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Homology and Derived Functors

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1. *(Co-) homology of (co-) chain complexes*

A **(chain) complex** (over R) is a diagram

$$C = \{C_n, \partial_n\} = \cdots \longrightarrow C_n \xrightarrow{\partial_n} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

of R-modules with the property that

$$\partial_{n-1} \circ \partial_n = 0$$

The R-module C_i is the i^{th} **graded piece** of the complex. The operators ∂_n are **boundary maps**.

The **homology**

$$H_*(C) = \{H_n(C) : n \geq 0\}$$

of the complex C is the collection of quotients

$$H_n(C) = \ker \partial_n / \mathrm{Im}\, \partial_{n+1}$$

of the n-**cycles** $\ker \partial_i$ by the n-**boundaries** $\mathrm{Im}\, \partial_{n+1}$. Thus, the homology is a measure of the non-exactness of C. Dually, a **cochain complex** is a diagram

$$C = \{C_n, \delta_n\} = 0 \longrightarrow C_0 \xrightarrow{\delta_0} C_1 \xrightarrow{\delta_1} \cdots$$

of R-modules with the property that

$$\delta_n \circ \delta_{n-1} = 0$$

The R-module C_i is the i^{th} **graded piece** of the complex. The **cohomology**

$$H^*(C) = \{H^i(C) : i \geq 0\}$$

of the complex C is the collection of quotients

$$H^n(C) = \ker \delta_n / \mathrm{Im}\, \delta_{n-1}$$

of the n-**cocycles** $\ker \delta_n$ by the n-**coboundaries** $\mathrm{Im}\, \delta_{n-1}$. The operators δ_n are **coboundary maps**.

Much of what can be said for chain complexes and homology carries over to co-chain complexes and cohomology simply by reversing arrows, and vice-versa. Thus, we may discuss just one of the two cases and leave the other as an exercise.

A **chain map** or **complex map**

$$f : \{C_i, \partial_i\} \longrightarrow \{C'_i, \partial'_i\}$$

is a collection $f = \{f_i\}$ with

$$f_i : C_i \longrightarrow C_i' \qquad\qquad f_i \circ \partial_{i+1} \;=\; \partial_i \circ f_{i+1}$$

That is, all squares in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & C_0 \arrow[r, "\partial_0"] \arrow[d, "f_0"] & 0 \\
\cdots \arrow[r] & C_1' \arrow[r, "\partial_1"] & C_0' \arrow[r, "\partial_0'"] & 0
\end{tikzcd}$$

commute. For a chain map $f : C \longrightarrow C'$ is a chain map, there are **induced maps on homology**

$$H_n(f) : H_n(C) \longrightarrow H_n(C')$$

defined by

$$H_n(f)(\zeta + \partial_{n+1} C_{n+1}) \;=\; f_n \zeta + \partial'_{n+1} C'_{n+1} \qquad\qquad (\text{with } \partial_n \zeta = 0)$$

The defining property of 'chain map' assures that this is well-defined. A short exact sequence

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

of complexes is **exact** when the associated short exact sequences

$$0 \longrightarrow C_i' \longrightarrow C_i \longrightarrow C_i'' \longrightarrow 0$$

are all exact sequences of R-modules. The most basic result here is

[1.0.1] Theorem: A short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of complexes gives rise to a natural **long exact sequence in homology**

$$\cdots \longrightarrow H_2(C) \xrightarrow{\partial_2} H_1(A) \xrightarrow{H_1(f)} H_1(B) \xrightarrow{H_1(g)} H_1(C) \xrightarrow{\partial_1} H_0(A) \xrightarrow{H_0(f)} H_0(B) \xrightarrow{H_0(g)} H_0(C) \longrightarrow 0$$

[1.0.2] Remark: The **connecting homomorphisms** ∂_i will be defined in the course of the proof. (We tolerate the two different uses of the symbol '∂').

