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Topological vectorspaces

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This little part is the first introduction to the notion of *topological vectorspace* in greatest generality. This would only be motivated after one is already acquainted with Hilbert spaces, Banach spaces, Fréchet spaces, and perhaps to understand that other important examples don't fall into these classes.

Some basic concepts are introduced which do not require the presence of a metric. Further, some concepts which would *appear* to depend upon having a metric are given sense in this more general context.

The last point is that even in this generality *finite-dimensional* topological vectorspaces have just one possible topology. This has immediate consequences for maps to and from finite-dimensional topological vectorspaces.

All this set-up works perfectly well with very mild hypotheses on the scalars involved.

After this set-up, we are ready to look at the Baire Category Theorem and pursuant results: Banach-Steinhaus (Uniform Boundedness), Open Mapping, Closed Graph, etc.

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1. A natural non-Fréchet space of functions

There are many natural spaces of functions that are most definitely *not* Fréchet spaces. Most often these spaces have some sort of support condition.

Let

$$C_c^o(\mathbf{R}) = \{\text{compactly-supported continuous } \mathbf{C}\text{-valued functions on } \mathbf{R}\}$$

This is a strictly smaller space than the space $C^o(\mathbf{R})$ of *all* continuous functions on $\mathbf{R}$, which we saw *was* Fréchet. Of course, we can express this function space as an *ascending union*

$$C_c^o(\mathbf{R}) = \bigcup_{N=1}^{\infty} \{f \in C_c^o(\mathbf{R}) : \mathrm{spt} f \subset [-N, N]\}$$

Note, too, that each space

$$C_N^o = \{f \in C_c^o(\mathbf{R}) : \mathrm{spt} f \subset [-N, N]\} \subset C^o[-N, N]$$

is strictly smaller than the space $C^o[-N, N]$ of *all* continuous functions on the interval $[-N, N]$, since functions in C_N^o must have values 0 at the endpoints $\pm N$. Still, C_N^o is a *closed* subspace of the Banach space $C^o[-N, N]$ (with sup norm), since a sup-norm limit of functions vanishing at $\pm N$ must also vanish there. Thus, each individual C_N^o *is* a Banach space.

Also, observe that for $0 < M < N$ the space C_M^o is a *closed* subspace of C_N^o (with sup norm), since the property of vanishing off $[-M, M]$ is preserved under sup-norm limits.

But for $0 < M < N$ the space C^o_M is *nowhere dense* in C^o_N, since an open ball of radius $\varepsilon > 0$ around any function in C^o_N contains many functions with non-zero values off $[-M, M]$.

Thus, the *set* $C^o_c(\mathbf{R})$ is an ascending union of a countable collection of subspaces, each closed in its successor, but nowhere-dense there.

Any acceptable topology on $C^o_c(\mathbf{R})$ should give subspace C^o_M its natural (Banach-space) topology. *Then* $C^o_c(\mathbf{R})$ *is a countable union of nowhere-dense subsets.* By the Baire category theorem the topology on $C^o_c(\mathbf{R})$ cannot be given by a complete metric. In particular, it cannot be Fréchet.

Nevertheless, the space $C^o_c(\mathbf{R})$ and many similarly-constructed spaces *do* have a reasonable structure. In particular, being the ascending union of a countable collection of Fréchet spaces, each closed in the next, is a structure of **LF-space**. [1]

Remark: The space of integrals against regular Borel measures on a σ-compact [2] topological space X can be construed (either *defined* to be or *proven* [3] to be, depending on one's choice) to be the collection of continuous linear maps $C^o_c(X) \longrightarrow \mathbf{C}$. This gives considerable motivation to understand the topology of $C^o_c(X)$, and, thus, to understand non-Fréchet spaces.

Remark: A similar argument proves that the space $\mathrm{C}^\infty_c(\mathbf{R}^n)$ of **test functions** (compactly-supported infinitely differentiable functions) on $\mathbf{R}^n$ cannot be Fréchet. These functions play a critical role in the study of *distributions* or *generalized functions*, so we must be able to accommodate spaces that are not Fréchet. Again, $\mathrm{C}^\infty_c(\mathbf{R}^n)$ is an LF-space.

