Diffusion processes and parabolic differential equations with boundary conditions

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Outline

- Preliminaries — Diffusion Processes; Itô's formula; Martingale.
- Diffusion Processes and Parabolic Equations.
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Preliminaries — Diffusions

A *diffusion process* X_t in $\mathbb{R}^n$ can be described as a continuous Markovian process or the solution of a stochastic differential equation

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt \tag{1}$$

with the initial condition $X_0 = x_0 \in \mathbb{R}^n$, where B_t is a Brownian motion in $\mathbb{R}^n$, $\sigma(x) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $b(x) : \mathbb{R}^n \to \mathbb{R}^n$ are measurable.

Existence and Uniqueness of (1)?

In order for (1) to have a unique strong solution, we also assume that

(i) $\sup_{x\in\mathbb{R}^n} \|\sigma(x)\| < A$ and $\sup_{x\in\mathbb{R}^n} \|b(x)\| < A$;

(ii) $\|\sigma(x) - \sigma(y)\| + \|b(x) - b(y)\| \leq A\|x - y\|$.

hold for some constant $A < \infty$ and any $x, y \in \mathbb{R}^n$. Here $\|\cdot\|$ denotes the Euclidean norm in either $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$.

Itô's Formula Let X_t satisfy (1) and $f(t,x) \in C^{1,2}([0,\infty) \times \mathbb{R}^n)$ such that f, $\partial_t f$,$\partial_{x_i} f$ and $\partial_{x_i x_j} f$ $(1 \le i,j \le n)$ are bounded pointwise by $Ae^{B\|x\|}$ for some A and B, then $f(t,X_t)$ satisfies the following stochastic differential equation

$$df(t,X_t) = \langle \sigma \nabla_x f, dB_t \rangle + (\partial_t f + \langle b, \nabla_x f \rangle + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma\sigma^T)_{ij} \partial_{x_i x_j} f) dt \ . \quad (2)$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$.

Remark

1 Intuitively, one can roughly think of $dB_t = (dt)^{\frac{1}{2}}$ since $|B_t|$ grows at the order $\sqrt{t}$ by central limit theorem. Then $(dB_t)^2 = dt$ and $dB_t dt = 0$.

2 $\mathcal{L}_x := \frac{1}{2} \sum_{i,j=1}^n a_{ij} \partial_{x_i x_j} + \sum_{i=1}^n b_i \partial_{x_i} = \frac{1}{2} \nabla_x \cdot (a \nabla_x) + \tilde{b} \cdot \nabla_x$, where $a = \sigma\sigma^T$, is then the infinitesimal generator of the diffusion X_t.

More Assumptions

We here have no intent to pursue the most general assumptions. The following assumptions just make the calculations work.

(iii) $\sigma_{ij}(x)$ and $b_i(x)$, $1 \leq i, j \leq n$, are smooth functions.

(iv) $\lambda\|\theta\| \leq \langle \theta a, \theta \rangle \leq \lambda^{-1}\|\theta\|$, for any $\theta \in \mathbb{R}^n$ and some constant $0 < \lambda < 1$.

Martingale

An adapted process Y_t on a probability space $(\Omega, \mathcal{F}_t, P)$ is called a *martingale* if for any $s < t$, it satisfies

$$E(Y_t|\mathcal{F}_s) = Y_s \quad a.s.$$

Remark

1. Using a property of conditional probability $E(E(X|\mathcal{F})) = EX$, it follows easily that $EY_t = EY_s = EY_0$.
2. $\int_0^t \langle \sigma \nabla_x f, dB_s \rangle$ is a martingale, therefore, $E \int_0^t \langle \sigma \nabla_x f, dB_s \rangle = 0$.

Diffusion Processes and Parabolic Equations — Feynman-Kac Formula

Feynman-Kac Formula Let $f(x), \Psi(x) \in C_0^\infty(\mathbb{R}^n)$ and $X_t, \mathcal{L}_x$ as above. Then the function

$$u(s,x) = E[\Psi(X_T)e^{-\int_s^T c(X_t)dt}|X_s = x] \tag{3}$$

is a bounded $C^\infty([0, T] \times \mathbb{R}^n)$ solution of the partial differential equation subject to a terminal condition

$$\begin{cases} u_s = -\mathcal{L}_x u + cu, & \text{for } s < T, \\ u(T,x) = \Psi(x). \end{cases} \tag{4}$$

Conversely, if $u(s,x)$ is a bounded $C^{1,2}([0, T] \times \mathbb{R}^n)$ solution of the (4), then $u(s,x)$ can be represented as (3).

