

(February 26, 2005)

## Homework 16, due Wed Mar 02

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[16.1] Let  $p$  be the smallest prime dividing the order of a finite group  $G$ . Show that a subgroup  $H$  of  $G$  of index  $p$  is necessarily *normal*.

[16.2] Let  $T \in \text{End}_k(V)$  for a finite-dimensional  $k$ -vectorspace  $V$ , with  $k$  a field. Let  $W$  be a  $T$ -stable subspace. Prove that the minimal polynomial of  $T$  on  $W$  is a divisor of the minimal polynomial of  $T$  on  $V$ . Define a natural action of  $T$  on the quotient  $V/W$ , and prove that the minimal polynomial of  $T$  on  $V/W$  is a divisor of the minimal polynomial of  $T$  on  $V$ .

[16.3] Let  $T \in \text{End}_k(V)$  for a finite-dimensional  $k$ -vectorspace  $V$ , with  $k$  a field. Suppose that  $T$  is *diagonalizable* on  $V$ . Let  $W$  be a  $T$ -stable subspace of  $V$ . Show that  $T$  is diagonalizable on  $W$ .

[16.4] Let  $T \in \text{End}_k(V)$  for a finite-dimensional  $k$ -vectorspace  $V$ , with  $k$  a field. Suppose that  $T$  is *diagonalizable* on  $V$ , with *distinct eigenvalues*. Let  $S \in \text{End}_k(V)$  commute with  $T$ , in the natural sense that  $ST = TS$ . Show that  $S$  is diagonalizable on  $V$ .

[16.5] Let  $T \in \text{End}_k(V)$  for a finite-dimensional  $k$ -vectorspace  $V$ , with  $k$  a field. Suppose that  $T$  is *diagonalizable* on  $V$ . Show that  $k[T]$  contains the projectors to the eigenspaces of  $T$ .

[16.6] Let  $V$  be a complex vector space with a (positive definite) inner product. Show that  $T \in \text{End}_k(V)$  cannot be a normal operator if it has any non-trivial Jordan block.

[16.7] Show that a positive-definite hermitian  $n$ -by- $n$  matrix  $A$  has a unique positive-definite square root  $B$  (that is,  $B^2 = A$ ).

[16.8] Given a square  $n$ -by- $n$  complex matrix  $M$ , show that there are unitary matrices  $A$  and  $B$  such that  $AMB$  is *diagonal*.

[16.9] Given a square  $n$ -by- $n$  complex matrix  $M$ , show that there is a unitary matrix  $A$  such that  $AM$  is *upper triangular*.

[16.10] Let  $Z$  be an  $m$ -by- $n$  complex matrix. Let  $Z^*$  be its conjugate-transpose. Show that

$$\det(1_m - ZZ^*) = \det(1_n - Z^*Z)$$