

(February 19, 2005)

# Homology and Derived Functors

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## 1. (Co-) homology of (co-) chain complexes

A **(chain) complex** (over  $R$ ) is a diagram

$$C = \{C_n, \partial_n\} = \dots \rightarrow C_n \xrightarrow{\partial_n} \dots \rightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

of  $R$ -modules with the property that

$$\partial_{n-1} \circ \partial_n = 0$$

The  $R$ -module  $C_i$  is the  $i^{\text{th}}$  **graded piece** of the complex. The operators  $\partial_n$  are **boundary maps**.

The **homology**

$$H_*(C) = \{H_n(C) : n \geq 0\}$$

of the complex  $C$  is the collection of quotients

$$H_n(C) = \ker \partial_n / \text{Im } \partial_{n+1}$$

of the  $n$ -**cycles**  $\ker \partial_i$  by the  $n$ -**boundaries**  $\text{Im } \partial_{n+1}$ . Thus, the homology is a measure of the non-exactness of  $C$ .

Dually, a **cochain complex** is a diagram

$$C = \{C_n, \delta_n\} = 0 \rightarrow C_0 \xrightarrow{\delta_0} C_1 \xrightarrow{\delta_1} \dots$$

of  $R$ -modules with the property that

$$\delta_n \circ \delta_{n-1} = 0$$

The  $R$ -module  $C_i$  is the  $i^{\text{th}}$  **graded piece** of the complex.

The **cohomology**

$$H^*(C) = \{H^i(C) : i \geq 0\}$$

of the complex  $C$  is the collection of quotients

$$H^n(C) = \ker \delta_n / \text{Im } \delta_{n-1}$$

of the  $n$ -**cocycles**  $\ker \delta_n$  by the  $n$ -**coboundaries**  $\text{Im } \delta_{n-1}$ . The operators  $\delta_n$  are **coboundary maps**.

*Much of what can be said for chain complexes and homology carries over to co-chain complexes and cohomology simply by reversing arrows, and vice-versa. Thus, we may discuss just one of the two cases and leave the other as an exercise.*

A **chain map** or **complex map**

$$f : \{C_i, \partial_i\} \rightarrow \{C'_i, \partial'_i\}$$

is a collection  $f = \{f_i\}$  with

$$f_i : C_i \rightarrow C'_i$$

$$f_i \circ \partial_{i+1} = \partial_i \circ f_{i+1}$$

That is, all squares in the diagram

$$\begin{array}{ccccccc} \dots & \rightarrow & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \\ \dots & \rightarrow & C'_1 & \xrightarrow{\partial_1} & C'_0 & \xrightarrow{\partial'_0} & 0 \end{array}$$

commute.

If  $f : C \rightarrow C'$  is a chain map, then there are **induced maps on homology**

$$H_n(f) : H_n(C) \rightarrow H_n(C')$$

defined by

$$H_n(f)(\zeta + \partial_{n+1}C_{n+1}) = f_n\zeta + \partial'_{n+1}C'_{n+1}$$

for  $\partial_n\zeta = 0$ . The defining property of ‘chain map’ assures that this is well-defined.

A short exact sequence

$$0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$$

of complexes is **exact** if the associated short exact sequences

$$0 \rightarrow C'_i \rightarrow C_i \rightarrow C''_i \rightarrow 0$$

are all exact sequences of  $R$ -modules.

The most basic result here is

**Theorem:** A short exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} B \rightarrow 0$$

of complexes gives rise to a natural **long exact sequence in homology**

$$\dots \rightarrow H_2(A) \xrightarrow{H_2(f)} H_2(B) \xrightarrow{H_2(g)} H_2(C) \xrightarrow{\partial_2} H_1(A) \xrightarrow{H_1(f)} H_1(B) \xrightarrow{H_1(g)} H_1(C) \xrightarrow{\partial_1} H_0(A) \xrightarrow{H_0(f)} H_0(B) \xrightarrow{H_0(g)} H_0(C) \rightarrow 0$$

**Remark:** The **connecting homomorphisms**  $\partial_i$  will be defined in the course of the proof. (We tolerate the two different uses of the symbol ‘ $\partial$ ’).