Transition Density

Assume that X_t has a *transition density* $p(s, x, t, y)$, i.e., for $x, y \in \mathbb{R}^n$ and $s \leq t$,

$$P(X_t = dy | X_s = x) = p(s, x, t, y)dy.$$

Let $c(x) \equiv 0$ in Feynman-Kac formula. Then $E[\Psi(X_T)|X_s = x] = \int \Psi(y)p(s,x,t,y)dy$ as a function of s and x solves (4). In particular, we have

$$\int \Psi(y)\partial_s p(s,x,t,y)dy = -\int \Psi(y)\mathcal{L}_x p(s,x,t,y)dy,$$

for any Ψ. Thus we get the *backwards equation* for X_t,

$$\partial_s p = -\mathcal{L}_x p. \tag{5}$$

Chapman-Kolmogorov Equation For $x, z \in \mathbb{R}^n$ and $s \le t \le r$,

$$\int_{\mathbb{R}^n} p(s, x, t, y) p(t, y, r, z) dy = p(s, x, r, z).$$

Forwards Equation

Taking derivative with respect to t, we get

$$\int_{\mathbb{R}^n} [\partial_t p(s,x,t,y)p(t,y,r,z) + p(s,x,t,y)\partial_t p(t,y,r,z)]dy = 0.$$

using (5) and integration by parts,

$$\int_{\mathbb{R}^n} (\partial_t p(s,x,t,y) - \mathcal{L}_y^* p(s,x,t,y))p(t,y,r,z)dy = 0,$$

from which we can see the *forwards equation*

$$\partial_t p = \mathcal{L}_y^* p. \tag{6}$$

Here
$\mathcal{L}_y^* = \frac{1}{2}\sum_{i,j=1}^n a_{ij}\partial_{y_i y_j} + \sum_{i=1}^n(-b_i + \sum_{j=1}^n \partial_{y_j} a_{ij})\partial_{y_i} + (\frac{1}{2}\sum_{i,j=1}^n \partial_{y_i y_j} a_{ij} - \sum_{i=1}^n \partial_{y_i} b_i)$
is the dual operator of $\mathcal{L}$.

Stopped Diffusion Processes and Dirichlet Conditions — Stopped Diffusion Processes

Given a bounded smooth domain G in $\mathbb{R}^n$ and X_t a diffusion process as in the previous section starting at some point in G, a *stopped diffusion process* Y_t is defined as $Y_t = X_{t \wedge \tau}$ where $\tau = \inf\{t : X_t \notin G\}$.

Intuitively, Y_t is X_t before X_t hits ∂G (the boundary of G) and Y_t does not change after X_t hits ∂G.

Assume that there exists a function $p_1(s, x, t, y)$ such that

$$p_1(s, x, t, y)dy = P(Y_t = dy | Y_s = x) = P(X_t = dy, t < \tau_s | X_s = x),$$

where $\tau_s = \inf\{r > s : X_r \notin G\}$, $s < t$ and $x, y \in G$.

Dirichlet Conditions

Consider the following PDE

$$\begin{cases} \partial_s u = -\mathcal{L}_x u \quad \text{for } x \in G, s < T, \\ u(s,x) = 0 \quad \text{for } x \in \partial G, s < T, \\ u(T,x) = \Psi(x) \quad \text{for } x \in G, \end{cases} \tag{7}$$

where $T > 0$ is a constant, $\mathcal{L}$ is as before and $\Psi(x) \in C_0^\infty(G)$. As in Feynman-Kac formula, the solution can be represented as

$$u(s,x) = E(\Psi(Y_T)1_{\{T<\tau_s\}}|Y_s = x) = \int_G \Psi(y)p_1(s,x,T,y)dy.$$

Backwards Equation

By the previous slide, we get immediately

$$\partial_s p_1 = -\mathcal{L}_x p_1 \quad \text{for } x \in G, s < t,$$

and it is easy to see that

$$p_1(s, x, t, y) = 0 \qquad \text{for } x \in \partial G,$$

since the process Y_t gets stuck on ∂G.

Dynkin's Formula

Dynkin's Formula Let X_t be as in the first section and ζ a stopping time with $E^x[\zeta] < \infty$. f is $C^2(\mathbb{R}^n)$ with compact support. Dynkin's formula holds as follows:

$$E^x[f(X_\zeta)] = f(x) + E^x[\int_0^\zeta \mathcal{L}f(X_s)ds]. \tag{8}$$

Remark This is a result of Itô's formula and martingale optional stopping time theorem.

