Solution to FM 5022 Homework 2

Problem 17.1

From the book (*Page 397 - 398*), we have

$$
\begin{aligned}
\Delta &= \frac{f_{11} - f_{10}}{S_0 u - S_0 d} \\
\Gamma &= \frac{[(f_{22} - f_{21})/(S_0 u^2 - S_0)] - [(f_{21} - f_{20})/(S_0 - S_0 d^2)]}{h} \\
\Theta &= \frac{f_{21} - f_{00}}{2\Delta t}
\end{aligned}
$$

So, Δ, Γ and Θ can be determined by a single binomial tree. ν can be obtained by making a small change in the volatility $\Delta\sigma$, in the volatility and constructing a new binomial tree to obtain a new value of the option.

$$\nu = \frac{f^* - f}{\Delta\sigma}.$$

Rio is calculated by making a small change to the interest rate and recomputing the option price using a new tree.

Problem 17.9

Monte Carlo simulation is to simulate sample values for the derivative security by simulating pathes for the underlying variables. It works by moving forward from time t to time T, so it's impossible to determine whether early exercise is optimal since the range of paths which might occur after time t have not been investigated.

Problem 17.24

There might be many ways to represent the samples, you just need to show one of them.
Let x_1, x_2, x_3 follow standard Normal distribution $N(0,1)$, since $cov(\epsilon_i, \epsilon_j) = \rho_{ij}$, we can define

$$
\begin{aligned}
\epsilon_1 &= x_1 \\
\epsilon_2 &= \rho_{12} x_1 + x_2 \sqrt{1 - \rho_{12}^2} \\
\epsilon_3 &= \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3
\end{aligned}
$$

Then by

$$
\begin{aligned}
cov(\epsilon_2, \epsilon_3) &= \rho_{12}\alpha_1 + \sqrt{1 - \rho_{12}^2}\alpha_2 = \rho_{23} \\
cov(\epsilon_1, \epsilon_3) &= \alpha_1 = \rho_{13} \\
var(\epsilon_3) &= \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1.
\end{aligned}
$$

Then solve for α_1, α_2 and α_3, we have

$$
\begin{aligned}
\alpha_1 &= \rho_{13} \\
\alpha_2 &= \frac{\rho_{23} - \rho_{13}\rho 12}{\sqrt{1 - \rho_{12}^2}} \\
\alpha_3 &= \sqrt{1 - \alpha_1^2 - \alpha_2^2}.
\end{aligned}
$$

Problem 18.1

The standard deviation of the daily change in each asset is $1,000,000 * 1\% = 1000$. The variance is

$$1000^2 + 1000^2 + 2 \times 0.3 \times 1000 \times 1000 = 2,600,000.$$

Then standard deviation is $\sqrt{2,600,000} = 1,612.45$.
The standard deviation of 5-day change is $1,612.45 \times \sqrt{5} = 3,605.55$.
Since $N(-2.33) = 0.01$, the 5-day 99% value-at-risk is $2.33 \times 3,605.55 = 8,401$.

Problem 18.5

Since the factors X, Y are uncorrelated, and $\Delta_X = 6, \Delta_Y = -4$, then the daily variance of the portfolio $\Delta_X X + \Delta_Y Y$ is

$$\Delta_X \sigma_X^2 + \Delta_Y^2 \sigma_Y^2 = 15424.$$

Recall that $N(-1.282) = 0.90$, the 5-day 90% value-at-risk is

$$\sqrt{15424} \times \sqrt{5} \times 1.282 = 356.01.$$

Problem 18.11

(Similar to Appendix)
Since a cash flow of \$ 1000 received in 6.5 years into a position in a 5-year bond and a position in a 7-year bond, then by interpolation

$$6\% + (7\% - 6\%)/(7 - 5) * (6.5 - 5) = 6.75\%.$$

Then the present value of \$ 1000 received in 6.5 years is

$$1000/(1.0675)^6.5 = 654.05.$$

Similarly, the interpolated volatility of a 6.5-year zero coupon bond is therefore 0.56% per day. Suppose we allocate α of the present value to the 5-year bond and $1 - \alpha$ of the present value to the 7-year bond. Then by matching the variances, we have

$$0.56^2 = 0.50^2\alpha^2 + 0.58^2(1 - \alpha)^2 + 2 \times 0.6 \times 0.50 \times 0.58\alpha(1 - \alpha)$$

then $\alpha = 0.074243$.
This means that a value of $0.074243 \times 654.05 = 48.56$ is allocated to the 5-year bond and a value of $0.925757 \times 654.05 = 605.49$ is allocated to the 7-year bond.
Then the equivalent 5-year and 7-year cash flows are $48.56 \times 1.06^5 = 64.98$ and $605.49 \times 1.07^7 = 972.28$.