } x \in \partial G. \end{cases} \tag{19}$$

Next, we use integration by parts to find the forwards equation as we did in Section 2. Since (17) is a martingale, in particular, if we can choose $f(t,x) \equiv g(x)$ to be independent of t for some $g \in C_0^\infty(\mathbb{R}^n)$ such that $\nabla_x g \cdot \overrightarrow{n} \geq 0$ on ∂G, then

$$g(Z_t) - \int_0^t \chi_G \mathcal{L} g(Z_u)du - \int_0^t \nabla g \cdot \overrightarrow{n}(x(u))d\xi(u)$$

is a martingale. If $Z_0 = x$, then it follows immediately that

$$E[g(Z_t)] = g(x) + E[\int_0^t \chi_G \mathcal{L} g(Z_u)du] + E[\int_0^t \nabla g \cdot \overrightarrow{n}(x(u))d\xi(u)].$$

Let us choose g such that $\nabla_x g \cdot \overrightarrow{n} \equiv 0$ for $x \in \partial G$. Notice that $E[g(Z_t)] = \int_G g(y)p_2(0,x,t,y)dy$. Take derivative with respect to t in the above equation, and then we get

$$\begin{aligned} & \int_G g(y)\partial_t p_2(0,x,t,y)dy = \int_G \mathcal{L} g(y) p_2(0,x,t,y)dy \\ = \; & \frac{1}{2}\int_G \nabla_y \cdot (a\nabla_y g(y))p_2(0,x,t,y)dy + \int_G \tilde{b} \cdot \nabla_y g(y) p_2(0,x,t,y)dy \\ = \; & -\frac{1}{2}\int_{\partial G} g(y) a \nabla_y p_2(0,x,t,y) \cdot \overrightarrow{n} dS_y + \frac{1}{2}\int_G g(y)\nabla_y(a\nabla_y p_2(0,x,t,y))dy \\ & + \int_{\partial G} g(y)p_2(0,x,t,y)\tilde{b} \cdot \overrightarrow{n} dS_y - \int_G g(y)\nabla_y(p_2(0,x,t,y)\tilde{b})dy \end{aligned}$$

Choose g_k's compactly supported in G and $g_k \to \delta_y$ for $y \in G$, it follows from the above equation that $p_2(0,x,t,y)$ satisfies

$$\partial_t p_2(0,x,t,y) = \mathcal{L}_y^* p_2(0,x,t,y), \quad \text{for } y \in G,$$

where $\mathcal{L}_y^*$ is the same as (13). Choose g_k's such that $\text{supp} g_k \cap G \to \emptyset$ and $g_k dS_y \to$ a point mass at $y \in \partial G$, it follows that

$$(-\frac{1}{2} a \nabla_y p_2(0,x,t,y) + p_2(0,x,t,y)\tilde{b}) \cdot \overrightarrow{n} = 0 \ \text{ for } y \in \partial G.$$

Thus $p_2(s,x,t,y) = p_2(0,x,t-s,y)$ satisfies the *forwards equation*

$$\begin{cases} \partial_t p_2 = \mathcal{L}_y^* p_2 & \text{for } y \in G, \ t > s \\ (-\frac{1}{2} a \nabla_y p_2 + p_2 \tilde{b}) \cdot \overrightarrow{n} = 0 & \text{for } y \in \partial G. \end{cases} \tag{20}$$

We can also check consistency between the Chapman-Kolmogorov equation and backwards/forwards equation (19)/(20) in the same manner as in Section 2.

As a conclude of this section, we go back to the special case of 1-dim RBM in Section 3.1. In that case, $\mathcal{L}^* = \frac{1}{2}\Delta$, $a = 1$ and $\tilde{b} = 0$. (20) is reduced to (15), the forwards equation for 1-dim RBM.

4 Periodic Conditions

In this section, we want to consider 1-dimensional parabolic differential equations with periodic boundary conditions:

$$\begin{cases} \partial_t u = -\mathcal{L} u & \text{for } x \in (0,1), t > 0, \\ u(t,0) = u(t,1) & \text{for } t > 0. \end{cases} \tag{21}$$

(21) can be extended to the whole real line by periodicity. In particular, if $\mathcal{L} = \frac{1}{2}\sigma(x)^2 \frac{d^2}{dx^2} + b(x)\frac{d}{dx}$ for $0 \le x \le 1$, we can extend $\mathcal{L}$ to $\bar{\mathcal{L}} = \frac{1}{2}\bar{\sigma}(x)^2 \frac{d^2}{dx^2} + \bar{b}(x)\frac{d}{dx}$ where $\bar{\sigma}(x) = \sigma(x - \lfloor x \rfloor)$ and $\bar{b}(x) = b(x - \lfloor x \rfloor)$ for any $x \in \mathbb{R}$. If we also let $\bar{u}(t,x) = u(t, x - \lfloor x \rfloor)$, then $\bar{u}$ satisfies

$$\partial_t \bar{u} = -\bar{\mathcal{L}} \bar{u}, \quad \text{for } t > 0, x \in \mathbb{R}.$$

By section 1, the transition density $\bar{p}(s,x,t,y)$ of the diffusion process $\bar{X}_t$ defined by

$$d\bar{X}_t = \bar{\sigma}(\bar{X}_t) dB_t + \bar{b}(\bar{X}_t) dt$$

satisfies the backwards equation $\partial_s \bar{p} = -\bar{\mathcal{L}}_x \bar{p}$ and the forwards equation $\partial_t \bar{p} = \bar{\mathcal{L}}_y^* \bar{p}$.

Consider a diffusion R_t on a circle $\mathbb{R}/\mathbb{Z}$ defined by $R_t = \bar{X}_t - \lfloor \bar{X}_t \rfloor$. Then the transition density $p_3(s,x,t,y)$ $(0 \le x, y \le 1)$ of R_t can be expressed as

$$p_3(s,x,t,y) = \sum_{n=-\infty}^{\infty} \bar{p}(s,x,t,y+n) = \sum_{n=-\infty}^{\infty} \bar{p}(s,x+n,t,y).$$

From this and backwards/forwards equations for $\bar{X}_t$, p_3 satisfies $\partial_s p_3 = -\mathcal{L}_x p_3$ and $\partial_t p_3 = \mathcal{L}_y^* p_3$. And the periodic boundary conditions for p_3 are obvious. Thus the *backwards equation* for R_t is

$$\begin{cases} \partial_s p_3 = -\mathcal{L}_x p_3, & \text{for } s < t, x, y \in [0,1], \\ p_3(s,0,t,y) = p_3(s,1,t,y) & \text{for } s < t, y \in [0,1]. \end{cases} \tag{22}$$

The *forwards equation* for R_t is

$$\begin{cases} \partial_t p_3 = \mathcal{L}_y^* p_3, & \text{for } t > s, x, y \in [0,1], \\ p_3(s,x,t,0) = p_3(s,x,t,1) & \text{for } t > s, y \in [0,1]. \end{cases} \tag{23}$$

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