

Invariant Calculus with VESSIOT

Irina Kogan (Yale University)

August 21, 2002

- *Invariant Euler-Lagrange Equations and the Invariant Variational Bicomplex*, I. Kogan, P.J.Olver.
- MAPLE package VESSIOT for calculus on the jet bundles. Utah State University, I.Anderson et al.

Group action on jet bundles

$$\boxed{G \curvearrowright M \Rightarrow G \curvearrowright \{p\text{-dimensional submanifolds of } M\} \Rightarrow G \curvearrowright J^k(M, p)}$$

Jet spaces: $J^k = J^k(M, p)$ are bundles over M .

The fiber over $z \in M$ consists of the equivalence classes of p -dim. submanifolds with k -th order contact at z .

Local coordinates on J^k :

$\mathbf{x} = (x^1, \dots, x^p)$ – independent variables,

$\mathbf{u} = (u^1, \dots, u^q)$ – dependent variables,

$\mathbf{u}^{(\mathbf{k})} = (u_J^\alpha, \alpha = 1 \dots q, J = \text{multi-index}, |J| \leq k)$ – “derivatives”

$$J = (j_1, \dots, j_p) \quad 0 \leq j_i, \quad |J| = j_1 + \dots + j_p, \quad (1)$$

$$J = (j_1, \dots, j_k)^S, \quad 1 \leq j_i \leq p, \quad |J| = k \quad (2)$$

Projections: $\pi_k^n: J^n \rightarrow J^k$ for $k \leq n$:

$$J^\infty \dots \rightarrow J^k \rightarrow J^{k-1} \rightarrow \dots \rightarrow J^1 \rightarrow J^0 = M$$

Jets of submanifolds:

If $N: u^\alpha = f^\alpha(\mathbf{x})$ then $j^k(N): u^\alpha = f^\alpha(\mathbf{x})$, $u_J^\alpha = \frac{\partial^k f^\alpha}{\partial^{j_1} x^1 \dots \partial^{j_p} x^p}$.

Prolongation of the action:

$$g \cdot j^k(N) = j^k(g \cdot N).$$

Contact structure on $J^\infty(M, p)$

Basis of horizontal sub-bundle	Basis of vertical sub-bundle
--------------------------------	------------------------------

Tangent

total derivatives: $\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q u_i^\alpha \frac{\partial}{\partial u^\alpha}$ $+ \sum_{\alpha, J} u_{Ji}^\alpha \frac{\partial}{\partial u_J^\alpha}$	vertical derivatives $\frac{\partial}{\partial u^\alpha},$ $\frac{\partial}{\partial u_J^\alpha}, \alpha = 1 \dots q$
--	--

Cotangent

horizontal one-forms dx^1, \dots, dx^p	contact one-forms $\theta^\alpha = du^\alpha - \sum_{i=1}^p u_i^\alpha dx^i,$ $\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^p u_{Ji}^\alpha dx^i.$
---	---

Bigrading of exterior differential algebra:

Grading: $\Lambda^* = \bigoplus \Lambda^k$, where $\Lambda^k = \sum \underbrace{\text{one form} \wedge \cdots \wedge \text{one form}}_{k \text{ times}}$.

$$d: \Lambda^k \rightarrow \Lambda^{k+1}, \quad d \circ d = 0 \implies \text{de Rham complex.}$$

Bigrading: $\Lambda^* = \bigoplus \Lambda^{s,t}$, where $\Lambda^{s,t} = \sum \underbrace{\text{hor. 1-form} \wedge \cdots \wedge \text{hor. 1-form}}_{s \text{ times}} \wedge \underbrace{\text{cont. 1-form} \wedge \cdots \wedge \text{cont. 1-form}}_{t \text{ times}}$

$$d: \Lambda^{s,t} \rightarrow \Lambda^{s+1,t} \oplus \Lambda^{s,t+1} \implies d = d_H + d_V$$

$$d^2 = (d_H + d_V)^2 = 0 \implies d_H^2 = 0, d_V^2 = 0, d_H \circ d_V = -d_V \circ d_H$$

↓

Bicomplex

Variational Bicomplex

Tulczyjew, Tsujishita, Vinogradov, Anderson

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \vdots \\
 d_V \uparrow & & d_V \uparrow & & d_V \uparrow & & d_V \uparrow \\
 \Lambda^{0,2} & \xrightarrow{d_H} & \Lambda^{1,2} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Lambda^{p-1,2} & \xrightarrow{d_H} & \Lambda^{p,2} & \xrightarrow{I} & \mathcal{F}^2 \\
 d_V \uparrow & & d_V \uparrow & & \uparrow & & d_V \uparrow & & d_V \uparrow & & \uparrow \\
 \Lambda^{0,1} & \xrightarrow{d_H} & \Lambda^{1,1} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Lambda^{p-1,1} & \xrightarrow{d_H} & \Lambda^{p,1} & \xrightarrow{I} & \mathcal{F}^1 \\
 d_V \uparrow & & d_V \uparrow & & \uparrow & & d_V \uparrow & & d_V \uparrow & & \uparrow \\
 \Lambda^{0,0} & \xrightarrow{d_H} & \Lambda^{1,0} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Lambda^{p-1,0} & \xrightarrow{d_H} & \Lambda^{p,0} & & \nearrow \partial_V
 \end{array}$$

$I: \Lambda^{p,s} \rightarrow \mathcal{F}^s = \Lambda^{p,s}/\text{Im } d_H$ - integration by parts operator

$\partial_v = I \circ d_V$ - variational derivative.

