Math 8652, Homework set #4 Solutions

1. In class it was shown that

$$\int y\, p_n(X_n, dy) = E[X_{n+1}|\mathcal{F}_n],$$

thus $d_n(X_n) = E[X_{n+1}|\mathcal{F}_n] - X_n$ and is, in particular, measurable with respect to $\mathcal{F}_n$. Therefore M_n is $\mathcal{F}_n$- measurable.

Since

$$|d_i(X_i)| \leq E[|X_{i+1}||\mathcal{F}_n] + |X_i|,$$

we have that $E[|M_n|] < \infty$ for each n provided $E[|X_i|]$ is finite for each i. Then, since $d_{i-1}(X_{i-1})$ is $\mathcal{F}_{i-1}$ measurable, we have:

$$\begin{aligned}
E[M_{n+1}|\mathcal{F}_n] &= E\left[X_{n+1} - \sum_{i=0}^{n+1} d_{i-1}(X_{i-1}) \middle| \mathcal{F}_n \right] \\
&= E[X_{n+1}|\mathcal{F}_n] - \sum_{i=0}^{n+1} E[d_{i-1}(X_{i-1})|\mathcal{F}_n] \\
&= E[X_{n+1}|\mathcal{F}_n] - \sum_{i=0}^{n+1} d_{i-1}(X_{i-1}) \\
&= E[X_{n+1}|\mathcal{F}_n] - d_n(X_n) - \sum_{i=0}^{n} d_{i-1}(X_{i-1}) \\
&= E[X_{n+1}|\mathcal{F}_n] - E[X_{n+1}|\mathcal{F}_n] + X_n - \sum_{i=0}^{n} d_{i-1}(X_{i-1}) \\
&= X_n - \sum_{i=0}^{n} d_{i-1}(X_{i-1}) \\
&= M_n.
\end{aligned}$$

Thus, M_n is a martingale. To show $M_n/n \to 0$ a.s., it suffices to show that the martingale M_n has bounded increments. Observe that under the assumption $|X_{n+1} - X_n| \leq 1$ we have

$$\begin{aligned}
|M_{n+1} - M_n| &= |X_{n+1} - X_n - d(X_n)| \\
&\leq |X_{n+1} - X_n| + |d(X_n)| \\
&\leq 1 + |d(X_n)| \\
&= 1 + |E[X_{n+1}|\mathcal{F}_n] - X_n| \\
&= 1 + |E[X_{n+1} - X_n|\mathcal{F}_n]] \\
&\leq 1 + E[|X_{n+1} - X_n||\mathcal{F}_n] \\
&\leq 1 + 1 \\
&= 2.
\end{aligned}$$

2. There are a few different ways to do this.

Method 1: Let X_n denote the total amount of work in the queue at time n. That is to say, X_n is the amount of time it would take the server to finish serving the queue (including the customer currently in service) if there were no more arrivals after time n.

Let $\{Y_n\}$ be i.i.d. Bernoulli random variables with parameter p which take the value 1 if there is an arrival at time n and 0 otherwise. Let $\{T_n\}$ be i.i.d. (and independent of Y_i for each i) geometric random variables of parameter p_1, where T_n represents the amount of time a customer arriving at time n will need to be serviced once he reaches the front of the line (the amount of time the customer has to wait in line is not counted here). Thus the total amount of work in the queue at time n is given by

$$X_n = X_{n-1} - 1_{\{X_{n-1}>0\}} + Y_n T_n.$$

Then for any measurable B we have

$$\begin{aligned} P(X_n \in B|\sigma(X_0,\ldots,X_{n-1})) &= P(X_{n-1} - 1_{\{X_{n-1}>0\}} + Y_n T_n \in B|\sigma(X_0,\ldots,X_{n-1})) \\ &= P(X_{n-1} - 1_{\{X_{n-1}>0\}} \in B - Y_n T_n|\sigma(X_0,\ldots,X_{n-1})) \\ &= P(X_{n-1} - 1_{\{X_{n-1}>0\}} \in B - Y_n T_n) \\ &= P(X_{n-1} - 1_{\{X_{n-1}>0\}} \in B - Y_n T_n|\sigma(X_n)) \end{aligned}$$

so the process is Markov.

The transition probabilities are given by

$$p(i,j) = \begin{cases} 1-p & \text{if } j = i-1 \text{ and } i > 0; \\ 1-p & \text{if } i = j = 0; \\ pp_1(1-p_1)^j & \text{if } j \ge i > 0; \\ pp_1(1-p_1)^{j-1} & \text{if } i = 0, j \ge 0; \\ 0 & \text{otherwise.} \end{cases}$$

Method 2: In this model, we will let X_n be the number of people waiting in line at the instant the nth customer enters service. Thus X_n is zero if the customer in service is the only customer in the system. In this model, X_{n+1} is equal to X_n plus the number of arrivals during the last service time minus one (one person will enter service), provided either X_n or the number of arrivals is not zero.

