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Phragmén-Lindelöf Theorems

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The paper that gave its name to these results is

E. Phragmén, E. Lindelöf, *Sur une extension d'un principe classique de l'analyse*, Acta Math. **31** (1908), 381–406.

The context is that the maximum modulus principle^[1] in complex analysis does not apply to *unbounded* regions. That is, holomorphic functions on an unbounded region may be bounded by 1 on the edges but be violently unbounded in the interior.

The simplest example is $f(z) = e^{e^z}$. Obviously for z real and going to $+\infty$ this function blows up. Indeed,

$$\left| e^{e^{x+iy}} \right| = e^{\operatorname{Re}(e^{x+iy})} = e^{e^x \cdot \cos y}$$

Thus, for fixed $y = \operatorname{Im} z$ with $\cos y > 0$, the function blows up as $x = \operatorname{Re} z \rightarrow +\infty$. On the other hand, for $\cos y = 0$ the function is *bounded*. Thus, on the strip $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, the function e^{e^z} is bounded on the edges but blows up as $x \rightarrow +\infty$.

This example suggests *growth conditions* under which a bound of 1 on the edges implies the same bound throughout the strip.

[1.0.1] Theorem: Let f be a holomorphic function on the horizontal half-strip

$$\{z : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \text{ and } 0 \leq x\}$$

If

$$|f(z)| \ll e^{e^{C \cdot \operatorname{Re} z}} \quad (\text{for some constant } 0 \leq C < 1)$$

then $|f(z)| \leq 1$ on the edges of the half-strip implies $|f(z)| \leq 1$ in the interior, as well.

Proof: Unsurprisingly, the proof is a reduction to the usual maximum modulus principle. Take any fixed D in the range

$$C < D < 1$$

The function

$$F_\varepsilon(z) = f(z)/e^{\varepsilon e^{D \cdot z}} \quad (\text{for } \varepsilon > 0)$$

is certainly bounded by 1 on the edges of the half-strip, and in the interior goes to 0 uniformly in y as $x \rightarrow +\infty$, for fixed $\varepsilon > 0$. (The uniform decay in the interior is where the modification with D is used.) Thus, on a rectangle

$$R_T = \{z : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \text{ and } 0 \leq x \leq T\}$$

for sufficiently large $T > 0$ depending upon ε , the function F_ε is bounded by 1 on the edge. The usual maximum modulus principle implies that F_ε is bounded by 1 throughout. That is, for each fixed z_o in the half-strip,

$$|f(z_o)| \leq e^{\varepsilon \cdot e^{D \operatorname{Re} z_o}} \quad (\text{for all } \varepsilon > 0)$$

We can let $\varepsilon \rightarrow 0^+$, giving $|f(z_o)| \leq 1$. ///

[1] The *maximum modulus principle* in complex analysis is that a holomorphic function f on a bounded region in \mathbb{C} with $|f(z)| \leq 1$ on the edges is bounded by 1 in the interior, as well.

[1.0.2] **Remark:** Analogous theorems on strips of other widths follow by using $e^{c \cdot e^z}$ with suitable constants c .

An analogous theorem on a full strip, rather than half-strip, follows by using a function like $e^{\cosh z}$ in place of e^{e^z} , as follows.

[1.0.3] **Theorem:** Let f be a holomorphic function on the full horizontal strip

$$\left\{z : -\frac{\pi}{2} \leq \operatorname{Im} z \leq \frac{\pi}{2}\right\}$$

If

$$|f(z)| \ll e^{\cosh C \cdot \operatorname{Re} z} \quad (\text{for some constant } 0 \leq C < 1)$$

then $|f(z)| \leq 1$ on the edges of the strip implies $|f(z)| \leq 1$ in the interior, as well.

Proof: Again, reduce to the maximum modulus principle. Fix D in the range $C < D < 1$. The function

$$F_\varepsilon(z) = f(z)/e^{\varepsilon \cosh Dz} \quad (\text{for } \varepsilon > 0)$$

is bounded by 1 on the edges of the strip, and in the interior goes to 0 uniformly in y as $x \rightarrow \pm\infty$, for fixed $\varepsilon > 0$. Thus, on a rectangle

$$R_T = \left\{z : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \text{ and } -T \leq x \leq T\right\} \quad (\text{for large } T > 0, \text{ depending upon } \varepsilon)$$

the function F_ε is bounded by 1 on the edge. The usual maximum modulus principle implies that F_ε is bounded by 1 throughout. That is, for each fixed z_o in the half-strip,

$$|f(z_o)| \leq e^{\varepsilon \cosh D \operatorname{Re} z_o} \quad (\text{for all } \varepsilon > 0)$$

We can let $\varepsilon \rightarrow 0^+$, giving $|f(z_o)| \leq 1$. ///

The details of various adjustments can be made to disappear by strengthening the hypotheses:

[1.0.4] **Corollary:** Let f be a holomorphic function on a strip or half-strip, with a bound

$$|f(z)| \ll e^{|z|^A} \quad (\text{for some } A > 0)$$

If $|f(z)| \leq 1$ on the edges of the (half-)strip, then $|f(z)| \leq 1$ in the interior, as well. ///

[1.0.5] **Remark:** Further variations are easily possible, by additional adjustments of functions. For example, *polynomial growth* of a function f on the edges of a strip or half-strip can be accommodated by considering $f(z)/(z - z_o)^M$ for z_o outside the strip, and large M .