Proof: For now, we only show how to define the connecting homomorphisms

$$H_n(C) \xrightarrow{\partial_n} H_{n-1}(A)$$

For $c_n \in C_n$ with $\partial_n c_n = 0$, the surjectivity of $g : B_n \longrightarrow C_n$ assures that there is $b_n \in B_n$ so that $g b_n = c_n$. The fact that g is a chain map assures that

$$g(\partial_n b_n) \;=\; \partial_n (g b_n) \;=\; \partial_n c_n \;=\; 0$$

Exactness assures that there is $x_{n-1} \in A_{n-1}$ so that

$$f x_{n-1} \;=\; \partial_n b_n$$

We define the connecting homomorphism *on homology* by

$$\partial_n(c_n + \partial_{n+1}C_{n+1}) \;=\; x_{n-1} + \partial_n A_n$$

Checking that this is well-defined (on homology) is a non-trivial but standard exercise left to the reader, as is the diagram-chase verification of exactness at the three different types of joints in the long sequence. ///

A **chain homotopy** $\theta : f \longrightarrow g$ from one complex map $f : C \longrightarrow C'$ to another $g : C \longrightarrow C'$ is a collection $\theta = \{\theta_i\}$ with

$$\theta_i : C_i \longrightarrow C'_{i+1}$$

and so that

$$\partial'_{i+1} \circ \theta_i + \theta_{i-1} \circ \partial_i \;=\; f - g$$

[1.0.3] Proposition: For chain-homotopic maps f and g maps $C \longrightarrow C'$, the induced maps on homology are identical.

Proof: For such a chain homotopy θ, with $\partial_n \zeta = 0$,

$$f_n\zeta - g_n\zeta \;=\; \partial'_{n+1} \circ \theta_n\,\zeta + \theta_{n-1} \circ \partial_n\,\zeta \;=\; \partial'_{n+1} \circ \theta_n\,\zeta$$

since $\partial_n\,\zeta = 0$. That is,

$$f_n\zeta - g_n\zeta \;=\; \partial'_{n+1}(\theta_n\,\zeta)$$

giving 0 in homology. ///

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2. *A small example*

We can give a small but non-trivial examples of the utility of the long exact sequence in (co-) homology arising from a short exact sequence of (co-) chain complexes. The question does not explicitly mention (co-) homology, but the discussion shows that the issues are genuinely homological in nature.

Let R be a not-necessarily-commutative ring, and

$$B \xrightarrow{q} C \longrightarrow 0$$

a surjection of R-modules. Let T be an R-endomorphism of B which stabilizes $\ker q$ so descends to an R-endomorphism of C. Let $u \in C$ be an element such that $Tu = 0$. The question we wish to address is an **extension problem**: *Is there an element $\tilde{u}$ of B such that $q\tilde{u} = u$ and still $T\tilde{u} = 0$? Is $\tilde{u}$ unique? Such $\tilde{u}$ would be an* **extension** *of u.*

[2.0.1] Claim: In the situation just above, if T is injective and surjective on $\ker q$, then there is a unique extension $\tilde{u}$ of u.

Proof: Let $A = \ker q$, so that

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{q} C \longrightarrow 0$$

is a short exact sequence of R-modules. Consider the exact sequence of complexes of R-modules

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "T"] & B \arrow[r, "q"] \arrow[d, "T"] & C \arrow[r] \arrow[d, "T"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[r, "q"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Viewing a complex (M,T)

$$0 \xrightarrow{i} M \xrightarrow{T} M \xrightarrow{z} 0$$

attached to an R-module M and R-endomorphism T as being a chain complex, the associated homology is

$$H_0(M,T) \;=\; \ker z/\mathrm{Im}T \;=\; M/\mathrm{Im}T \qquad\qquad H_1(M,T) \;=\; \ker T/\mathrm{Im}\, i \;=\; \ker T$$

and higher homology is all 0. Therefore, the long exact sequence in homology shortens to

$$0 \longrightarrow H_1(A,T) \longrightarrow H_1(B,T) \longrightarrow H_1(C,T) \longrightarrow H_0(A,T) \longrightarrow H_0(B,T) \longrightarrow H_0(C,T) \longrightarrow 0$$

which is

$$0 \longrightarrow \ker_A T \longrightarrow \ker_B T \longrightarrow \ker_C T \longrightarrow A/\mathrm{Im}_A T \longrightarrow B/\mathrm{Im}_B T \longrightarrow C/\mathrm{Im}_C T \longrightarrow 0$$