*Proof:* For now, we only show how to define the connecting homomorphisms

$$H_n(C) \xrightarrow{\partial_n} H_{n-1}(A)$$

For  $c_n \in C_n$  with  $\partial_n c_n = 0$ , the surjectivity of  $g : B_n \rightarrow C_n$  assures that there is  $b_n \in B_n$  so that  $gb_n = c_n$ . Then the fact that  $g$  is a chain map assures that

$$g(\partial_n b_n) = \partial_n(gb_n) = \partial_n c_n = 0$$

Exactness assures that there is  $x_{n-1} \in A_{n-1}$  so that

$$fx_{n-1} = \partial_n b_n$$

We define the connecting homomorphism *on homology* by

$$\partial_n(c_n + \partial_{n+1}C_{n+1}) = x_{n-1} + \partial_n A_n$$

Checking that this is well-defined (on homology) is a non-trivial but standard exercise left to the reader, as is the diagram-chase verification of exactness at the three different types of joints in the long sequence. ///

A **chain homotopy**  $\theta : f \rightarrow g$  from one complex map  $f : C \rightarrow C'$  to another  $g : C \rightarrow C'$  is a collection  $\theta = \{\theta_i\}$  with

$$\theta_i : C_i \rightarrow C'_{i+1}$$

and so that

$$\partial'_{i+1} \circ \theta_i + \theta_{i-1} \circ \partial_i = f - g$$

**Proposition:** If  $f$  and  $g$  are chain homotopic chain maps  $C \rightarrow C'$ , then the induced maps on homology are identical. Indeed, let  $\theta$  be a such a chain homotopy. Then with  $\partial_n \zeta = 0$  we have

$$\begin{aligned} f_n \zeta - g_n \zeta &= \partial'_{n+1} \circ \theta_n \zeta + \theta_{n-1} \circ \partial_n \zeta = \\ &= \partial'_{n+1} \circ \theta_n \zeta \end{aligned}$$

since  $\partial_n \zeta = 0$ . That is,

$$f_n \zeta - g_n \zeta = \partial'_{n+1}(\theta_n \zeta)$$

giving 0 in homology.

## 2. A small example

We can give a small but non-trivial examples of the utility of the long exact sequence in (co-) homology arising from a short exact sequence of (co-) chain complexes. The question addressed does not explicitly mention (co-) homology, but the discussion makes clear that the issues are genuinely homological in nature.

Let  $R$  be a not-necessarily-commutative ring, and

$$B \xrightarrow{q} C \rightarrow 0$$

a surjection of  $R$ -modules. Let  $T$  be an  $R$ -endomorphism of  $B$  which stabilizes  $\ker q$  so descends to an  $R$ -endomorphism of  $C$ . Let  $u \in C$  be an element such that  $Tu = 0$ . The question we wish to address is an **extension problem**: *Is there an element  $\tilde{u}$  of  $B$  such that  $q\tilde{u} = u$  and still  $T\tilde{u} = 0$ ? Is  $\tilde{u}$  unique? Such  $\tilde{u}$  would be an **extension** of  $u$ .*

- In the situation just above, if  $T$  is injective and surjective on  $A$ , then there is a unique extension  $\tilde{u}$  of  $u$ .

*Proof:* Let  $A = \ker q$ , so that

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{q} C \rightarrow 0$$

is a short exact sequence of  $R$ -modules. Consider the exact sequence of complexes of  $R$ -modules

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A & \xrightarrow{i} & B & \xrightarrow{q} & C \rightarrow 0 \\ & & T\downarrow & & T\downarrow & & T\downarrow \\ 0 & \rightarrow & A & \xrightarrow{i} & B & \xrightarrow{q} & C \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Viewing a complex  $(M, T)$

$$0 \xrightarrow{i} M \xrightarrow{T} M \xrightarrow{z} 0$$

attached to an  $R$ -module  $M$  and  $R$ -endomorphism  $T$  as being a chain complex, the associated homology is

$$H_0(M, T) = \ker z / \text{Im } T = M / \text{Im } T$$

$$H_1(M, T) = \ker T / \text{Im } i = \ker T$$

and higher homology is all 0. Therefore, the long exact sequence in homology shortens to

$$0 \rightarrow H_1(A, T) \rightarrow H_1(B, T) \rightarrow H_1(C, T) \rightarrow H_0(A, T) \rightarrow H_0(B, T) \rightarrow H_0(C, T) \rightarrow 0$$

which is

$$0 \rightarrow \ker_A T \rightarrow \ker_B T \rightarrow \ker_C T \rightarrow A / \text{Im}_A T \rightarrow B / \text{Im}_B T \rightarrow C / \text{Im}_C T \rightarrow 0$$

