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exam 09solns

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[exam 09.1] Write an explicit isomorphism

$$\mathbf{Z}/a \otimes_{\mathbf{Z}} \mathbf{Z}/b \to \mathbf{Z}/\gcd(a,b)$$

and verify that it is what is claimed.

First, we know that monomial tensors generate the tensor product, and for any $x, y \in \mathbf{Z}$

$$x \otimes y = (xy) \cdot (1 \otimes 1)$$

so the tensor product is generated by $1 \otimes 1$. Next, we claim that $g = \gcd(a,b)$ annihilates every $x \otimes y$, that is, $g \cdot (x \otimes y) = 0$. Indeed, let r, s be integers such that $ra + sb = g$. Then

$$g \cdot (x \otimes y) = (ra + sb) \cdot (x \otimes y) = r(ax \otimes y) = s(x \otimes by) = r \cdot 0 + s \cdot 0 = 0$$

So the generator $1 \otimes 1$ has order dividing g. To prove that that generator has order *exactly* g, we construct a bilinear map. Let

$$B : \mathbf{Z}/a \times \mathbf{Z}/b \to \mathbf{Z}/g$$

by

$$B(x \times y) = xy + g\mathbf{Z}$$

To see that this is well-defined, first compute

$$(x + a\mathbf{Z})(y + b\mathbf{Z}) = xy + xb\mathbf{Z} + ya\mathbf{Z} + ab\mathbf{Z}$$

Since

$$xb\mathbf{Z} + ya\mathbf{Z} \subset b\mathbf{Z} + a\mathbf{Z} = \gcd(a,b)\mathbf{Z}$$

(and $ab\mathbf{Z} \subset g\mathbf{Z}$), we have

$$(x + a\mathbf{Z})(y + b\mathbf{Z}) + g\mathbf{Z} = xy + xb\mathbf{Z} + ya\mathbf{Z} + ab\mathbf{Z} + \mathbf{Z}$$

and well-definedness. By the defining property of the tensor product, this gives a unique linear map β on the tensor product, which on monomials is

$$\beta(x \otimes y) = xy + \gcd(a,b)\mathbf{Z}$$

The generator $1 \otimes 1$ is mapped to 1, so the image of $1 \otimes 1$ has order $\gcd(a,b)$, so $1 \otimes 1$ has order divisible by $\gcd(a,b)$. Thus, having already proven that $1 \otimes 1$ has order at most $\gcd(a,b)$, this must be its order.

In particular, the map β is injective on the cyclic subgroup generated by $1 \otimes 1$. That cyclic subgroup is the whole group, since $1 \otimes 1$. The map is also surjective, since $\cdot 1 \otimes 1$ hits $r \bmod \gcd(a,b)$. Thus, it is an isomorphism. ///

[exam 09.2] Let $\varphi : R \to S$ be commutative rings with unit, and suppose that $\varphi(1_R) = 1_S$, thus making S an R-algebra. For an R-module N prove that $\mathrm{Hom}_R(S, N)$ is (*yet another*) good definition of *extension of scalars* from R to S, by checking that for every S-module M there is a natural isomorphism

$$\mathrm{Hom}_R(\mathrm{Res}^S_R M, N) \approx \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, N)$$

where $\mathrm{Res}^S_R M$ is the R-module obtained by forgetting S, and letting $r \in R$ act on M by $r \cdot m = \varphi(r)m$. (*Do* prove naturality in M, also.)

Let

$$i : \mathrm{Hom}_R(\mathrm{Res}^S_R M, N) \to \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, N)$$

be defined for $\varphi \in \mathrm{Hom}_R(\mathrm{Res}^S_R M, N)$ by

$$i(\varphi)(m)(s) = \varphi(s \cdot m)$$

This makes *some* sense, at least, since M is an S-module. We must verify that $i(\varphi) : M \to \mathrm{Hom}_R(S, N)$ is S-linear. Note that the S-module structure on $\mathrm{Hom}_R(S, N)$ is

$$(s \cdot \psi)(t) = \psi(st)$$

where $s, t \in S$, $\psi \in \mathrm{Hom}_R(S, N)$. Then we check:

$$\left(i(\varphi)(sm)\right)(t) = i(\varphi)(t \cdot sm) = i(\varphi)(stm) = i(\varphi)(m)(st) = \left(s \cdot i(\varphi)(m)\right)(t)$$

which proves the S-linearity.