2. First definitions: topological vectorspaces

For the moment, the 'scalars' certainly need not be the real or complex numbers, need not be locally compact, and need not even be commutative. Let k be a division ring. Any k-module V is a *free* k-module. [4] *We will substitute 'k-vectorspace' for 'k-module' in what follows.*

Suppose that k has a **norm** $|\,|$. That is, $|\,|$ is a non-negative real-valued function on k so that

$$|x| = 0 \implies x = 0$$

$$|xy| = |x||y|$$

$$|x + y| \le |x| + |y|$$

for all $x, y \in k$. Further, we suppose that with regard to the metric

$$d(x, y) = |x - y|$$

the topological space k is *complete* and *non-discrete.* The latter assertion is that, for every $\varepsilon > 0$ there is $x \in k$ so that

$$0 < |x| < \varepsilon$$

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[1] This stands for *limit of Fréchet.* The topology on the union is best described as a *colimit*, as we will do a bit later.

[2] As usual, *σ-compact* means that the space is a countable union of compacts.

[3] This is the Riesz-Markov-Kakutani theorem.

[4] The proof of this free-ness is the same as the proof that a vector space over a (commutative) field is free, that is, has a basis. The argument is often called the *Lagrange replacement principle*, and if suitably set up works as well for infinite-dimensional vector spaces, as long as we grant ourselves the Axiom of Choice or equivalent.

A **topological vector space** V (over k) is a k-vectorspace V with a topology on V in which *points are closed*, and so that the scalar multiplication

$$x \times v \to xv \text{ for } x \in k \text{ and } v \in V$$

and the vector addition

$$v \times w \to v + w \text{ for } v, w \in V$$

are *continuous.*

For two subsets X, Y of V, let

$$X + Y = \{x + y : x \in X, y \in Y\}$$

Also, write

$$-X = \{-x : x \in X\}$$

The following idea is completely elementary, but at the same time indispensable. Given an open neighborhood U of 0 in a topological vectorspace V, the continuity of vector addition assures that there is an open neighborhood U' of 0 so that

$$U' + U' \subset U$$

Since $0 \in U'$, necessarily $U' \subset U$ also. This can be repeated to give, for any positive integer n, an open neighborhood U_n of 0 so that

$$\underbrace{U_n + \ldots + U_n \subset U}_{n}$$

In a similar vein, for fixed $v \in V$ the map $V \to V$ by $x \to x + v$ is a *homeomorphism* as it is invertible by the obvious $x \to x - v$. Thus, *the open neighborhoods of* v *are of the form* $v + U$ *where* U *is an open neighborhood of* 0. In particular, giving a local basis at 0 specifies the whole topology on a topological vectorspace.

Lemma: Given a compact subset K of a topological vectorspace V, and given a closed subset C of V, if $K \cap C = \emptyset$ then there is an open neighborhood U of 0 in V so that

$$\text{closure}(K + U) \cap (C + U) = \emptyset$$

Proof: Take $x \in K$. Since C is closed, there is a neighborhood U_x of 0 so that the neighborhood $x + U_x$ of x does not meet C. By continuity of the vector addition

$$V \times V \times V \to V$$

$$(v_1, v_2, v_3) \to v_1 + v_2 + v_3$$

there is a smaller open neighborhood N_x of 0 so that

$$N_x + N_x + N_x \subset U_x$$

By replacing N_x by $N_x \cap -N_x$, which still is an open neighborhood of 0, we may also suppose that N_x is *symmetric* in the sense that $N_x = -N_x$.

Then, using this symmetry, we have

$$(x + N_x + N_x) \cap (C + N_x) = \emptyset$$

Since K is compact, there are finitely-many $x_1, \ldots, x_n$ so that

$$K \subset (x_1 + N_{x_1}) \cup \ldots \cup (x_n + N_{x_n})$$

Let

$$U = \bigcap_i N_{x_i}$$

Since the intersection is finite, this is an open set. Then

$$(K+U) \subset \bigcup_{i=1,\ldots,n} (x_i + N_{x_i} + U) \subset \bigcup_{i=1,\ldots,n} (x_i + N_{x_i} + N_{x_i})$$

All these sets are disjoint from $C+U$, by construction, since $U \subset N_{x_i}$ for all i.

Finally, since $C+U$ is a union of open sets $y+U$ for $y \in C$, it is open, so even the *closure* of $K+U$ does not meet $C+U$. ///

Corollary: A topological vectorspace is *Hausdorff*. (Take $K = \{x\}$ and $C = \{y\}$ in the lemma). ///

Corollary: The topological closure $\bar{E}$ of a subset E of a topological vectorspace V is obtained as

$$\bar{E} = \bigcap_U E + U$$

where U ranges over a local basis at 0.