Integration by parts operator, $\lambda \in \Lambda^{p,s}$:

$$I(\lambda) = \sum_{\alpha} \frac{1}{s} \theta^{\alpha} \wedge \left(\left\langle \lambda; \frac{\partial}{\partial u^{\alpha}} \right\rangle + \sum_J \left(-\frac{d}{dx_J} \right) \left\langle \lambda; \frac{\partial}{\partial u_J^{\alpha}} \right\rangle \right).$$

$$\lambda = L(\mathbf{x}, \mathbf{u}^{(n)}) d\mathbf{x} \xrightarrow{d_V} \sum_{\alpha, J} \frac{\partial L}{\partial u_J^{\alpha}} \theta_J^{\alpha} \wedge d\mathbf{x} \xrightarrow{I} \sum_{\alpha, J} E^{\alpha}(L) \theta^{\alpha} \wedge d\mathbf{x}$$

$E^{\alpha}(L) = 0$, $\alpha = 1, \dots, q$ – Euler-Lagrange equations.

$\Lambda^{p,0}$ –Lagrangians $\supset d_H \Lambda^{p-1,0} = (\text{loc}) = \ker \partial_V$ –trivial Lagrangians.

\mathcal{F}^1 –source forms $\supset \partial_V \Lambda^{p,0} = (\text{loc}) = \ker \partial_V$ –Euler-Lagrange forms.

Invariant structure on $J^\infty(M, p)$.

Invariant horizontal basis	Invariant vertical basis
----------------------------	--------------------------

Tangent

invariant total diff. operators $\tilde{\mathcal{D}}_1, \dots, \tilde{\mathcal{D}}_p$ $span \{ \tilde{\mathcal{D}}_i \} = span \{ \frac{d}{dx_i} \}$	invariant vertical diff. operators $\tilde{\mathcal{C}}^\alpha, \tilde{\mathcal{C}}_j^\alpha$ $span \{ \tilde{\mathcal{C}}^\alpha \} \neq span \{ \frac{\partial}{\partial u^\alpha} \}$ unless the action is projective
--	---

Cotangent

invariant “horizontal” one-forms $\tilde{\omega}^1, \dots, \tilde{\omega}^p$ $\tilde{H} = span \{ \tilde{\omega}^i \} \neq H = span \{ dx^i \}$ unless the action is projective	invariant contact one-forms $\tilde{\theta}^\alpha, \tilde{\theta}_j^\alpha$ $span \{ \tilde{\theta}^\alpha, \tilde{\theta}_j^\alpha \} = span \{ \theta^\alpha, \theta_j^\alpha \}$
--	--

A generating set of differential invariants.

Thm. (Tresse) $\exists\{I^1, \dots, I^n\}$ - invariant functions on $J^\infty(M, p)$, s. t. any invariant function

$$I = F\left(\dots, \tilde{\mathcal{D}}_J(I^l), \dots\right)$$

\Downarrow

non-free differential algebra (with syzygies):

$$S\left(\dots, \tilde{\mathcal{D}}_J(I^l), \dots\right) = 0$$

Problems:

- Finding a (minimal) set of generating invariants.
- Finding a (minimal) set of generating syzygies.
- Finding invariant basis for differential forms.

Solution: invariantization via moving frame method (Olver, Fels).

Euclidean group $SE(2) = SO(2) \ltimes R^2$ acts on plane curves

$u = u(x)$:

$$\boxed{x \mapsto \cos(\phi)x - \sin(\phi)u + a, \quad u \mapsto \sin(\phi)x + \cos(\phi)u + b.}$$

- a generating set: $\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$, $\tilde{\mathcal{D}} = \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}} \frac{d}{dx}$,
- any invariant $I = F(\kappa, \kappa_s, \kappa_{ss}, \dots)$, no syzygies,
- $ds = \sqrt{1+u_x^2} dx$ is not an invariant form:
 $g^* ds = ds + \text{contact form.}$
- invariant “horizontal” form: $\tilde{\omega} = ds + \frac{u_x}{\sqrt{1+u_x^2}} \theta$.
- invariant contact forms: $\tilde{\theta} = \frac{\theta}{\sqrt{1+u_x^2}}$, $\tilde{\theta}_1 = \frac{(1+u_x^2)\theta_x - u_x u_{xx} \theta}{(1+u_x^2)^2}, \dots$