Thus, if $T : A \longrightarrow A$ is surjective, $H_0(A,T) = A/\mathrm{Im}_A T = 0$, and

$$\ker_B T \longrightarrow \ker_C T \longrightarrow A/\mathrm{Im}_A T \;=\; 0$$

is exact, so the natural map $\ker_B T \longrightarrow \ker_C T$ is surjective. And, if $H_1(A,T) = \ker_A T = 0$, then

$$0 \;=\; \ker_A T \longrightarrow \ker_B T \longrightarrow \ker_C T$$

is exact, so the natural map $\ker_B T \longrightarrow \ker_C T$ is injective. ///

There are several similar questions which use the same short exact sequence of complexes

$$0 \longrightarrow (A,T) \longrightarrow (B,T) \longrightarrow (C,T) \longrightarrow 0$$

attached to the exact sequence of modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and an endomorphism T. The associated long exact sequence in homology

$$0 \longrightarrow H_1(A,T) \longrightarrow H_1(B,T) \longrightarrow H_1(C,T) \longrightarrow H_0(A,T) \longrightarrow H_0(B,T) \longrightarrow H_0(C,T) \longrightarrow 0$$

is more explicitly

$$0 \longrightarrow \ker_A T \longrightarrow \ker_B T \longrightarrow \ker_C T \longrightarrow A/\mathrm{Im}_A T \longrightarrow B/\mathrm{Im}_B T \longrightarrow C/\mathrm{Im}_C T \longrightarrow 0$$

and then one may choose various pieces of this long exact sequence and create questions which are immediately interpretable as asking about vanishing of one or more of the six modules appearing.

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3. *Projectives and injectives*

Let R be a ring. An R-module P is **projective** if every diagram

$$\begin{tikzcd} & P \arrow[d] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

can be extended to a commutative diagram

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

Always a *free* module is projective, *at least if the ring has a unit.* Quite generally, *sums* of projectives are projective. An R-module I is **injective** if every diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & N \\ & I & \end{tikzcd}$$

can be extended to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[dl, dashed] \\ & I & \end{tikzcd}$$

Quite generally, *products* of injectives are injectives. For example, it is not too hard to show that $\mathbb{Q}/\mathbb{Z}$ is an injective module in the category of $\mathbb{Z}$-modules.

Note that the notions of projective and injective make sense in more general categories, not merely categories of modules, since their definitions are diagram-theoretic.

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4. *Resolutions*

Let M an R-module. A **(left) resolution** of M is an exact sequence

$$\ldots \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

Similarly, a **(right) resolution** of M is an exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \ldots$$

If each P_i in a *left* resolution is *projective*, then the resolution is called a **projective resolution**. Likewise, if every P_i is *free*, then the resolution is termed a **free resolution**. Since free implies projective, construction of a free resolution for any M will show that every M admits a projective resolution. We construct a free (left) resolution as follows. Let P_0 be the free R-module on the *set* M, and $P_0 \longrightarrow M$ the natural surjection, with kernel K_0. Let P_1 be the free R-module on the set K_0 with natural surjection $P_1 \longrightarrow K_0$, so that

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is exact. Continuing inductively we obtain a free resolution, hence a projective resolution. A category in which every object is the quotient of a projective, and (hence) has a projective resolution, is said to **have enough projectives**. The argument just given shows that any category of all modules over a ring with unit (so that free implies projective) *has enough projectives.*

If each I_i in a *right* resolution is *injective*, then the resolution is called an **injective resolution**. A category in which every object has in injection to an injective, and (hence) has an injective resolution, is said to

have enough injectives. The simplest example of a category *with enough injectives* is the category of finite-dimensional vector spaces over a field. The category of *torsion* $\mathbb{Z}$-modules is a less trivial example of a category with enough injectives.