Forwards Equation

By Dynkin's formula

$$E^x[f(Y_t)] = f(x) + E^x[\int_0^t \mathcal{L}f(Y_s)1_{\{Y_s \in G\}}ds].$$

On the other hand,

$$E^x[f(Y_t)] = \int_G f(y)p_1(0, x, t, y)dy + E^x[f(Y_\tau)1_{\{t \geq \tau\}}].$$

Equate the RHS, take derivative with respect to t, use integration by parts, and then we obtain

Forwards Equation

$$
\begin{aligned}
& \int_G f(y)\partial_t p_1(0,x,t,y)dy + \partial_t(E^x[f(Y_\tau)1_{\{t\geq\tau\}}]) \\
= \quad & \frac{1}{2}\int_{\partial G}[p_1(0,x,t,y)a\nabla_y f(y) - f(y)a\nabla_y p_1(0,x,t,y)]\cdot\overrightarrow{n}\,dS_y \\
& + \int_G f(y)\mathcal{L}_y^* p_1(0,x,t,y))dy \\
& + \int_{\partial G} f(y)p_1(0,x,t,y)\tilde{b}\cdot\overrightarrow{n}\,dS_y
\end{aligned}
$$

Forwards Equation

Compare the integral in G, we obtain

$$\partial_t p_1(0,x,t,y) = \mathcal{L}_y^* p_1(0,x,t,y) \quad \text{for } y \in G,$$

Compare the integral of $\Lambda_y f$ on G, we obtain

$$p_1(0,x,t,y) = 0 \quad \text{for } y \in \partial G.$$

These are the *forwards equation* for Y_t.

Remark We can also check the consistency between backwards/forwards equation and Chapman-Kolmogorov by the same manner.

Reflected Diffusion Process and Neumann Conditions — 1-dim Reflected Brownian Motion

Z_t, *1-dimensional Brownian motion reflected at* 0, can be defined as one of the following three

$$(a)\ |B_t|, \quad (b)\ \max_{s=0}^{t} B_s - B_t, \quad \text{or } (c)\ B_t + L_t^0,$$

where B_t is a standard 1-dimensional Brownian motion, and L_t^0 is the local time of B_t at 0 defined as

$$L_t^0 = \lim_{\epsilon \to 0} \frac{\int_0^t 1_{\{B_s \in (-\epsilon, \epsilon)\}} ds}{2\epsilon}, \ \text{a.s.}$$

Remark $(a), (b), (c)$ are path-wise different, but distributionally the same.

Forwards equation for 1-dim RBM

Let $p(s,x,t,y)$ be the transition density of the standard Brownian motion B_t and $p_2(s,x,t,y)$ be the transition density of RBM Z_t. By definition (a),

$$p_2(s,x,t,y) = \frac{1}{2}p(s,x,t,y) + \frac{1}{2}p(s,x,t,-y).$$

It is well known that $\partial_t p = \frac{1}{2}\Delta_y p$. The forwards equation follows immediately

$$\begin{cases} \partial_t p_2(s,x,t,y) = \frac{1}{2}\Delta_y p_2(s,x,t,y) & \text{for } y > 0 \text{ and } t > s, \\ \partial_y p_2(s,x,t,0) = 0 \end{cases} \tag{9}$$

We here generalize (c) to give a definition of Reflected Diffusion Process. Let G be a bounded smooth domain in $\mathbb{R}^n$ and $\{\overrightarrow{n}(x) : x \in \partial G\}$ be the inner normal vectors on ∂G. Z_t is a *reflecting diffusion process* in G if, for any smooth function $f(t, x)$ with compact support satisfying $\nabla_x f \cdot \overrightarrow{n} \geq 0$ on ∂G, we have

$$f(t, Z_t) - \int_0^t (\partial_u f + \mathcal{L} f)(u, Z_u) \chi_G(Z_u) du \tag{10}$$

is a submartingale.
Then there is a continuous non-decreasing process $\xi(t)$ which increases only when $Z_t \in \partial G$ and has the property that

$$f(t, Z_t) - \int_0^t \chi_G (\partial_u f + \mathcal{L} f)(u, Z_u) du - \int_0^t \nabla_x f \cdot \overrightarrow{n}(u, Z_u) d\xi(u) \tag{11}$$

is a martingale for all smooth function f.