$SE(2) = SO(2) \ltimes R^2$ acts on surfaces $u = u(x, y)$ in \mathbb{R}^3 :

$$\boxed{x \mapsto \cos(\phi)x - \sin(\phi)y + a, \quad y \mapsto \sin(\phi)x + \cos(\phi)y + b, \quad u \mapsto u}$$

- a minimal generating set: $I = u$, $I_{xx} = \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{u_x^2 + u_y^2}$,

$$\tilde{\mathcal{D}}_1 = \frac{1}{\sqrt{u_x^2 + u_y^2}} \left(u_y \frac{d}{dx} - u_x \frac{d}{dy} \right), \quad \tilde{\mathcal{D}}_2 = \frac{1}{\sqrt{u_x^2 + u_y^2}} \left(u_x \frac{d}{dx} + u_y \frac{d}{dy} \right).$$

- generating syzygies:

$$\tilde{\mathcal{D}}_1 I = 0,$$

$$I_{xx}^2 - I_{xx} \tilde{\mathcal{D}}_2 I + 2 \left(\tilde{\mathcal{D}}_1 \tilde{\mathcal{D}}_2 I \right)^2 - (\tilde{\mathcal{D}}_2 I) \left(\tilde{\mathcal{D}}_1^2 \tilde{\mathcal{D}}_2 I - \tilde{\mathcal{D}}_2 I_{xx} \right) = 0,$$

$$\left(\tilde{\mathcal{D}}_1 \tilde{\mathcal{D}}_2 I \right) \left(I_{xx} + \tilde{\mathcal{D}}_2^2 I \right) + \left(\tilde{\mathcal{D}}_2 I \right) \left(\tilde{\mathcal{D}}_2 \tilde{\mathcal{D}}_1 \tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1 \tilde{\mathcal{D}}_2^2 \right) I = 0$$

- $[\tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2] = \frac{I_{xx}}{\tilde{\mathcal{D}}_2 I} \tilde{\mathcal{D}}_1 + \frac{\tilde{\mathcal{D}}_1 \tilde{\mathcal{D}}_2 I}{\tilde{\mathcal{D}}_2 I} \tilde{\mathcal{D}}_2$

- invariant horizontal forms:

$$\tilde{\omega}_1 = \frac{1}{\sqrt{u_x^2 + u_y^2}} (u_y dx - u_x dy), \quad \tilde{\omega}_2 = \frac{1}{\sqrt{u_x^2 + u_y^2}} (u_x dx + u_y dy).$$

Application: symmetry reduction.

Thm. (S.Lie)

- (almost) any symmetric system of differential equations can be written in terms of differential invariants.
- (almost) any symmetric variational problem can be written in terms of differential invariants and invariant differential forms.

Invariant Variational Problems:

$$\text{for } \forall g \in G: \mathcal{L}[u] = \int_{\Omega} L(\mathbf{x}, \mathbf{u}^{(n)}) d\mathbf{x} = \int_{g \cdot \Omega} g^* [L(\mathbf{x}, \mathbf{u}^{(n)}) d\mathbf{x}]$$

$$\mathcal{L}[u] = \int L(\mathbf{x}, \mathbf{u}^{(n)}) d\mathbf{x} \quad \Leftrightarrow \quad \mathcal{L}[u] = \int \tilde{L}(\dots, \tilde{\mathcal{D}}_J I^l, \dots) \tilde{\omega}_1 \wedge \dots \wedge \tilde{\omega}_p$$

$$\downarrow E^\alpha = \sum_J \left(-\frac{d}{dx_J} \right) \frac{\partial}{\partial u_J^\alpha} \quad \downarrow \quad ?$$

$$\begin{aligned} \Delta^\alpha(\mathbf{x}, \mathbf{u}^{(n)}) = 0, & \quad \Leftrightarrow \quad \tilde{\Delta}^\alpha \left(\tilde{\mathcal{D}}_{J_1} I^1, \dots, \tilde{\mathcal{D}}_{J_N} I^N \right) = 0, \\ \alpha = 1, \dots, q. & \quad \quad \quad \alpha = 1, \dots, N. \end{aligned}$$

problem and examples: P. Griffiths, I. Anderson;

general formula: P. Olver, I. Kogan; V. Itskov.

Euclidean invariant Lagrangian:

$$\mathcal{L}[u] = \int \frac{u_{xx}^2}{2(1+u_x^2)^{5/2}} dx \quad \Longleftrightarrow \quad \int \frac{1}{2} \kappa^2 ds$$

$$\downarrow E = \left(\frac{d}{dx}\right)^2 \frac{\partial}{\partial u_{xx}} - \left(\frac{d}{dx}\right) \frac{\partial}{\partial u_x} \quad \downarrow \quad ?$$

$$\Delta = 0 \quad \Longleftrightarrow \quad \kappa_{ss} + \frac{1}{2} \kappa^3 = 0$$

$$\Delta = \frac{1}{2} \frac{30 u_2^3 u_1^2 - 5 u_2^3 - 20 u_2 u_1 u_3 - 20 u_2 u_1^3 u_3 + 2 u_4 + 4 u_4 u_1^2 + 2 u_4 u_1^4}{(1 + u_1^2)^{(9/2)}}.$$

Invariantization via moving frames.