Let $\{T_n\}$ be i.i.d. geometric (parameter p_1) random variables which represent the service time of the nth cusomer. For $k = 1, 2, \ldots$, let $\{Y_k^n\}_{n=0}^\infty$ be i.i.d. (and independent of T_i for all i) binomial random variables with parameters k and p. Then

$$X_{n+1} = X_n + Y_{T_{n-1}}^n - 1_{\{X_n Y_{T_{n-1}}^n > 0\}}.$$

Since X_{n+1} depends only on X_n, an argument similar to method 1 can be used to prove that the process is Markov.

A comment must be made regarding the event when X_n and $Y_{T_{n-1}}^n$ are both zero. In this case, we must wait longer for the nth customer to enter service. Since customers arrive one at a time the next customer to arrive will begin service immediately and X_{n+1} will automatically be zero. Thus the above model is valid even in this case.

The transition probabilities are given by

$$p_n(i,j) = \begin{cases} \sum_{k=0}^{\infty} P(T_{n-1} = k, Y_k^n = j) & \text{if } i = 0, j > 0; \\ \sum_{k=0}^{\infty} P(T_{n-1} = k, Y_k^n = j - i + 1) & \text{if } i > 0, j \geq 0; \\ 0 & \text{otherwise,} \end{cases}$$

which is to say

$$p(i,j) = \begin{cases} \sum_{k=0}^{\infty} p_1(1-p_1)^{k-1}\binom{k}{j} p^j (1-p)^{k-j} & \text{if } i = 0, j > 0; \\ \sum_{k=0}^{\infty} p_1(1-p_1)^{k-1}\binom{k}{j-i+1} p^{j-i+1}(1-p)^{k-j+i-1} & \text{if } 0 \leq j \leq i-1; \\ 0 & \text{otherwise.} \end{cases}$$

Method 3: Here we will let X_n represent the number of customers in the system (in the queue plus the one in service) at time n. Let $\{Y_n\}$ be i.i.d. Bernoulli random variables of parameter p and let $\{T_n\}$ be i.i.d. (independent of the Y's) Bernoulli random variables of parameter p_1. Think of Y_n as the indicator of the event of an arrival at time n and T_n as the indicator of the event of a service completion. We claim that

$$X_{n+1} = X_n + Y_n - T_n 1_{\{X_n > 0\}}.$$

In order to verify this, we must check, for Z a geometric random variable with parameter p_1, that $P(Z = k | Z > k-1) = p_1$ for k=1,2,.... This is easily verified. This means that no matter how long the current customer has been in service at a given time n, the customer has probability p_1 of finishing service at time n.

Again, one can check that this is a Markov chain, using the techniques of Method 1. The transition probabilities here are given by

$$p(i,j) = \begin{cases} p & \text{if } i = 0, j = 1; \\ 1-p & \text{if } i = 0, j = 0; \\ (1-p)p_1 & \text{if } j = i-1, i > 0; \\ pp_1 + (1-p)(1-p_1) & \text{if } j = i > 0; \\ p(1-p_1) & \text{if } j = i+1, i > 0; \\ 0 & \text{otherwise.} \end{cases}$$

(*) Optional part. This is most easily done in the context of method 3. Define $\alpha = \log(1 + p(e-1))$ and $\beta = \log(1 + p_1(e-1))$. Oberserve that

$$E(e^{Y_n}) = pe + (1-p) = e^{\alpha} \text{ for all } n$$

and

$$E(e^{T_n 1_{\{X_n > 0\}}} | \mathcal{F}_n) = \begin{cases} p_1 e + (1-p_1) & \text{if } X_n > 0; \\ 1 & \text{if } X_n = 0 \end{cases} = e^{\beta 1_{\{X_n > 0\}}}.$$

Define

$$M_n = \exp\{X_n - \sum_{i=1}^{n}(\alpha - \beta 1_{\{X_n>0\}})\}.$$

Then M_n is $\sigma(X_1, \ldots, X_n)$-measurable and $E[|M_n|] < \infty$ for each n. Also,

$$\begin{aligned}
E[M_{n+1}|\mathcal{F}_n] &= E\left[\exp\left\{X_n + 1 - \sum_{i=1}^{n+1}(\alpha - \beta 1_{\{X_n>0\}})\right\}\middle|\mathcal{F}_n\right] \\
&= E\left[\exp\left\{X_n + Y_n - T_n 1_{\{X_n>0\}} - \alpha + \beta 1_{\{X_n>0\}} - \sum_{i=1}^{n}(\alpha - \beta 1_{\{X_n>0\}})\right\}\middle|\mathcal{F}_n\right] \\
&= \exp\left\{X_n - \sum_{i=1}^{n}(\alpha - \beta 1_{\{X_n>0\}})\right\} E[\exp\{Y_n - T_n 1_{\{X_n>0\}} - \alpha + \beta 1_{\{X_n>0\}}\}|\mathcal{F}_n] \\
&= M_n e^{-\alpha} e^{\beta 1_{\{X_n>0\}}} E[\exp\{Y_n - T_n 1_{\{X_n>0\}}\}|\mathcal{F}_n] \\
&= M_n e^{-\alpha} e^{\beta 1_{\{X_n>0\}}} E[e^{Y_n}] E[e^{-T_n 1_{\{X_n>0\}}}|\mathcal{F}_n] \\
&= M_n e^{-\alpha} e^{\beta 1_{\{X_n>0\}}} e^{\alpha} e^{-\beta 1_{\{X_n>0\}}} \\
&= M_n.
\end{aligned}$$