Thus, if  $T : A \rightarrow A$  is surjective,  $H_0(A, T) = A / \text{Im}_A T = 0$ , and

$$\ker_B T \rightarrow \ker_C T \rightarrow A / \text{Im}_A T = 0$$

is exact, so the natural map  $\ker_B T \rightarrow \ker_C T$  is surjective. And, if  $H_1(A, T) = \ker_A T = 0$ , then

$$0 = \ker_A T \rightarrow \ker_B T \rightarrow \ker_C T$$

is exact, so the natural map  $\ker_B T \rightarrow \ker_C T$  is injective. ///

There are several similar questions which use the same short exact sequence of complexes

$$0 \rightarrow (A, T) \rightarrow (B, T) \rightarrow (C, T) \rightarrow 0$$

attached to the exact sequence of modules

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

and an endomorphism  $T$ . The associated long exact sequence in homology

$$0 \rightarrow H_1(A, T) \rightarrow H_1(B, T) \rightarrow H_1(C, T) \rightarrow H_0(A, T) \rightarrow H_0(B, T) \rightarrow H_0(C, T) \rightarrow 0$$

is more explicitly

$$0 \rightarrow \ker_A T \rightarrow \ker_B T \rightarrow \ker_C T \rightarrow A / \text{Im}_A T \rightarrow B / \text{Im}_B T \rightarrow C / \text{Im}_C T \rightarrow 0$$

and then one may choose various pieces of this long exact sequence and create questions which are immediately interpretable as asking about vanishing of one or more of the six modules appearing.

### 3. Projectives and injectives

Let  $R$  be a ring. An  $R$ -module  $P$  is **projective** if every diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ M & \rightarrow & N & \rightarrow & 0 \end{array}$$

can be extended to a commutative diagram

$$\begin{array}{ccccc} & & P & & \\ & \swarrow & \downarrow & & \\ M & \rightarrow & N & \rightarrow & 0 \end{array}$$

Always a *free* module is projective, *at least if the ring has a unit*. Quite generally, *sums* of projectives are projective.

An  $R$ -module  $I$  is **injective** if every diagram

$$\begin{array}{ccccc} 0 & \rightarrow & M & \rightarrow & N \\ & & \downarrow & & \\ & & I & & \end{array}$$

can be extended to a commutative diagram

$$\begin{array}{ccccc} 0 & \rightarrow & M & \rightarrow & N \\ & & \downarrow & \swarrow & \\ & & I & & \end{array}$$

Quite generally, *products* of injectives are injectives. For example, it is not too hard to show that  $\mathbf{Q}/\mathbf{Z}$  is an injective module in the category of  $\mathbf{Z}$ -modules.

Note that the notions of projective and injective make sense in arbitrary categories, not merely categories of modules, since their definitions are purely diagram-theoretic.

## 4. Resolutions

Let  $M$  an  $R$ -module. A **(left) resolution** of  $M$  is an exact sequence

$$\dots \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

Similarly, a **(right) resolution** of  $M$  is an exact sequence

$$0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \dots$$

If each  $P_i$  in a *left* resolution is *projective*, then the resolution is called a **projective resolution**. Likewise, if every  $P_i$  is *free*, then the resolution is termed a **free resolution**. Since free implies projective, construction of a free resolution for any  $M$  will show that every  $M$  admits a projective resolution. We construct a free (left) resolution as follows. Let  $P_0$  be the free  $R$ -module on the set  $M$ , and  $P_0 \rightarrow M$  the natural surjection, with kernel  $K_0$ . Let  $P_1$  be the free  $R$ -module on the set  $K_0$  with natural surjection  $P_1 \rightarrow K_0$ , so that

$$P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

is exact. Continuing inductively we obtain a free resolution, hence a projective resolution. A category in which every object is the quotient of a projective, and (hence) has a projective resolution, is said to **have enough projectives**. The argument just given shows that any category of all modules over a ring with unit (so that free implies projective) *has enough projectives*.

If each  $I_i$  in a *right* resolution is *injective*, then the resolution is called an **injective resolution**. A category in which every object has in injection to an injective, and (hence) has an injective resolution, is said to **have enough injectives**. The simplest example of a category *with enough injectives* is the category of finite-dimensional vector spaces over a field. The category of *torsion*  $\mathbf{Z}$ -modules is a less trivial example of a category with enough injectives.