(Fels, Olver)

Thm. (Ovsyannikov) The action of G is loc. effective on open subsets

↓

$\exists n \leq \dim G$ (the order of stabilization), s. t. the prolonged action is loc. free on an open dense $\mathcal{V}^n \subset J^n$.

↓

- $\dim \mathcal{O}_{\mathbf{z}^{(n)}} = \dim G, \quad \forall \mathbf{z}^{(n)} \in \mathcal{V}^n$
- \exists (local) cross-section \mathcal{K}^n :
 - $\text{codim } \mathcal{K}^n = \dim \mathcal{O}_{\mathbf{z}^{(n)}}$,
 - \mathcal{K}^n is transversal to $\mathcal{O}_{\mathbf{z}^{(n)}}$ for $\forall \mathbf{z}^{(n)} \in \mathcal{V}^n$,
 - $\mathcal{K}^n \cap \mathcal{O}_{\mathbf{z}^{(n)}}$ consists of at most one point \iff regular action.

Define $\rho : \mathcal{V}^n \rightarrow G$, by the condition $\rho(\mathbf{z}^{(n)}) \cdot \mathbf{z}^{(n)} \in \mathcal{K}^n$

$$\boxed{\rho(g \cdot \mathbf{z}^{(n)})g \cdot \mathbf{z}^{(n)} = \rho(z) \cdot \mathbf{z}^{(n)}; \text{ freeness} \implies \rho(g \cdot \mathbf{z}^{(n)}) = \rho(\mathbf{z}^{(n)})g^{-1}}$$

\Downarrow

ρ is a G -equivariant map.

c.-s. and ρ can be lifted to J^k for $k > n$:

$$\rho(\mathbf{z}^{(k)}) = \rho\left(\pi_n^k(\mathbf{z}^{(k)})\right) \in G, \quad \mathcal{K}^k = \left(\pi_n^k\right)^{-1} \mathcal{K}^n \subset J^k, \quad \mathcal{K} = \left(\pi_n^\infty\right)^{-1} \mathcal{K}^n$$

Projection on the c.-s. $\gamma(\mathbf{z}) = \rho(\mathbf{z}) \cdot \mathbf{z} : J^\infty \rightarrow \mathcal{K}$ for $\mathbf{z} \in J^\infty$.

Generalized definition of a moving frame.

$G \curvearrowright M$; a moving frame is an equivariant smooth map $\rho: M \rightarrow G$.

$$\begin{array}{ccc} G & \xrightarrow{R_{g^{-1}}} & G \\ \rho \uparrow & & \uparrow \rho \\ M & \xrightarrow{g} & M \end{array}$$

Thm. \exists moving frame \Leftrightarrow the action is regular and free.

Invariantization ι :

- **functions** $f: J^\infty \rightarrow \mathbb{R}$: $\boxed{\iota(f)(\mathbf{z}) = f(\rho(\mathbf{z}) \cdot \mathbf{z})}$

$\{\mathcal{X}^i = \iota(x^i), I_j^\alpha = \iota(u_j^\alpha)\} \supset$ complete set of functionally indep. inv.

- **differential forms**: $\boxed{\iota(\Omega)|_{\mathbf{z}} = [\rho(\mathbf{z})]^* (\Omega|_{\rho(\mathbf{z}) \cdot \mathbf{z}})}$

ω – 1-form, v – vector field: $\langle \iota(\omega)|_{\mathbf{z}}; v|_{\mathbf{z}} \rangle = \langle \omega|_{\rho(\mathbf{z}) \cdot \mathbf{z}}; [\rho_{\mathbf{z}}]_* (v|_{\mathbf{z}}) \rangle$

$\tilde{\theta}_j^\alpha = \iota(\theta_j^\alpha)$, $\alpha = 1, \dots, q$ - inv. basis of contact 1-forms.

$\tilde{\omega}^i = \iota(dx^i)$, $i = 1, \dots, p$ - inv. basis of “horizontal” 1-forms.

- 1-st order inv. **diff. operators** = inv. vector fields.

Basis – dual to $\{\tilde{\omega}^i, \tilde{\theta}_j^\alpha\}$.

Basis of total inv. operators $\langle \tilde{\omega}^i; \tilde{\mathcal{D}}_j \rangle = \delta_j^i$.