Neumann Conditions

For fixed $T > 0$ and $\Psi \in C^{\infty}(\overline{G})$, consider

$$\begin{cases} \partial_s u = -\mathcal{L}_x u & \text{for } x \in G, t < T \\ \nabla_x u \cdot \overrightarrow{n} = 0 & \text{for } x \in \partial G, t < T \\ u(T, x) = \Psi(x) & \text{for } x \in \overline{G}. \end{cases} \tag{12}$$

The classical solution can be represented as

$$u(s, x) = E[\Psi(Z_T)|Z_s = x] = \int_{\overline{G}} \Psi(y) p_2(s, x, T, y) dy.$$

where $p_2(s, x, t, y)$ is the transition density of Z_t.

Backwards Equation

Substitute the representation back into the equations, and the *backwards equation* for Z_t follows immediately

$$\begin{cases} \partial_s p_2 = -\mathcal{L}_x p_2 & \text{for } x \in G, \\ \nabla_x p_2 \cdot \overrightarrow{n} = 0 & \text{for } x \in \partial G. \end{cases} \tag{13}$$

Forwards Equation

Let $f(t,x) \equiv g(x)$. By definition,

$$g(Z_t) - \int_0^t \chi_G \mathcal{L} g(Z_u) du - \int_0^t \nabla g \cdot \overrightarrow{n}(Z_u) d\xi(u)$$

is a martingale. Also choose g such that $\nabla_x g \cdot \overrightarrow{n} \equiv 0$ for $x \in \partial G$. Then,

$$\begin{aligned} E[g(Z_t)] &= g(x) + E[\int_0^t \chi_G \mathcal{L} g(Z_u) du] \\ &= \int_G g(y) p_2(0,x,t,y) dy \end{aligned}$$

Forwards Equation

Taking derivative with respect to t and using integration by parts, we obtain

$$
\begin{aligned}
& \int_G g(y)\partial_t p_2(0,x,t,y)dy = \int_G \mathcal{L}g(y)p_2(0,x,t,y)dy \\
= \quad & -\frac{1}{2}\int_{\partial G} g(y)a\nabla_y p_2(0,x,t,y)\cdot \overrightarrow{n}\, dS_y + \frac{1}{2}\int_G g(y)\nabla_y(a\nabla_y p_2(0,x,t,y))dy \\
& + \int_{\partial G} g(y)p_2(0,x,t,y)\tilde{b}\cdot \overrightarrow{n}\, dS_y - \int_G g(y)\nabla_y(p_2(0,x,t,y)\tilde{b})dy
\end{aligned}
$$

Forwards Equation

Therefore, we get the *forwards equation* of Z_t

$$\begin{cases} \partial_t p_2 = \mathcal{L}_y^* p_2 \quad \text{for } y \in G, \ t > s \\ (-\frac{1}{2} a \nabla_y p_2 + p_2 \tilde{b}) \cdot \overrightarrow{n} = 0 \ \text{ for } y \in \partial G. \end{cases} \tag{14}$$

Remark

1 This is not Neumann condition, but still consistent with 1-dim RBM ($\mathcal{L} = \Delta, \tilde{b} = 0$).

2 It is easy to check the consistency between backwards/forwards equations and Chapman-Kolmogorov equation.

Diffusion on a Circle and Periodic Boundary Conditions — Periodic Conditions

Consider 1-dimensional parabolic differential equations with periodic boundary conditions:

$$\begin{cases} \partial_t u = -\mathcal{L}u & \text{for } x \in (0,1), t > 0, \\ u(t,0) = u(t,1) & \text{for } t > 0. \end{cases} \tag{15}$$

where $\mathcal{L} = \frac{1}{2}\sigma(x)^2 \frac{d^2}{dx^2} + b(x)\frac{d}{dx}$ for $0 \leq x \leq 1$ and $\sigma(0) = \sigma(1), b(0) = b(1)$.

Extension to Real line

We can extend $\mathcal{L}$ to $\bar{\mathcal{L}} = \frac{1}{2}\bar{\sigma}(x)^2 \frac{d^2}{dx^2} + \bar{b}(x)\frac{d}{dx}$ where $\bar{\sigma}(x) = \sigma(x - \lfloor x \rfloor)$ and $\bar{b}(x) = b(x - \lfloor x \rfloor)$ for any $x \in \mathbb{R}$. If we also let $\bar{u}(t,x) = u(t, x - \lfloor x \rfloor)$, then $\bar{u}$ satisfies

$$\partial_t \bar{u} = -\bar{\mathcal{L}}\bar{u}, \quad \text{for } t > 0, x \in \mathbb{R}.$$

The transition density $\bar{p}(s,x,t,y)$ of the diffusion process $\bar{X}_t$ defined by

$$d\bar{X}_t = \bar{\sigma}(\bar{X}_t)dB_t + \bar{b}(\bar{X}_t)dt$$

satisfies the backwards equation $\partial_s \bar{p} = -\bar{\mathcal{L}}_x \bar{p}$ and the forwards equation $\partial_t \bar{p} = \bar{\mathcal{L}}_y^* \bar{p}$.

