Complex analysis examples 10

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If you want feedback from me on your treatment of these examples, please get your work to me by Friday, Mar 27, preferably as a PDF emailed to me.

10.1 Show that there is a well-defined, holomorphic function $1/\sqrt{1+z^4}$ on the region $|z| > 2$. Show that
$$\int_\gamma \frac{dz}{\sqrt{1+z^4}} = 0,$$
where $\gamma$ traces out $|z| = 2$.

10.2 Let $\gamma$ be a simple closed path counter-clockwise encircling 0, 2, and not enclosing $-2$. Let $\delta$ be a simple closed path counter-clockwise encircling $-2,0,$ and not enclosing 2. Show that there is a holomorphic function $1/\sqrt{z(z^2-4)}$ on the annulus $1 < |z-1| < 3$, and a holomorphic function $1/\sqrt{z(z^2-4)}$ on the annulus $1 < |z+1| < 3$. Show that the two
periods
$$\int_\gamma \frac{dz}{\sqrt{z(z^2-4)}} \quad \int_\delta \frac{dz}{\sqrt{z(z^2-4)}}$$
are linearly independent over $\mathbb{R}$.

10.3 Show that for irrational $\alpha \in \mathbb{R}$, the set $\{m+n\alpha : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

10.4 Let $v_1, \ldots, v_n$ be linearly independent vectors in $\mathbb{R}^n$, and $L = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_n$ the lattice generated by them. Let $\mathbb{R}^n$ has its usual inner product and associated metric. For $r > 0$ let $B_r$ be the ball of radius 0 centered at 0 $\in \mathbb{R}^n$. Show that for small-enough $r > 0$ we have $B_r \cap L = \{0\}$.

10.5 Let $L$ be a lattice in $\mathbb{R}^n$, that is, the $\mathbb{Z}$-module generated by $n$ vectors linearly independent over $\mathbb{R}$. Prove that
$$\sum_{\lambda \neq \lambda \in L} \frac{1}{|\lambda|^n}$$
is absolutely convergent for $\text{Re}(s) > n$, where $|\cdot|$ is the usual length in $\mathbb{R}^n$. (Do not invoke any non-existent integral tests in several variables, despite the fact that the idea of such gives a good heuristic.)

10.6 Recall that we need finite growth order $|f(z)| \ll e^{|z|^N}$ as $|z| \to +\infty$ in a strip $a \leq \text{Re}(z) \leq b$, for some $N$, before we can invoke the Phragmén-Lindelöf theorem. Use the integral representation of $\zeta(s)$ via $\theta(y)$, and properties of $\Gamma(s)$, to show that it has finite order of growth in $-1 \leq \text{Re}(s) \leq 2$.

10.7 Show that $f(x,y) = (x + iy)^\ell e^{-\pi(y^2+x^2)}$ is multiplied by $i^{-\ell}$ by Fourier transform
$$\hat{f}(\xi,\eta) = \int_{\mathbb{R}^2} e^{-2\pi i(\xi x + \eta y)} f(x,y) \, dx, \, dy$$
Hint: rewrite this in terms of $z = x + iy$ and $\bar{z}$, and another complex variable $w = \xi + i\eta$ and $\bar{w}$, and look for a chance to differentiate under the integral defining the Fourier transform.

10.8 Define a harmonic theta function $\Theta_\ell(y)$ by
$$\Theta_\ell(y) = \begin{cases} 
\frac{1}{4} \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} (m+in)^\ell e^{-\pi y(m^2+n^2)} & \text{for } \ell > 0 \\
\frac{1}{4} \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} (m-in)^\ell e^{-\pi y(m^2+n^2)} & \text{for } \ell < 0 
\end{cases}$$
Show that this is identically 0 unless $\ell$ is divisible by 4, and prove the functional equation

$$\Theta_\ell(1/y) = y \cdot \Theta_\ell(y)$$

[10.9] Let $\chi(\alpha) = (\alpha/|\alpha|)^\ell$ for $\alpha \in \mathbb{C}^\times$. The associated Hecke $L$-function on the Gaussian integers $\mathbb{Z}[i]$ is

$$L(s, \chi) = \frac{1}{\#\mathbb{Z}[i]} \sum_{0 \neq \alpha \in \mathbb{Z}[i]} \frac{\chi(\alpha)}{|\alpha|^{2s}}.$$ 

Show that this is identically 0 unless $\ell$ is divisible by 4. Prove that $L(s, \chi_\ell)$ has an analytic continuation and functional equation and has the integral representation

$$\pi^{-(s + \frac{\ell}{2})} \Gamma \left( s + \frac{\ell}{2} \right) L(s, \chi) = \int_0^\infty y^{s + \frac{\ell}{2}} \Theta_\ell(y) \frac{dy}{y} \quad \text{(for Re}(s) > 1)$$

[10.10] With $\chi_\ell(\alpha) = (\alpha/|\alpha|)^\ell$, and the $L$-functions $L(s, \chi)$ as in the previous example, express $L(4, \chi_{-8})$ as a polynomial in $L(2, \chi_{-4})$.

[10.11] Show how to achieve the effect of replacing a quartic by a cubic in an elliptic integral: exhibit a change of variables so that

$$\int_a^b \frac{dx}{\sqrt{x^4 - 1}} = \int_A^B \frac{dy}{\sqrt{4y^3 + 6y^2 + 4y + 1}}$$

[10.12] Fix a lattice $L$. Express

$$f(z) = \frac{1}{z^4} + \sum_{0 \neq \lambda \in L} \frac{1}{(z - \lambda)^4}$$

in terms of $\wp(z)$ and $\wp'(z)$.

[10.13] Express $\wp(2z)$ in terms of $\wp(z)$.

[10.14] Show that

$$\theta(z) = \sum_{v \in \mathbb{Z}^*} e^{\pi i |v|^2 \cdot z} \quad \text{(with } z \in \mathfrak{H})$$

is an elliptic modular form of weight 4 for the congruence subgroup $\Gamma_0$. 

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Paul Garrett: Complex analysis examples 10 (March 19, 2015)