Invariantization steps:

1. Pull back a form (a function) by the action of $g \in G$,
2. Replace g with $\rho(\mathbf{z})$.

Euclidean action on the plane curves:

$$x \mapsto \bar{x} = \cos(\phi)x - \sin(\phi)u + a, \quad u \mapsto \bar{u} = \sin(\phi)x + \cos(\phi)u + b$$

$$u_x \mapsto \bar{u}_x = \frac{\sin(\phi) + \cos(\phi)u_x}{\cos(\phi) - \sin(\phi)u_x}, \quad u_{xx} \mapsto \bar{u}_{xx} = \frac{u_{xx}}{(\cos(\phi) - \sin(\phi)u_x)^3}$$

loc. free and transitive on $J^1 \Rightarrow$ c.-s. $\mathcal{K}^1 = \{x = 0, u = 0, u_x = 0\}$

↓

moving frame is the solution of $\bar{x} = 0, \bar{u} = 0, \bar{u}_x = 0$:

$$\phi = -\arctan(u_x), \quad a = -\frac{x + u_x u}{\sqrt{u_x^2 + 1}}, \quad b = \frac{u_x x - u}{\sqrt{u_x^2 + 1}}$$

↓

$$g^* u_{xx} = \bar{u}_{xx} = \frac{u_{xx}}{(\cos(\phi) - \sin(\phi)u_x)^3} \Rightarrow I_2 = \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

$$g^*(dx) = \cos(\phi)dx - \sin(\phi)du \Rightarrow \tilde{\omega} = \frac{dx + u_x du}{\sqrt{1+u_x^2}} = \sqrt{1+u_x^2} dx + \frac{u_x}{\sqrt{1+u_x^2}} \theta.$$

Structure equations.

$$d\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^r \rho^*(\mu^\kappa) \wedge \iota[\mathbf{v}_\kappa(\Omega)]$$

$\mathbf{v}_1, \dots, \mathbf{v}_r$ – basis for infinitesimal generators of G -action

$\mathbf{v}_\kappa(\Omega)$ – Lie derivative of Ω with respect to \mathbf{v}_κ .

μ_1, \dots, μ_r – dual basis of invariant differential forms on G

Invariant bigrading $\Lambda^* = \bigoplus \tilde{\Lambda}^{s,t}$.

Unless the action is projective $\tilde{\Lambda}^{s,t} \neq \Lambda^{s,t}$ and for $s \geq 1$:

$$d: \tilde{\Lambda}^{s,t} \rightarrow \tilde{\Lambda}^{s+1,t} \oplus \tilde{\Lambda}^{s,t+1} \oplus \tilde{\Lambda}^{s-1,t+2} \Rightarrow d = d_{\tilde{H}} + d_{\tilde{V}} + d_W$$

$$d^2 = (d_{\tilde{H}} + d_{\tilde{V}} + d_W)^2 = 0$$

$$d_{\tilde{H}}^2 = 0, \quad d_{\tilde{V}}^2 + d_{\tilde{H}}d_W + d_Wd_{\tilde{H}} = 0, \quad d_{\tilde{H}} \circ d_{\tilde{V}} = -d_{\tilde{V}} \circ d_{\tilde{H}}, \quad d_W^2 = 0$$

Algorithm for computing structure equations.

input: – dependent and independent variables $[x^1, \dots, x^p], [u^1, \dots, u^q]$,
– infinitesimal generators $V = [\mathbf{v}_1, \dots, \mathbf{v}_r]$,
– coordinate cross-section $\mathcal{K} = [z^{i_1} = c^1, \dots, z^{i_r} = c^r]$,
– order of prolongation k .

output: – $d_{\tilde{V}} \tilde{\omega}^i, \quad d_{\tilde{H}} \tilde{\omega}^i = \sum c_{jk}^i \tilde{\omega}^j \wedge \tilde{\omega}^k, \quad [\tilde{\mathcal{D}}_k, \tilde{\mathcal{D}}_j] = - \sum c_{jk}^i \tilde{\mathcal{D}}_i,$
– $d_{\tilde{V}} \tilde{\theta}_J^\alpha, \quad d_{\tilde{H}} \tilde{\theta}_J^\alpha, \quad \tilde{\mathcal{D}}_i (\tilde{\theta}_J^\alpha),$
– $d_{\tilde{V}} \mathcal{X}^i, \quad d_{\tilde{V}} I_J^\alpha, \quad d_{\tilde{H}} \mathcal{X}^i, \quad d_{\tilde{H}} I_J^\alpha, \quad \tilde{\mathcal{D}}_j (\mathcal{X}^i), \quad \tilde{\mathcal{D}}_j (I_J^\alpha).$

operations: – differentiation and linear algebra.

comments: – explicit formulae for moving frame, invariants and invariant forms are not needed.