Proof: In the lemma, take $K = \{x\}$ and $C = \bar{E}$ for a point x of V not in C. Then we obtain an open neighborhood U of 0 so that $x+U$ does not meet $\bar{E}+U$. The latter contains $E+U$, so certainly $x \notin E+U$. That is, for x not in the closure, there is an open U containing 0 so that $x \notin E+U$. This proves the assertion. ///

It is convenient to know that Hausdorff-ness of topological vectorspaces follows from the weaker assumption that points are closed.

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3. Quotients and linear maps

We continue to suppose that the *scalars* k are a *non-discrete complete normed division ring.*

For two topological vectorspaces V, W over k, a function

$$f : V \to W$$

is **(k-)linear** if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all $\alpha, \beta \in k$ and $x, y \in V$. Almost without exception we will be interested only in **continuous linear maps**, meaning linear maps which are continuous with respect to the topologies on V, W. The **kernel** $\ker f$ of a linear map is

$$\ker f = \{v \in V : f(v) = 0\}$$

Being the inverse image of a closed set by a continuous map, it is *closed.* It is easy to check that it is a k-subspace of V.

For a *closed* k-subspace H of a topological vectorspace V. We wish to form the *quotient* V/H as topological vectorspace, with k-linear quotient map $q : V \to V/H$ given as usual by

$$q : v \to v + H$$

The **quotient topology** on E is the *finest* topology on E such that the quotient map $q : V \longrightarrow E$ is continuous, namely, a subset E of V/H is open if and only if $q^{-1}(E)$ is open. It is easy to check that this is a topology.

Remark: If the subspace H is *not* closed, then the quotient topology on the quotient V/H is *not* Hausdorff [5] Non-Hausdorff spaces are not topological vector spaces in our sense. For our purposes, we do not want non-Hausdorff spaces.

Further, *unlike* more general topological quotient maps, we have

Proposition: For a closed subspace H of a topological vector space V, the quotient map $q : V \longrightarrow V/H$ is *open*, that is, carries open sets to open sets.

Proof: Let U be open in V. Then

$$q^{-1}(q(U)) = q^{-1}(U + H) = U + H = \bigcup_{h\in H} h + U$$

This is a union of opens, so is open. ///

Corollary: If W is a k-subspace of a topological vectorspace V and if W is a *closed* subset of V, then the quotient V/W, with the quotient topology, is a topological vectorspace. That is, in the quotient topology points are closed.

Proof: The *algebraic* quotient exists without any topological hypotheses on W. Since W is closed, and since vector addition is a homeomorphism, $v + W$ is closed as well. Thus, its complement $V - (v + W)$ is open, so $q(V - (v + W))$ is open, by definition of the quotient topology. Thus, the complement

$$q(v) = v + W = q(v + W) = V/W - q(V - (v + W))$$

of the open set $q(V - (v + W))$ is closed. ///

Corollary: Let $f : V \to X$ be a linear map so that $f(W) = \{0\}$. Let $\bar{f}$ be the induced map $\bar{f} : V/W \to X$ defined by $\bar{f}(v + W) = f(v)$. Then f is continuous if and only if $\bar{f}$ is continuous.

Proof: Certainly if $\bar{f}$ is continuous then $f = \bar{f} \circ q$ is continuous. The converse follows from the fact that q is *open.* ///

That is, a continuous linear map $f : V \to X$ *factors through* any quotient V/W where W is a closed subspace contained in the kernel of f.

4. More topological features

Now we can consider the notions of **balanced subset, absorbing subset** and also **directed set, Cauchy net**, and *completeness.* We continue to suppose that the 'scalars' k are a *non-discrete complete normed division ring.*

A subset E of V is **balanced** if for every $x \in k$ with $|x| \le 1$ we have $xE \subset E$.

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[5] That the quotient V/H by a not-closed subspace H is not Hausdorff is easy to see, using the definition of the quotient topology, as follows. Let v be in the closure of H but not in H. Then every neighborhood U of v meets H. Every neighborhood of $v + H$ in the quotient is of the form $v + H + U$ for some neighborhood U of v in V, and includes 0. That is, even though the image of v in the quotient is not 0, every neighborhood of that image includes 0.

Lemma: Let U be a neighborhood of 0 in a topological vectorspace V over k. Then U contains a *balanced* neighborhood N of 0.

