Quiz 1 Solutions

Math 1572H, 26 January 2006

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1. [4 points, 1 per part] For parts (a) through (d), write down what you feel is the most logical substituion to use in order to do the integral. For example, your answer may be $u = 16x^2$. You are *not* expected to do the itegration.

(a) $\int x\sqrt{x^2+1}\,dx$ (b) $\int \tan^2 x \cos x\,dx$ (c) $\int \frac{x^3}{\sqrt{9-x^2}}\,dx$ (d) $\int \frac{x^3}{\sqrt{9+x^2}}\,dx$

Solution: For part (a), the easiest substitution to use would be $u = x^2 + 1$. For part (b), the integrand can be simplified:

$$\tan^2 x \cos x = \frac{\sin^2 x}{\cos^2 x}\cos x = \frac{\sin^2 x}{1-\sin^2 x}\cos x.$$

In light of this simplification, we can pick $u = \sin x$. Parts (c) and (d) are both trig substitution integrals, where we pick $x = 3\sin\theta$ for (c) and $x = 3\tan\theta$ for part (d). Other answers than the ones given here are possible.

2. [2 points] Evaluate the integral

$$\int \sin^2 x \cos^3 x\,dx.$$

Solution: We split the cosine term into two parts, so that the integral reads

$$\int \sin^2 x \cos^2 x \cos x\,dx.$$

Then, we use the identity $\sin^2 x + \cos^2 x = 1$ to rewrite the cosine term in terms of sines:

$$\int \sin^2 x(1-\sin^2 x)\cos x\,dx.$$

Making the substituion $u = \sin x$ yields $du = \cos x\,dx$, which translates the integral to

$$\int u^2(1-u^2)\,du.$$

The antiderivative is easily found to be $u^3/3 - u^5/5$. Translating back to x, we get an answer of

$$\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

3. [4 points] Evaluate the integral

$$\int \frac{(x+3)\,dx}{\sqrt{-x^2+4x+5}}.$$

Solution: We first complete the square underneath the square root in the denominator to get

$$-x^2 + 4x + 5 = -(x^2 - 4x) + 5 = -(x^2 + 4x + 4) + 4 + 5 = -(x + 2)^2 + 9.$$

This allows us to rewrite the integral as

$$\int \frac{x + 3}{\sqrt{9 - (x + 2)^2}}\, dx.$$

We let $x + 2 = 3\sin\theta$. This gives $dx = 3\cos\theta\, d\theta$ and $\theta = \arcsin((x + 2)/3)$. We can then translate the integral to

$$\int \frac{3\sin\theta + 1}{\sqrt{9 - 9\sin^2\theta}} 3\cos\theta\, d\theta.$$

We use the indentity $1 - \sin^2\theta = \cos^2\theta$. Doing the integral gives

$$\begin{aligned}
\int \frac{3\sin\theta + 1}{\sqrt{9 - 9\sin^2\theta}} 3\cos\theta\, d\theta &= \int \frac{3\sin\theta + 1}{3\cos\theta} 3\cos\theta\, d\theta \\
&= \int (3\sin\theta + 1)\, d\theta \\
&= -3\cos\theta + \theta + C \\
&= -3\cos(\arcsin((x + 2)/3)) + \arcsin((x + 2)/3) + C \\
&= -\sqrt{9 - (x + 2)^2} + \arcsin((x + 2)/3) + C
\end{aligned}$$