I. Preliminary computations:

- V^k := k -th prolongations of vectors V ; $d := \dim J^k = p + qC_{p+k}^p$,
 $L := (r \times d)$ -matrix of the coefficients of V^k ,
 $L_\xi := (r \times p)$ -submatrix of L , corresponding to x^i ,
 $L_\phi := (r \times (d - p))$ -submatrix of L , corresponding to u_j^α
 $L_c := r \times (d - p)$ -matrix of contact forms such that

$$\{L_c\}_{J;\kappa}^\alpha = \mathbf{v}_\kappa(\theta_J^\alpha) = d_V \phi_{J;\kappa}^\alpha - \sum_{i=1}^p u_{J_i}^\alpha d_V \xi_\kappa^i$$

 $L_r := (r \times r)$ -submatrix of L of columns i_1, \dots, i_r defining \mathcal{K} .
- **if** $\det L_r|_{\mathcal{K}} = 0$ **then** *cross-section is invalid*; stop.
else define the process of invariantization:
 $\iota(\text{expression}) := \text{subs} \left(\{dx^i = \tilde{\omega}^i, \theta_J^\alpha = \tilde{\theta}_J^\alpha, z^{i_1} = c^1, \dots, z^{i_r} = c^r, \right.$
for the rest of variables $x^i = \mathcal{X}^i, u_j^\alpha = I_j^\alpha\}, \text{expression}$)

- matrices of variables and forms:

$K := [z^{i_1}, \dots, z^{i_r}] (1 \times r)$ - is matrix of normalized coordinates.

$Z := (1 \times d)$ - matrix of all coordinates,

$I := \iota(Z)$ is $(1 \times d)$ -matrix of invariants.

$\tilde{\omega} := [\tilde{\omega}^1, \dots, \tilde{\omega}^p]$ – invariant horizontal forms

$\theta := [\theta_J^\alpha, \alpha = 1, \dots, q, |J| \leq k]$ – $(1 \times d)$ -matrix of contact forms.

$\tilde{\theta} := [\tilde{\theta}_J^\alpha, \alpha = 1, \dots, q, |J| \leq k] = \iota(\theta)$ -invariant contact forms.

II. Structure equations:

- $d_{\tilde{H}} \tilde{\omega} = \iota [(d_H K) \wedge L_r^{-1} (d_H L_\xi)]$
- $d_{\tilde{V}} \tilde{\omega} = \iota [(d_H K) \wedge L_r^{-1} (d_V L_\xi) + (d_V K) \wedge L_r^{-1} (d_H L_\xi)]$
- $d_{\tilde{H}} \tilde{\theta} = \iota [d_H \theta - (d_H K) \wedge L_r^{-1} L_c] = \sum_i \tilde{\mathcal{D}}_i(\tilde{\theta}) \wedge \tilde{\omega}^i$
- $d_{\tilde{V}} \tilde{\theta} = \iota [(d_V K) \wedge L_r^{-1} L_c]$

III. Differentials of fundamental invariants.

$$\begin{aligned}
 d_{\tilde{H}} I &= \iota [d_H Z - (d_H K) L_r^{-1} L] = \sum_i \tilde{\mathcal{D}}_i(I) \wedge \tilde{\omega}^i \\
 d_{\tilde{V}} I &= \iota [d_V Z - (d_V K) L_r^{-1} L]
 \end{aligned}$$

SE(2) acting on curves in R^2 .

```
> restart;with(Vessiot):read(IVBproc);with(linalg):  
> coord_frame([x],[u],fr1);
```

frame name : fr1

```
> vectE:=[[1,0],[0,1],[-u[0],x]];  
sectionE:=[x=0,u[0]=0,u[1]=0];
```

vectE := [[1, 0], [0, 1], [-u₀, x]]

sectionE := [x = 0, u₀ = 0, u₁ = 0]

```
> ivb(vectE,sectionE,3);
```

INVARIANT HORIZONTAL FORMS :

invariant vertical differentials

$$[dv Dx = u_2 Dx \wedge Cu_{[0]}]$$

invariant horizontal differentials

[0]

INVARIANT CONTACT FORMS

invariant vertical differentials :

$$\left[\begin{array}{l} dv Cu_{[0]} = u_2 Cu_{[0]} \wedge Cu_{[1]}, \\ dv Cu_{[1]} = u_3 Cu_{[0]} \wedge Cu_{[1]}, \\ dv Cu_{[2]} = u_4 Cu_{[0]} \wedge Cu_{[1]}, \\ dv Cu_{[3]} = u_5 Cu_{[0]} \wedge Cu_{[1]} - 6 u_2 Cu_{[1]} \wedge Cu_{[2]} \end{array} \right]$$

invariant horizontal differentials :

$$\left[\begin{array}{l} dh Cu_{[0]} = -u_2^2 Dx \wedge Cu_{[0]} + Dx \wedge Cu_{[1]}, \\ dh Cu_{[1]} = -u_3 u_2 Dx \wedge Cu_{[0]} + Dx \wedge Cu_{[2]}, \\ dh Cu_{[2]} = -u_4 u_2 Dx \wedge Cu_{[0]} - 3 u_2^2 Dx \wedge Cu_{[1]} + Dx \wedge Cu_{[3]} \end{array} \right]$$