The map j in the other direction is described, for $\Phi \in \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, N)$, by

$$j(\Phi)(m) = \Phi(m)(1_S)$$

where 1_S is the identity in S. Verify that these are mutual inverses, by

$$i(j(\Phi))(m)(s) = j(\Phi)(s \cdot m) = \Phi(sm)(1_S) = \left(s \cdot \Phi(m)\right)(1_S) = \Phi(m)(s \cdot 1_S) = \Phi(m)(s)$$

as hoped. (Again, the equality

$$\left(s \cdot \Phi(m)\right)(1_S) = \Phi(m)(s \cdot 1_S)$$

is the definition of the S-module structure on $\mathrm{Hom}_R(S, N)$.) In the other direction,

$$j(i(\varphi))(m) = i(\varphi)(m)(1_S) = \varphi(1 \cdot m) = \varphi(m)$$

Thus, i and j are mutual inverses, so are isomorphisms.

For naturality, let $f : M \to M'$ be an S-module homomorphism. Add indices to the previous notation, so that

$$i_{M,N} : \mathrm{Hom}_R(\mathrm{Res}^S_R M, N) \to \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, N)$$

is the isomorphism discussed just above, and $i_{M',N}$ the analogous isomorphism for M' and N. We must show that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_R(\mathrm{Res}^S_R M, N) & \xrightarrow{\;i_{M,N}\;} & \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, N) \\
{\scriptstyle -\circ f}\uparrow & & \uparrow{\scriptstyle -\circ f} \\
\mathrm{Hom}_R(\mathrm{Res}^S_R M', N) & \xrightarrow{\;i_{M',N}\;} & \mathrm{Hom}_S(M', \mathrm{Hom}_R(S, N)
\end{array}$$

commutes, where $- \circ f$ is pre-composition with f. (We use the same symbol for the map $f : M \to M'$ on the modules whose S-structure has been forgotten, leaving only the R-module structure.) Starting in the lower left of the diagram, going up then right, for $\varphi \in \mathrm{Hom}_R(\mathrm{Res}^S_R M', N)$,

$$\left(i_{M,N} \circ (- \circ f)\ \varphi\right)(m)(s) = \left(i_{M,N}(\varphi \circ f)\right)(m)(s) = (\varphi \circ f)(s \cdot m) = \varphi(f(s \cdot m))$$

On the other hand, going right, then up,

$$\left((- \circ f) \circ i_{M',N}\ \varphi\right)(m)(s) = \left(i_{M',N}\ \varphi\right)(fm)(s) = \varphi(s \cdot fm) = \varphi(f(s \cdot m))$$

since f is S-linear. That is, the two outcomes are the same, so the diagram commutes, proving functoriality in M, which is a part of the naturality assertion. ///

[exam 09.3] Let

$$M = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \qquad N = \mathbf{Z} \oplus 4\mathbf{Z} \oplus 24\mathbf{Z} \oplus 144\mathbf{Z}$$

What are the elementary divisors of $\bigwedge^2(M/N)$?

First, note that this is *not* the same as asking about the structure of $(\bigwedge^2 M)/(\bigwedge^2 N)$. Still, we can address that, too, after dealing with the question that *was* asked.

First,

$$M/N = \mathbf{Z}/\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/24\mathbf{Z} \oplus \mathbf{Z}/144\mathbf{Z} \approx \mathbf{Z}/4 \oplus \mathbf{Z}/24 \oplus \mathbf{Z}/144$$

where we use the obvious slightly lighter notation. Generators for M/N are

$$m_1 = 1 \oplus 0 \oplus 0 \quad m_2 = 0 \oplus 1 \oplus 0 \quad m_3 = 0 \oplus 0 \oplus 1$$

where the 1s are respectively in $\mathbf{Z}/4$, $\mathbf{Z}/24$, and $\mathbf{Z}/144$. We know that $e_i \wedge e_j$ *generate* the exterior square, for the 3 pairs of indices with $i < j$. Much as in the computation of $\mathbf{Z}/a \otimes \mathbf{Z}/b$, for e in a $\mathbf{Z}$-module E with $a \cdot e = 0$ and f in E with $b \cdot f = 0$, let r, s be integers such that

$$ra + sb = \gcd(a, b)$$

Then

$$\gcd(a, b) \cdot e \wedge f = r(ae \wedge f) + s(e \wedge bf) = r \cdot 0 + s \cdot 0 = 0$$

Thus, $4 \cdot e_1 \wedge e_2 = 0$ and $4 \cdot e_1 \wedge e_3 = 0$, while $24 \cdot e_2 \wedge e_3 = 0$. If there are no further relations, then we could have

$$\bigwedge^2(M/N) \approx \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/24$$

(so the elementary divisors would be $4, 4, 24$.)

