(*April 30, 2005*)

Homework 20, due Wed Apr 20

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[20.1] Prove the *expansion by minors* formula for determinants, namely, for an n-by-n matrix A with entries a_{ij}, letting A^{ij} be the matrix obtained by deleting the i^{th} row and j^{th} column, for any fixed row index i,

$$\det A = (-1)^i \sum_{j=1}^{n} (-1)^j \, a_{ij} \, \det A^{ij}$$

and symmetrically for expansion along a column.

[20.2] Let M and N be free R-modules of ranks r, s, where R is a commutative ring with identity. Prove that $M \otimes_R N$ is free of rank rs.

[20.3] Let M be a free R-module of rank r, where R is a commutative ring with identity. Let S be a commutative ring with identity containing R, such that $1_R = 1_S$. Prove that as an S module $M \otimes_R S$ is free of rank r.

[20.4] For finite-dimensional vectorspaces V, W over a field k, prove that there is a natural isomorphism

$$(V \otimes_k W)^* \approx V^* \otimes W^*$$

where $X^* = \mathrm{Hom}_k(X, k)$ for a k-vectorspace X.

[20.5] For a finite-dimensional k-vectorspace V, prove that the bilinear map

$$B : V \times V^* \to \mathrm{End}_k(V)$$

by

$$B(v \times \lambda)(x) = \lambda(x) \cdot v$$

gives an isomorphism $V \otimes_k V^* \to \mathrm{End}_k(V)$. Further, show that the composition of endormorphisms is the same as the map induced from the map on

$$(V \otimes V^*) \times (V \otimes V^*) \to V \otimes V^*$$

given by

$$(v \otimes \lambda) \times (w \otimes \mu) \to \lambda(w) v \otimes \mu$$

[20.6] Under the isomorphism of the previous problem, show that the linear map

$$\mathrm{tr} : \mathrm{End}_k(V) \to k$$

is the linear map

$$V \otimes V^* \to k$$

induced by the bilinear map $v \times \lambda \to \lambda(v)$.

[20.7] Prove that $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$ for two endomorphisms of a finite-dimensional vector space V over a field k, with trace defined as just above.

[20.8] Prove that tensor products are *associative*, in the sense that, for R-modules A, B, C, we have a *natural isomorphism*

$$A \otimes_R (B \otimes_R C) \approx (A \otimes_R B) \otimes_R C$$

In particular, *do* prove the *naturality*, at least the one-third part of it which asserts that, for every R-module homomorphism $f : A \to A'$, the diagram

$$\begin{tikzcd}
A \otimes_R (B \otimes_R C) \arrow[r, "\approx"] \arrow[d, "f\otimes(1_B\otimes 1_C)"] & (A \otimes_R B) \otimes_R C \arrow[d, "(f\otimes 1_B)\otimes 1_C"] \\
A' \otimes_R (B \otimes_R C) \arrow[r, "\approx"] & (A' \otimes_R B) \otimes_R C
\end{tikzcd}$$

commutes, where the two horizontal isomorphisms are those determined in the first part of the problem. (This is **functoriality** in A.) (One might also consider maps $g : B \to B'$ and $h : C \to C'$, but these behave similarly, so there's no real compulsion to worry about them, apart from awareness of the issue.)