Lie derivatives w.r.t inv. diff. operators :

$$\left[\begin{array}{l} D_1 Cu_{[0]} = -u_2^2 Cu_{[0]} + Cu_{[1]} \\ D_1 Cu_{[1]} = -u_3 u_2 Cu_{[0]} + Cu_{[2]} \\ D_1 Cu_{[2]} = -u_4 u_2 Cu_{[0]} - 3 u_2^2 Cu_{[1]} + Cu_{[3]} \\ D_1 Cu_{[3]} = -u_5 u_2 Cu_{[0]} - 4 u_3 u_2 Cu_{[1]} - 6 u_2^2 Cu_{[2]} + Cu_{[4]} \end{array} \right]$$

DIFFERENTIALS OF THE FUNDAMENTAL INVARIANTS

vertical

$$\left[\begin{array}{l} dv u_2 = Cu_{[2]} \\ dv u_3 = -3 u_2^2 Cu_{[1]} + Cu_{[3]} \end{array} \right]$$

horizontal

$$\left[\begin{array}{l} dh u_2 = u_3 Dx \\ dh u_3 = (-3 u_2^3 + u_4) Dx \end{array} \right]$$

Lie derivatives w.r.t inv. diff. operators :

$$\left[\begin{array}{l} D_1 u_2 = u_3 \\ D_1 u_3 = -3 u_2^3 + u_4 \end{array} \right]$$

SE(2) acting on independent variables for surfaces in \mathbb{R}^3 .

```
> coord_frame([x,y],[u],fr1):
> vectors:=[[1,0,0],[0,1,0],[-y,x,0]];
section:=[x=0,y=0,u[1,0]=0];
> ivb(vectors,section,2);
```

INVARIANT HORIZONTAL FORMS :

invariant vertical differentials

$$\left[dv Dx = \frac{Dy \wedge Cu_{[1,0]}}{u_{0,1}}, dv Dy = -\frac{Dx \wedge Cu_{[1,0]}}{u_{0,1}} \right]$$

invariant horizontal differentials

$$\left[dh Dx = -\frac{u_{2,0} Dx \wedge Dy}{u_{0,1}}, dh Dy = -\frac{u_{1,1} Dx \wedge Dy}{u_{0,1}} \right]$$

Lie algebra of invariant differential operators :

```
> invD := [[Lie_alg, InvH, [p]],  
[seq(invDstr[t],t=1..p*(p^2-p)/2)]];  
>  
Lie_alg_init(invD,[Delta],[omega]):frameInformation():  
invD := [[Lie_alg, InvH, [2]], [[1, 2, 1],  $\frac{u_{2,0}}{u_{0,1}}$ ], [[1, 2, 2],  $\frac{u_{1,1}}{u_{0,1}}$ ]]  
frame name : InvH  
library name : Koszul_frame  
Frame Jet Variables :  
[z1, z2]  
Frame Labels  
[Δ1 , Δ2 ]
```

CoFrame Labels

$[\omega_1, \omega_2]$

Horizontal Coframe Labels

\square

Vertical Coframe Labels

\square

$$ext_d :, \omega_1, -\frac{u_{2,0} \omega_1 \wedge \omega_2}{u_{0,1}}$$

$$ext_d :, \omega_2, -\frac{u_{1,1} \omega_1 \wedge \omega_2}{u_{0,1}}$$

Lie derivatives of invariants w.r.t inv. diff. operators :

$$\left[\begin{array}{ll} D_1 u_{0,0} = 0 & D_2 u_{0,0} = u_{0,1} \\ D_1 u_{0,1} = u_{1,1} & D_2 u_{0,1} = u_{0,2} \\ D_1 u_{2,0} = -2 \frac{u_{1,1} u_{2,0}}{u_{0,1}} + u_{3,0} & D_2 u_{2,0} = -2 \frac{u_{1,1}^2}{u_{0,1}} + u_{2,1} \\ D_1 u_{1,1} = \frac{u_{2,0}^2 - u_{2,0} u_{0,2}}{u_{0,1}} + u_{2,1} & D_2 u_{1,1} = \frac{u_{1,1} (u_{2,0} - u_{0,2})}{u_{0,1}} + u_{1,2} \\ D_1 u_{0,2} = 2 \frac{u_{1,1} u_{2,0}}{u_{0,1}} + u_{1,2} & D_2 u_{0,2} = 2 \frac{u_{1,1}^2}{u_{0,1}} + u_{0,3} \end{array} \right]$$

syzygies :

$$D_2 u_{2,0} - D_1 u_{1,1} = \frac{2 u_{1,1}^2 + u_{2,0}^2 - u_{2,0} u_{0,2}}{u_{0,1}}$$

$$D_2 u_{1,1} - D_1 u_{0,2} = -\frac{u_{1,1} (u_{2,0} + u_{0,2})}{u_{0,1}}$$

Invariant integration by parts for plane curves.

