

(April 15, 2005)

exam 08 solutions

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[exam 08 solutions.1] Let T be a hermitian operator on a finite-dimensional complex vector space V with a positive-definite inner product \langle, \rangle . Let P be an orthogonal projector to the λ -eigenspace V_λ of T . (This means that P is the identity on V_λ and is 0 on the orthogonal complement V_λ^\perp of V_λ .) Show that $P \in \mathbf{C}[T]$.

By the spectral theorem for hermitian operators, T is diagonalizable. So its minimal polynomial $f(x) \in \mathbf{C}[x]$ factors into linear factors without any repeats, and $f_\lambda(x) = f(x)/(x - \lambda)$ is relatively prime to $x - \lambda$, and (by the Euclidean-ness of $\mathbf{C}[x]$) there are polynomials $a_\lambda(x)$ such that

$$1 = \sum_{\lambda} a_\lambda(x) f_\lambda(x)$$

Let $P_\lambda = a_\lambda(T) f_\lambda(T) \in \mathbf{C}[T]$.

As usual, these P_λ s are mutually orthogonal projectors to the eigenspaces of T . Again, the argument is that

$$P_\lambda^2 = (a_\lambda(T) f_\lambda(T)) \left(1 - \sum_{\mu \neq \lambda} a_\mu(T) f_\mu(T) \right) = P_\lambda - \sum_{\mu \neq \lambda} 0 = P_\lambda$$

because $f(x)$ divides $f_\lambda(x) \cdot f_\mu(x)$ for $\mu \neq \lambda$. This shows that P_λ is a projector. Also, for $v \in V_\lambda$,

$$P_\lambda v = \left(1 - \sum_{\mu \neq \lambda} a_\mu(T) f_\mu(T) \right) (v) = v - \sum_{\mu \neq \lambda} a_\mu(T)(0) v = v$$

since for $\mu \neq \lambda$ the polynomial $f_\mu(x)$ has a factor of $x - \lambda$, and $(T - \lambda)v = 0$. Similarly, $P_\mu(v) = 0$ for $\mu \neq \lambda$ and $v \in V_\lambda$. This proves that these are projectors to the respective eigenspaces.

For hermitian T , eigenspaces for distinct eigenvalues are orthogonal. Thus, V_λ is orthogonal to $\sum_{\mu \neq \lambda} V_\mu$. Since $\sum_{\mu} V_\mu = V$, it must be that

$$V_\lambda^\perp = \sum_{\mu \neq \lambda} V_\mu$$

At last we can show that the orthogonal projector P to a given V_λ is, in fact, the P_λ constructed above. Indeed, P is 0 on V_μ for $\mu \neq \lambda$, since these are in the orthogonal complement to V_λ , and P is 1 on V_λ . Since the direct sum of the eigenspaces is the whole space,

$$v = \sum_{\mu} P_\mu v$$

Thus

$$P = P \circ 1 = P \circ \sum_{\text{all } \mu} P_\mu = \sum_{\text{all } \mu} P \circ P_\mu = 1 \cdot P_\lambda + \sum_{\mu \neq \lambda} 0 \cdot P_\mu = P_\lambda$$

Thus, $P = P_\lambda = a_\lambda(T) f_\lambda(T) \in \mathbf{C}[T]$. ///

[exam 08 solutions.2] Let T be a diagonalizable operator on a finite-dimensional vector space V over a field k . Show that there is a *unique* projector P to the λ -eigenspace V_λ of T such that $TP = PT$.

(The argument here repeats part of the previous argument, but worth repeating, with minor variations.) Since T is diagonalizable, its minimal polynomial $f(x) \in k[x]$ factors into linear factors without any repeats, and $g(x) = f(x)/(x - \lambda)$ is relatively prime to $x - \lambda$, and (by the Euclidean-ness of $k[x]$) there are polynomials $a(x), b(x) \in k[x]$ such that

$$1 = a(x)g(x) + b(x)(x - \lambda)$$

Let $P_\lambda = a(T)g(T) \in k[T]$, and $Q_\lambda = b(T)(T - \lambda)$.

As usual, P_λ, Q_λ are mutually orthogonal projectors, in the sense that they are projectors and their product is 0. And P_λ 's image is the λ eigenspace, and Q_λ 's image is a complementary subspace (in fact, the sum of all *other* eigenspaces, but we don't need that). The argument is that

$$P_\lambda^2 = a(T)g(T)(1 - b(T)(T - \lambda)) = P_\lambda - 0 = P_\lambda$$

because $f(x) = g(x) \cdot (x - \lambda)f_\mu(x)$. This shows that P_λ is a projector. And

$$Q_\lambda^2 = (1 - P_\lambda)^2 = 1 - 2P_\lambda + P_\lambda^2 = 1 - 2P_\lambda + P_\lambda = 1 - P_\lambda = Q_\lambda$$

so Q_λ is a projector. Also

$$P_\lambda Q_\lambda = P_\lambda(1 - P_\lambda) = P_\lambda - P_\lambda^2 = 0$$

For $v \in V_\lambda$,

$$P_\lambda v = (1 - b(T)(T - \lambda))(v) = v - 0 = v$$

since $(T - \lambda)v = 0$. And

$$Q_\lambda(v) = (1 - P_\lambda)(v) = 0$$

Finally, for any $v \in V$,

$$v = 1 \cdot v = (P_\lambda + Q_\lambda)v = P_\lambda v + Q_\lambda v$$

so V is the direct sum of the image of P_λ and the image of Q_λ .

The given projector P commutes with T so commutes with P_λ and Q_λ , since these are in $k[T]$. Thus, P stabilizes the 1-eigenspace of P_λ and stabilizes the 0-eigenspace for P_λ (the latter being the same as the 1-eigenspace of Q_λ). Thus,

$$P = P \circ 1 = P \circ (P_\lambda + Q_\lambda) = P \circ P_\lambda + P \circ Q_\lambda = 1 \cdot P_\lambda + 0 \cdot Q_\lambda = P_\lambda$$

as desired. ///

[exam 08 solutions.3] From just the most basic properties of determinants of matrices, show that the determinant of an upper-triangular matrix is the product of its diagonal entries. That is, show that

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & & \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & & 0 & a_{nn} \end{pmatrix} = a_{11}a_{22}a_{33} \dots a_{nn}$$

Let A be the given matrix, and let a_{ij} denote the ij^{th} entry, whether or not it's 0. In the formula

$$\det A = \sum_{\pi \in S_n} \sigma(\pi) a_{\pi(1),1} \dots a_{\pi(n),n}$$

(where π runs over the symmetric group S_n , and σ is the sign function on permutations), note that $a_{\pi(i),i} = 0$ unless $\pi(i) \leq i$. Thus, the π^{th} summand is 0 unless $\pi(i) \leq i$ for *all* i . Thus, $\pi(1) = 1$. By induction, $\pi(i) = i$ for all i . Thus, only the identity permutation appears (with non-zero summand) in the sum, which yields the asserted formula. ///