Thus, M_n is a positive martingale. Therefore we know that M_n converges almost surely. If $p < p_1$, then $\alpha < \beta$. Assuming this, suppose the queue did not empty infinitely often. Then there would exist an $N = N(\omega)$ so that $X_n > 0$ for all $n > N$. Thus, for any $M > 0$, we can find an n large enough, since $\alpha < \beta$, so that

$$-\sum_{i=1}^{n}(\alpha - \beta 1_{\{X_n>0\}}) = -n\alpha + \sum_{i=1}^{n}\beta 1_{\{X_n>0\}} > M.$$

But then $M_n > e^{X_n} e^{M}$. Hence, given any number M' large, we can find an n large enough so that $M_n > M'$. This contradicts the convergence of M_n. Therefore, $p < p_1$ assures that the queue will empty infinitely often. almost surely.

3. To prove it is a Markov chain, denote the increments of the simple random walk as X_n and observe

$$\begin{aligned}
P(S_{(n+1)\wedge\tau} = k|\mathcal{F}_n) &= P(S_{n\wedge\tau} + X_{n+1} 1_{\tau>n} = k|\mathcal{F}_n) \\
&= P(S_{n\wedge\tau} = k - X_{n+1} 1_{\tau>n}|\mathcal{F}_n) \\
&= P(S_{n\wedge\tau} = k - X_{n+1} 1_{\tau>n}|\sigma(S_n)) \\
&= P(S_{(n+1)\wedge\tau} = k|\sigma(S_n)),
\end{aligned}$$

where the third line follows because $S_{n\wedge\tau} = S_n$ in this model, hence $\sigma(S_{n\wedge\tau}) = \sigma(S_n)$.

Its transition probabilities are given by

$$p(i,j) = \begin{cases} 1 & \text{if } i = j = \pm L \\ 1/2 & \text{if } j = i-1, i \neq \pm L \\ 1/2 & \text{if } j = i+1, i \neq \pm L \\ 0 & \text{otherwise.} \end{cases}$$

Let the matrix of these transition probabilities be denoted by Π. The event $[\tau < k]$ is the same as the event $[X_{k-1} = \pm L]$ due to the absorbing boundaries. Thus,

$$P(\tau < k) = \Pi^{(k-1)}(0, L) + \Pi^{(k-1)}(0, -L),$$

which is given by

$$(0, \ldots, 0, 1, 0, \ldots, 0)\Pi^{k-1}(1, 0, \ldots, 0, 1)^T.$$

Another way to look at it is to let $f(x,k) = P^x(\tau < k)$. Then $f(x,k) = \frac{1}{2}f(x-1, k-1) + \frac{1}{2}f(x+1, k-1)$.

To show that τ is almost surely finite, we can divide time into intervals of length $2L$. In each time interval, the Markov chain can be absorbed by simply taking $2L$ steps to the right, an event which happens with probability 2^{-2L}. Thus, the probability that the chain is not absorbed in any one of these time intervals is at most $1 - 2^{-2L}$. Therefore, the probability of nonabsorption after k such time intervals is at most $(1 - 2^{-2L})^k$, which goes to zero as k goes to infinity.

Another way to do this, which involves no computation is as follows. The process S_n as described is a finite state Markov process. As shown in class, this means that there exists a recurrent state. Since there is a path from any interior point to L or $-L$ but there is no path from L or $-L$ to any interior point, we see that the interior points must all be transient. Indeed, $\rho_{xL} > 0$ but $\rho_{Lx} = 0$ for $-L+1 \leq x \leq L-1$. This violates a condition for recurrence. Therefore, L and $-L$ are the only possible recurrent states. The interior states must make up a transience class. Since there are only a finite number of transient states, the chain must eventually leave this transience class (which means finite transience classes are *non-essential*). Thus, the chain must hit L or $-L$ in finite time.

4. Let X_n indicate the stock price after the nth trading day. Let $\{Y_n\}$ be a collection of i.i.d. (and independent of the X's) Bernoulli random variables of parameter p. We have

$$X_{n+1} = X_n - 0.15 X_n Y_n + 0.01 X_n (1 - Y_n).$$

The process is Markov as X_{n+1} depends only on X_n and Y_n (which is independent of all X_n). A logical state space for this Markov chain is the rational numbers. The transition probabilities are given by

$$p(i,j) = \begin{cases} p & \text{if } j = 0.85i \\ 1-p & \text{it } j = 1.01i \\ 0 & \text{otherwise} \end{cases}.$$