## 5. Derived functors

A functor  $F$  from  $R$ -modules to  $R$ -modules is an **additive functor** if we have natural isomorphisms

$$F(M \oplus N) \approx FM \oplus FN$$

for all  $R$ -modules  $M, N$ . Let  $F$  be a (covariant) additive functor from  $R$ -modules to  $R$ -modules. It is easy to check that, given a complex

$$\dots \rightarrow C_n \xrightarrow{\partial_n} \dots \rightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

the image

$$\dots \rightarrow FC_n \xrightarrow{F\partial_n} \dots \rightarrow FC_1 \xrightarrow{F\partial_1} FC_0 \xrightarrow{F\partial_0} 0$$

of  $C$  under  $F$  is still a complex. A projective resolution

$$\dots \rightarrow P_n \xrightarrow{\partial_n} \dots \rightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \rightarrow 0$$

of an  $R$ -module  $M$  is exact, so is certainly a complex. Then application of  $F$  gives a complex

$$\dots \rightarrow FP_n \xrightarrow{F\partial_n} \dots \rightarrow FP_1 \xrightarrow{F\partial_1} FP_0 \xrightarrow{F\epsilon} FM \rightarrow 0$$

The associated **deleted complex**  $FP'$  is defined to be

$$\dots \rightarrow FP_n \xrightarrow{F\partial_n} \dots \rightarrow FP_1 \xrightarrow{F\partial_1} FP_0 \rightarrow 0$$

The  $n$ -th **left derived functor**  $L^n F$  of  $F$  evaluated on  $M$  is defined to be the  $n^{\text{th}}$  homology

$$L^n F(M) = H_n(FP')$$

of the deleted complex  $FP'$ .

Similarly, for a *left exact* (additive) functor  $F$ , let

$$0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow \dots$$

be an injective resolution of  $M$ . Apply  $F$  to obtain a complex

$$0 \rightarrow FM \rightarrow FI_0 \rightarrow FI_1 \rightarrow \dots$$

and then the deleted complex  $FI'$

$$0 \rightarrow FI_0 \rightarrow FI_1 \rightarrow \dots$$

The  $n$ -th **right derived functor**  $R^n F$  of  $F$  evaluated on  $M$  is the  $n^{\text{th}}$  cohomology  $H^n(FI')$  of the deleted complex  $FI'$ .

In both cases, for the derived functors to be well-defined *We must show that the indicated (co-)homology groups do not depend upon the choice of resolution.* We will treat the projective case only, as the injective case is identical except for the direction of arrows. The following assertion is more than we need.

Consider a diagram

$$\begin{array}{ccccccc} \dots & \rightarrow & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\epsilon} & M & \rightarrow & 0 \\ & & & & & & \downarrow f_{-1} & & \\ \dots & \rightarrow & C_1 & \xrightarrow{\partial'_1} & C_0 & \xrightarrow{\epsilon'} & M & \rightarrow & 0 \end{array}$$

where both rows are complexes, the  $P_i$  are *projective*, and the lower row is *exact*. Then there is a *chain complex map*  $f = \{f_i\}$  'extending'  $f_{-1}$  in the sense that we have commuting squares in

$$\begin{array}{ccccccc} \dots & \rightarrow & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\epsilon} & M & \rightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f_{-1} & & \\ \dots & \rightarrow & C_1 & \xrightarrow{\partial'_1} & C_0 & \xrightarrow{\epsilon'} & M & \rightarrow & 0 \end{array}$$

Further, any two such chain maps  $f$  and  $g$  extending  $f_{-1}$  are *chain homotopic*. Both assertions follow easily by induction, using the defining property of projectivity.

Now we want to show that a short exact sequence

$$0 \rightarrow M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow 0$$

gives rise ('naturally') to a long exact sequence

$$\begin{aligned} \dots \rightarrow L^2F(M^1) \rightarrow L^2F(M^2) \rightarrow L^2F(M^3) \rightarrow L^1F(M^1) \rightarrow L^1F(M^2) \rightarrow L^1F(M^3) \rightarrow \\ \rightarrow L^0F(M^1) \rightarrow L^0F(M^2) \rightarrow L^0F(M^3) \rightarrow 0 \end{aligned}$$