$G \curvearrowright J(\mathbb{R}^2, 1)$, $\tilde{\omega}$ – invariant differential form, $\tilde{\mathcal{D}}$ – dual invariant differential operator, κ – generating invariant, $\kappa_i = \left(\tilde{\mathcal{D}}\right)^i \kappa$.

$$\tilde{\lambda} = \tilde{L}(\kappa, \kappa_1, \dots, \kappa_n) \tilde{\omega}$$

$$d_{\tilde{\mathcal{V}}}\tilde{\lambda} = d_{\tilde{\mathcal{V}}}\tilde{L} \wedge \tilde{\omega} + \tilde{L} d_{\tilde{\mathcal{V}}}\tilde{\omega} = \sum_{i=0}^n \frac{\partial \tilde{L}}{\partial \kappa_i} (d_{\tilde{\mathcal{V}}}\kappa_i) \wedge \tilde{\omega} + \tilde{L} d_{\tilde{\mathcal{V}}}\tilde{\omega} \equiv (\text{mod } d_{\tilde{H}})$$

$$(d_{\tilde{\mathcal{V}}}\kappa_i) \wedge \tilde{\omega} = \left(d_{\tilde{\mathcal{V}}}\tilde{\mathcal{D}}\kappa_{i-1}\right) \wedge \tilde{\omega} = (d_{\tilde{\mathcal{V}}}d_{\tilde{H}}\kappa_{i-1}) - \kappa_i d_{\tilde{\mathcal{V}}}\tilde{\omega} = -(d_{\tilde{H}}d_{\tilde{\mathcal{V}}}\kappa_{i-1}) - \kappa_i d_{\tilde{\mathcal{V}}}\tilde{\omega}$$

$$\equiv \sum_{i=0}^n d_{\tilde{H}} \left(\frac{\partial \tilde{L}}{\partial \kappa_i}\right) (d_{\tilde{\mathcal{V}}}\kappa_{i-1}) - \left(\sum_{i=0}^n \frac{\partial \tilde{L}}{\partial \kappa_i} \kappa_i - \tilde{L}\right) d_{\tilde{\mathcal{V}}}\tilde{\omega}$$

$$= \sum_{i=0}^n \left(-\tilde{\mathcal{D}}\right) \left(\frac{\partial \tilde{L}}{\partial \kappa_i}\right) (d_{\tilde{\mathcal{V}}}\kappa_{i-1}) \wedge \tilde{\omega} - (\dots) d_{\tilde{\mathcal{V}}}\tilde{\omega}.$$

repeat!

$$\boxed{d_{\tilde{V}} \tilde{\lambda} \equiv \mathcal{E}(\tilde{L}) d_{\tilde{V}} \kappa \wedge \tilde{\omega} - \mathcal{H}(\tilde{L}) d_{\tilde{V}} \tilde{\omega}.}$$

$$\mathcal{E}(\tilde{L}) = \sum_{i=0}^n (-\tilde{\mathcal{D}})^i \frac{\partial \tilde{L}}{\partial \kappa_i}, \quad \mathcal{H}(\tilde{L}) = \sum_{i>j \geq 0}^n \kappa_{i-j} (-\tilde{\mathcal{D}})^j \frac{\partial \tilde{L}}{\partial \kappa_i} - \tilde{L}.$$

$$\lambda = L(x, u, u_1, \dots, u_m) dx \leftrightarrow \tilde{\lambda} = \tilde{L}(\kappa, \kappa_1, \dots, \kappa_n) \tilde{\omega}$$

$$d_V(dx) = 0 \quad d_{\tilde{V}}(\tilde{\omega}) = \mathcal{B}(\tilde{\theta}_0) \wedge \tilde{\omega}$$

$$d_V(u) = \theta_0 \quad d_{\tilde{V}}(\kappa) = \mathcal{A}(\tilde{\theta}_0)$$

$$\boxed{d_{\tilde{V}} \tilde{\lambda} \equiv [\mathcal{A}^* \mathcal{E}(\tilde{L}) - \mathcal{B}^* \mathcal{H}(\tilde{L})] \tilde{\theta}_0 \wedge \tilde{\omega}.}$$

Euclidean group $SE(2) = SO(2) \ltimes R^2$ acts on plane curves

$u = u(x)$:

$$\boxed{x \mapsto \cos(\phi)x - \sin(\phi)u + a, \quad u \mapsto \sin(\phi)x + \cos(\phi)u + b.}$$

- $d_{\tilde{V}}\kappa = \tilde{\theta}_2 = \frac{d}{ds}\theta_1 + \kappa^2\tilde{\theta}_0 = \left[\left(\frac{d}{ds}\right)^2 + \kappa^2\right]\tilde{\theta}_0,$

- $d_{\tilde{V}}\tilde{\omega} = -\kappa\tilde{\theta}_0 \wedge \tilde{\omega}.$

$$\boxed{d_{\tilde{V}}\lambda \equiv \left\{ \left[\left(\frac{d}{ds}\right)^2 + \kappa^2\right] \mathcal{E}(\tilde{L}) + \kappa\mathcal{H}(\tilde{L}) \right\} \tilde{\theta}_0 \wedge \tilde{\omega}}$$