To prove, in effect, that there are no further relations than those just indicated, we must construct suitable alternating bilinear maps. Suppose for $r, s, t \in \mathbf{Z}$

$$r \cdot e_1 \wedge e_2 + s \cdot e_1 \wedge e_3 + t \cdot e_2 \wedge e_3 = 0$$

Let

$$B_{12} : (\mathbf{Z}e_1 \oplus \mathbf{Z}e_2 \oplus \mathbf{Z}e_3) \times (\mathbf{Z}e_1 \oplus \mathbf{Z}e_2 \oplus \mathbf{Z}e_3) \to \mathbf{Z}/4$$

by

$$B_{12}(xe_1 + ye_2 + ze_3,\ \xi e_1 + \eta e_2 + \zeta e_3) = (x\eta - \xi y) + 4\mathbf{Z}$$

(As in earlier examples, since $4|4$ and $4|24$, this is *well-defined*.) By arrangement, this B_{12} is alternating, and induces a unique linear map β_{12} on $\bigwedge^2(M/N)$, with

$$\beta_{12}(e_1 \wedge e_2) = 1 \qquad \beta_{12}(e_1 \wedge e_3) = 0 \qquad \beta_{12}(e_2 \wedge e_3) = 0$$

Applying this to the alleged relation, we find that $r = 0 \bmod 4$. Similar contructions for the other two pairs of indices $i < j$ show that $s = 0 \bmod 4$ and $t = 0 \bmod 24$. This shows that we have all the relations, and

$$\bigwedge^2(M/N) \approx \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/24$$

as hoped/claimed. ///

Now consider the other version of this question. Namely, letting

$$M = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \qquad N = \mathbf{Z} \oplus 4\mathbf{Z} \oplus 24\mathbf{Z} \oplus 144\mathbf{Z}$$

compute the elementary divisors of $(\bigwedge^2 M)/(\bigwedge^2 N)$.

Let e_1, e_2, e_3, e_4 be the standard basis for $\mathbf{Z}^4$. Let $i : N \to M$ be the inclusion. We have shown that exterior powers of free modules are free with the expected generators, so M is free on

$$e_1 \wedge e_2, \;\; e_1 \wedge e_3, \;\; e_1 \wedge e_4, \;\; e_2 \wedge e_3, \;\; e_2 \wedge e_4, \;\; e_3 \wedge e_4$$

and N is free on

$$(1 \cdot 4)\, e_1 \wedge e_2, \;\; (1 \cdot 24)\, e_1 \wedge e_3, \;\; (1 \cdot 144)\, e_1 \wedge e_4, \;\; (4 \cdot 24)\, e_2 \wedge e_3, \;\; (4 \cdot 144)\, e_2 \wedge e_4, \;\; (24 \cdot 144)\, e_3 \wedge e_4$$

The inclusion $i : N \to M$ induces a natural map $\bigwedge^2 i : \bigwedge^2 \to \bigwedge^2 M$, taking $r \cdot e_i \wedge e_j$ (in N) to $r \cdot e_i \wedge e_j$ (in M). Thus, the quotient of $\bigwedge^2 M$ by (the image of) $\bigwedge^2 N$ is visibly

$$\mathbf{Z}/4 \oplus \mathbf{Z}/24 \oplus \mathbf{Z}/144 \oplus \mathbf{Z}/96 \oplus \mathbf{Z}/576 \oplus \mathbf{Z}/3456$$

The integers $4, 24, 144, 96, 576, 3456$ do not quite have the property $4|24|144|96|576|3456$, so are not elementary divisors. The problem is that neither $144|96$ nor $96|144$. The only primes dividing all these integers are 2 and 3, and, in particular,

$$4 = 2^2, \; 24 = 2^3 \cdot 3, \; 144 = 2^4 \cdot 3^2, \; 96 = 2^5 \cdot 3, \; 576 = 2^6 \cdot 3^2, \; 3456 = 2^7 \cdot 3^3,$$

From Sun-Ze's theorem,

$$\mathbf{Z}/(2^a \cdot 3^b) \approx \mathbf{Z}/2^a \oplus \mathbf{Z}/3^b$$

so we can rewrite the summands $\mathbf{Z}/144$ and $\mathbf{Z}/96$ as

$$\mathbf{Z}/144 \oplus \mathbf{Z}/96 \approx (\mathbf{Z}/2^4 \oplus \mathbf{Z}/3^2) \oplus (\mathbf{Z}/2^5 \oplus \mathbf{Z}/3) \approx (\mathbf{Z}/2^4 \oplus \mathbf{Z}/3) \oplus (\mathbf{Z}/2^5 \oplus \mathbf{Z}/3^2) \approx \mathbf{Z}/48 \oplus \mathbf{Z}/288$$

Now we do have $4|24|48|288|576|3456$, and

$$(\bigwedge^2 M)/(\bigwedge^2 N) \approx \mathbf{Z}/4 \oplus \mathbf{Z}/24 \oplus \mathbf{Z}/48 \oplus \mathbf{Z}/288 \oplus \mathbf{Z}/576 \oplus \mathbf{Z}/3456$$

is in elementary divisor form. ///