Evidently, one must create 'compatible' projective resolutions  $P^i$  of  $M^i$  to obtain a diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \rightarrow & P_1^1 & \rightarrow & P_0^1 & \rightarrow & M^1 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \rightarrow & P_1^2 & \rightarrow & P_0^2 & \rightarrow & M^2 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \rightarrow & P_1^3 & \rightarrow & P_0^3 & \rightarrow & M^3 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

with exact rows and columns. Further, we must require that upon application of  $F$  and taking deleted complexes  $(FP^i)'$ , we have a *short exact sequence of complexes*

$$0 \rightarrow (FP^1)' \rightarrow (FP^2)' \rightarrow (FP^3)' \rightarrow 0$$

For construction of 'compatible' projective resolutions, it suffices to prove the following. If  $P^1 \rightarrow M^1$  and  $P^3 \rightarrow M^3$  are surjections with  $P^i$  projective, then the diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ P^1 & \rightarrow & M^1 & \rightarrow & 0 \\ & & \downarrow & & \\ & & M^2 & \rightarrow & 0 \\ & & \downarrow & & \\ P^3 & \rightarrow & M^3 & \rightarrow & 0 \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

may be enlarged to a diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & P^1 & \rightarrow & M^1 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ P^1 \oplus P^3 & \rightarrow & M^2 & \rightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & P^3 & \rightarrow & M^3 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

with exact rows and columns. This is done as follows. The map  $P^1 \oplus P^3 \rightarrow M^2$  should be defined on the summand  $P^1$  via  $P^1 \rightarrow M^1 \rightarrow M^2$ , and on the summand  $P^3$  via the map  $P^3 \rightarrow M^3$  and invoking the projectivity.

It remains to show that if

$$0 \rightarrow P' \rightarrow P \rightarrow P'' \rightarrow 0$$

is an exact sequence of projective modules, and if  $F$  is a right exact functor from  $R$ -modules to  $R$ -modules, then

$$0 \rightarrow FP' \rightarrow FP \rightarrow FP'' \rightarrow 0$$

is also exact. Since  $P''$  is projective, the short exact sequence splits. Thus, there is *some* isomorphism

$$P \approx P' \oplus P''$$

so that the maps  $P \rightarrow P''$  and  $P' \rightarrow P$  induce the natural quotient map  $P' \oplus P'' \rightarrow P''$  and the natural inclusion  $P' \rightarrow P' \oplus P''$ , respectively. Then application of  $F$  gives the natural sequence of maps

$$FP' \rightarrow FP' \oplus FP'' \rightarrow FP''$$

by the additivity of  $F$ . Certainly

$$0 \rightarrow FP' \rightarrow FP' \oplus FP'' \rightarrow FP'' \rightarrow 0$$

is exact.

That is, the sequence

$$0 \rightarrow (FP^1)' \rightarrow (FP^2)' \rightarrow (FP^3)' \rightarrow 0$$

of deleted complexes is a short exact sequence of complexes, so gives rise to a long exact sequence as claimed, which is the long exact sequence for the left derived functors  $L^n F$ .

## 6. Acyclic resolutions

More generally, for a fixed right-exact (resp., left-exact) functor  $F$ , say that a module  $A$  is  $(F)$ -**acyclic** if all higher left (resp., right) derived functors of  $F$  annihilate  $A$ , i.e., if  $L^n F(A) = 0$  (resp.,  $R^n F(A) = 0$ ) for  $n > 0$ .

In fact, the proof that left (resp., right) derived functors' definitions do not depend upon the choice of projective (resp., injective) resolution shows that, if

$$0 \rightarrow S \rightarrow I_1 \xrightarrow{f_1} I_2 \xrightarrow{f_2} \dots$$

is injective and

$$0 \rightarrow S \rightarrow A_1 \xrightarrow{g_1} A_2 \xrightarrow{g_2} \dots$$

is merely  $F$ -acyclic, then we have a chain homotopy from

$$0 \rightarrow FS \rightarrow FI_1 \xrightarrow{Ff_1} FI_2 \xrightarrow{Ff_2} \dots$$

to

$$0 \rightarrow FS \rightarrow FA_1 \xrightarrow{Fg_1} FA_2 \xrightarrow{Fg_2} \dots$$

Therefore,

$$R^n F(S) \approx \ker Fg_n / \operatorname{im} Fg_{n-1}$$

That is, *if there is at least one injective resolution of  $S$ , then the right derived functors  $R^n F$  of  $F$  evaluated on  $S$  can be computed via any  $F$ -acyclic resolution.* The same argument, with arrows reversed, shows that *if there is at least one projective resolution of  $M$ , then the left derived functors  $L^n F$  of  $F$  can be computed via any  $F$ -acyclic resolution.*

The notions of 'injective' and 'projective' are 'extrinsic' to the extent that they depends upon the ambient category. From the definition of derived functors it follows immediately that injectives are acyclic for *any* right derived functors, and projectives are acyclic for *any* left derived functors. One certainly might imagine that this 'universal acyclicity' of is more than what is needed in order to prove acyclicity for specific functors.