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5. *Derived functors*

A functor F from R-modules to R-modules is an **additive functor** if we have natural isomorphisms

$$F(M \oplus N) \approx FM \oplus FN$$

for all R-modules M, N. Let F be a *(covariant) additive functor* from R-modules to R-modules. It is easy to check that, given a complex

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

the image

$$\cdots \longrightarrow FC_n \xrightarrow{F\partial_n} \cdots \xrightarrow{F\partial_2} FC_1 \xrightarrow{F\partial_1} FC_0 \xrightarrow{F\partial_0} 0$$

of C under F is still a complex. A projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

of an R-module M is exact, so is certainly a complex. Application of F gives a complex

$$\cdots \longrightarrow FP_n \xrightarrow{F\partial_n} \cdots \xrightarrow{F\partial_2} FP_1 \xrightarrow{F\partial_1} FP_0 \xrightarrow{F\epsilon} FM \longrightarrow 0$$

The associated **deleted complex** FP' is defined to be

$$\cdots \longrightarrow FP_n \xrightarrow{F\partial_n} \cdots \xrightarrow{F\partial_2} FP_1 \xrightarrow{F\partial_1} FP_0 \longrightarrow 0$$

The **n-th left derived functor** $\mathrm{L}^n F$ of F evaluated on M is defined to be the n^{th} homology

$$\mathrm{L}^n F(M) \;=\; H_n(FP')$$

of the deleted complex FP'. Similarly, for a *left exact* (additive) functor F, let

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \ldots$$

be an injective resolution of M. Apply F to obtain a complex

$$0 \longrightarrow FM \longrightarrow FI_0 \longrightarrow FI_1 \longrightarrow \ldots$$

and then the deleted complex FI'

$$0 \longrightarrow FI_0 \longrightarrow FI_1 \longrightarrow \ldots$$

The **n-th right derived** functor $\mathrm{R}^n F$ of F evaluated on M is the n^{th} cohomology $H^n(FI')$ of the deleted complex FI'.

In both cases, for the derived functors to be well-defined *we must show that the indicated (co-)homology groups do not depend upon the choice of resolution.* We will treat the projective case only, as the injective case is identical except for the direction of arrows. The following assertion is more than we need.

Given a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r, "\partial_1"] & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] \arrow[d, "f_{-1}"] & 0 \\
\cdots \arrow[r] & C_1 \arrow[r, "\partial_1'"] & C_0 \arrow[r, "\varepsilon'"] & M \arrow[r] & 0
\end{tikzcd}$$

where both rows are complexes, the P_i are *projective*, and the lower row is *exact*. Then there is a *chain complex map* $f = \{f_i\}$ extending f_{-1} in the sense that the squares commute in

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f_{-1}"] & 0 \\
\cdots \arrow[r] & C_1 \arrow[r, "\partial_1'"] & C_0 \arrow[r, "\varepsilon'"] & M \arrow[r] & 0
\end{tikzcd}$$

Further, any two such chain maps f and g extending f_{-1} are *chain homotopic*. Both assertions follow easily by induction, using the defining property of projectivity.

Now we want to show that a short exact sequence

$$0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^3 \longrightarrow 0$$

gives rise ('naturally') to a long exact sequence

$$\ldots \longrightarrow \mathrm{L}^2F(M^1) \longrightarrow \mathrm{L}^2F(M^2) \longrightarrow \mathrm{L}^2F(M^3) \longrightarrow \mathrm{L}^1F(M^1) \longrightarrow \mathrm{L}^1F(M^2) \longrightarrow \mathrm{L}^1F(M^3) \longrightarrow$$

$$\longrightarrow \mathrm{L}^0F(M^1) \longrightarrow \mathrm{L}^0F(M^2) \longrightarrow \mathrm{L}^0F(M^3) \longrightarrow 0$$

Evidently, one must create *compatible* projective resolutions P^i of M^i to obtain a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P_1^1 \arrow[r] \arrow[d] & P_0^1 \arrow[r] \arrow[d] & M^1 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_1^2 \arrow[r] \arrow[d] & P_0^2 \arrow[r] \arrow[d] & M^2 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_1^3 \arrow[r] \arrow[d] & P_0^3 \arrow[r] \arrow[d] & M^3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. Further, we must require that upon application of F and taking deleted complexes $(FP^i)'$, we have a *short exact sequence of complexes*

$$0 \longrightarrow (FP^1)' \longrightarrow (FP^2)' \longrightarrow (FP^3)' \longrightarrow 0$$

For construction of compatible projective resolutions, it suffices to prove the following. If $P^1 \longrightarrow M^1$ and