Diffusion on a Circle

We are ready to define a *diffusion* R_t *on a circle* as $R_t = \bar{X}_t - \lfloor \bar{X}_t \rfloor$.
The transition density $p_3(s, x, t, y)$ $(0 \leq x, y \leq 1)$ of R_t can be expressed as

$$p_3(s, x, t, y) = \sum_{n=-\infty}^{\infty} \bar{p}(s, x, t, y + n) = \sum_{n=-\infty}^{\infty} \bar{p}(s, x + n, t, y).$$

Backwards/Forwards Equations

The backwards/forwards equations of R_t follow from the ones of $\bar{X}_t$ easily. The *backwards equation* of R_t is

$$\begin{cases} \partial_s p_3 = -\mathcal{L}_x p_3, & \text{for } s < t, x, y \in [0,1], \\ p_3(s,0,t,y) = p_3(s,1,t,y) & \text{for } s < t, y \in [0,1]. \end{cases} \tag{16}$$

The *forwards equation* of R_t is

$$\begin{cases} \partial_t p_3 = \mathcal{L}_y^* p_3, & \text{for } t > s, x, y \in [0,1], \\ p_3(s,x,t,0) = p_3(s,x,t,1) & \text{for } t > s, y \in [0,1]. \end{cases} \tag{17}$$

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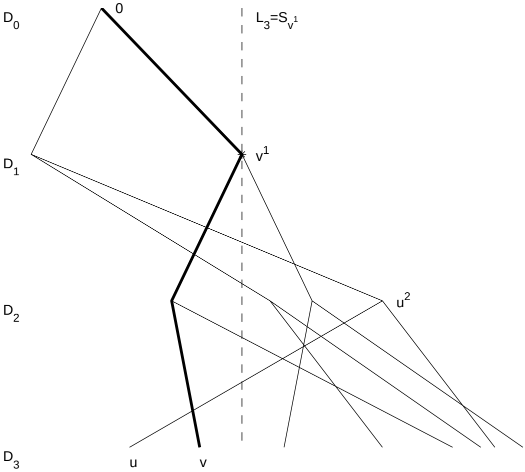
Consistent Minimal Displacement of Branching Random Walk



Dirichlet Problem

Consider $B_t + ct$ killed on the boundary of $(-c_-, c_+)$. Then $p(t,x) = P(B_t + ct = x, B_s + cs \in (-c_-, c_+))$ satisfies a Dirichlet problem, the forward equation of the stopped diffusion processes. Namely,

$$\begin{cases} \partial_t p = \mathcal{L}^* p \\ p(t, -c_-) = p(t, c_+) = 0 \\ p(0,x) = \delta_0 \end{cases}$$

Probability Estimate

Solve the previous PDE by separation of variables, we have

$$p(t,x) = e^{cx} e^{-\frac{c^2}{2}t} \sum_{k=1}^{\infty} d_k e^{-\frac{k^2\pi^2}{2(c_- + c_+)^2}t} \sin(\frac{x + c_-}{c_- + c_+} k\pi)$$

Therefore,

$$P(B_s + cs \in (-c_-, c_+) \text{for} 0 < s < n) = \int_{-c_-}^{c_+} p(n,x)dx \sim e^{-\frac{c^2}{2}n} e^{cc_+} e^{-\frac{\pi^2}{2(c_- + c_+)^2}n}$$

Limiting Behavior of L_n

Based on the previous slide, the expected number of walks staying within $(-c_-, c_+)$ is roughly

$$e^{cc_+} e^{-\frac{\pi^2}{2(c_- + c_+)^2} n},$$

from which we can see that if c_- and c_+ is chosen to be of order $n^{1/3}$, then there is a chance that the positive and negative exponents can be balanced with each other. This is the right order of the L_n.

Actually, we can calculate $\lim_{n\to\infty} \frac{L_3}{n^{1/3}}$ by more delicate observations. But this will not be discussed here.

End of Presentation

Thank you!