$P^3 \longrightarrow M^3$ are surjections with P^i projective, then the diagram

$$\begin{tikzcd} & 0 \arrow[d] & \\ P^1 \arrow[r] & M^1 \arrow[r] \arrow[d] & 0 \\ & M^2 \arrow[r] \arrow[d] & 0 \\ P^3 \arrow[r] & M^3 \arrow[r] \arrow[d] & 0 \\ & 0 & \end{tikzcd}$$

may be enlarged to a diagram

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] & \\ P^1 \arrow[r] \arrow[d] & M^1 \arrow[r] \arrow[d] & 0 \\ P^1 \oplus P^3 \arrow[r] \arrow[d] & M^2 \arrow[r] \arrow[d] & 0 \\ P^3 \arrow[r] \arrow[d] & M^3 \arrow[r] \arrow[d] & 0 \\ 0 & 0 & \end{tikzcd}$$

with exact rows and columns. This is done as follows. The map $P^1 \oplus P^3 \longrightarrow M^2$ should be defined on the summand P^1 via $P^1 \longrightarrow M^1 \longrightarrow M^2$, and on the summand P^3 via the map $P^3 \longrightarrow M^3$ and invoking the projectivity.

It remains to show that, for an exact sequence

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$$

of projective modules, and for a right-exact functor F from R-modules to R-modules, the sequence

$$0 \longrightarrow FP' \longrightarrow FP \longrightarrow FP'' \longrightarrow 0$$

is also exact. Since P'' is projective, the short exact sequence splits. Thus, there is *some* isomorphism

$$P \approx P' \oplus P''$$

so that the maps $P \longrightarrow P''$ and $P' \longrightarrow P$ induce the natural quotient map $P' \oplus P'' \longrightarrow P''$ and the natural inclusion $P' \longrightarrow P' \oplus P''$, respectively. Then application of F gives the natural sequence of maps

$$FP' \longrightarrow FP' \oplus FP'' \longrightarrow FP''$$

by the additivity of F. Certainly

$$0 \longrightarrow FP' \longrightarrow FP' \oplus FP'' \longrightarrow FP'' \longrightarrow 0$$

is exact. That is, the sequence

$$0 \longrightarrow (FP^1)' \longrightarrow (FP^2)' \longrightarrow (FP^3)' \longrightarrow 0$$

of deleted complexes is a short exact sequence of complexes, so gives rise to a long exact sequence as claimed, which is the long exact sequence for the left derived functors $\mathrm{L}^n F$. ///

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6. *Acyclic resolutions*

More generally, for a fixed right-exact (resp., left-exact) functor F, say that a module A is (F-)**acyclic** if all higher left (resp., right) derived functors of F annihilate A, i.e., if $\mathrm{L}^n F(A) = 0$ (resp., $\mathrm{R}^n F(A) = 0$) for $n > 0$.

In fact, the proof that left (resp., right) derived functors' definitions do not depend upon the choice of projective (resp., injective) resolution shows that, if

$$0 \longrightarrow S \longrightarrow I_1 \xrightarrow{f_1} I_2 \xrightarrow{f_2} \cdots$$

is injective and

$$0 \longrightarrow S \longrightarrow A_1 \xrightarrow{g_1} A_2 \xrightarrow{g_2} \cdots$$

is merely F-*acyclic*, then we have a chain homotopy from

$$0 \longrightarrow FS \longrightarrow FI_1 \xrightarrow{Ff_1} FI_2 \xrightarrow{Ff_2} \cdots$$

to

$$0 \longrightarrow FS \longrightarrow FA_1 \xrightarrow{Fg_1} FA_2 \xrightarrow{Fg_2} \cdots$$

Therefore,

$$\mathrm{R}^n F(S) \approx \mathit{ker}\, Fg_n / \mathrm{im}\, Fg_{n-1}$$

That is, *if there is at least one injective resolution of S, then the right derived functors $\mathrm{R}^n F$ of F evaluated on S can be computed via any F-acyclic resolution.* The same argument, with arrows reversed, shows that shows that *if there is at least one projective resolution of M, then the left derived functors $\mathrm{L}^n F$ of F can be computed via any F-acyclic resolution.*

The notions of *injective* and *projective* are *extrinsic* to the extent that they depends upon the ambient category. From the definition of derived functors it follows immediately that injectives are acyclic for *any* right derived functors, and projectives are acyclic for *any* left derived functors. One certainly might imagine that *universal acyclicity* of is than what is needed to prove acyclicity for specific functors.