Proof: By continuity of scalar multiplication, there is $\varepsilon > 0$ and a neighborhood U' of $0 \in V$ so that if $|x| < \varepsilon$ and $v \in U'$ then $xv \in U$. Since k is non-discrete, there is $x_o \in k$ with $0 < |x_o| < \varepsilon$. Since scalar multiplication by a non-zero element is a homeomorphism, $x_o U'$ is a neighborhood of 0 and $x_o U' \subset U$. Put

$$N = \bigcup_{|y| \leq 1} y x_o U'$$

Then, for $|x| \leq 1$, we have $|xy| \leq |y| \leq 1$, so

$$xN = \bigcup_{|y| \leq 1} x(y x_o U') \subset \bigcup_{|y| \leq 1} y x_o U' = N$$

This N is as desired. ///

A subset E of a (not necessarily topological) vectorspace V over k is **absorbing** if for every $v \in V$ there is $t_o \in \mathbf{R}$ so that $v \in \alpha E$ for every $\alpha \in k$ so that $|\alpha| \geq t_o$.

Lemma: Every neighborhood U of 0 in a topological vectorspace is *absorbing.*

Proof: We may as well shrink U so as to assure that U is balanced. By continuity of the map $k \to V$ given by $\alpha \to \alpha v$, there is $\varepsilon > 0$ so that $|\alpha| < \varepsilon$ implies that $\alpha v \in U$. By the non-discreteness of k, there is non-zero $\alpha \in k$ satisfying any such inequality. Then $v \in \alpha^{-1} U$, as desired. ///

Let S be a **poset**, that is, a set with a partial ordering $\geq$. We assume further that, given two elements $s, t \in S$, there is $z \in S$ so that $z \geq s$ and $z \geq t$. Then S is a **directed set**.

A **net** in V is a subset $\{x_s : s \in S\}$ of V indexed by a directed set S. A net $\{x_s : s \in S\}$ in a topological vectorspace V is a **Cauchy net** if, for every neighborhood U of 0 in V, there is an index s_o so that for $s, t \geq s_o$ we have $x_s - x_t \in U$. A net $\{x_s : s \in S\}$ is **convergent** if there is $x \in V$ so that, for every neighborhood U of 0 in V there is an index s_o so that for $s \geq s_o$ we have $x - x_s \in U$. Since points are closed, there can be *at most* one point to which a net converges. Thus, *a convergent net is Cauchy.* A topological vectorspace is **complete** if (also) every Cauchy net is convergent.

Lemma: Let Y be a vector subspace of a topological vector space X, and suppose that Y is *complete* when given the subspace topology from X. Then Y is a *closed* subset of X.

Proof: Let $x \in X$ be in the closure of Y. Let S be a local basis of opens at 0, where we take the partial ordering so that $U \geq U'$ if and only if $U \subset U'$. For each $U \in S$ choose

$$y_U \in (x + U) \cap Y$$

Then the net $\{y_U : U \in S\}$ converges to x, so is Cauchy. But then it must converge to a point in Y, so by uniqueness of limits of nets it must be that $x \in Y$. Thus, Y is closed. ///

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5. Finite-dimensional spaces

Now we look at the especially simple nature of finite-dimensional topological vectorspaces, and their interactions with other topological vectorspaces as well. Still we need only suppose that the scalar field k is a complete non-discrete normed division ring. The main point of this section is that *there is only one topology on a finite-dimensional space.* This has several important consequences.

Proposition: If the topological vectorspace V is one-dimensional, i.e., is a free module on one generator e, then the map $k \to V$ given by $x \to xe$ is a *homeomorphism.*

Proof: Since scalar multiplication is continuous, we need only show that the map is *open.* Given $\varepsilon > 0$, by the non-discreteness of k there is an element x_o of k so that $0 < |x_o| < \varepsilon$. Since V is Hausdorff, there is a neighborhood U of 0 so that $x_o \notin U$. We can shrink U if necessary to be able to assume that it is *balanced.* Take $x \in k$ so that $xe \in U$. If $|x| \geq |x_o|$ then $|x_o x^{-1}| \leq 1$, so that

$$x_o e = (x_o x^{-1})(xe) \in U$$

by the balanced-ness of U, contradiction. Thus, we see that

$$xe \in U \;\Rightarrow\; |x| < |x_o| < \varepsilon$$

This proves the claim. ///

Corollary: Fix $x_o \in k$. A not-identically-zero k-linear k-valued function f on V is *continuous* if and only if the **affine hyperplane**

$$H = \{v \in V : f(v) = x_o\}$$

is a *closed* subset of V.

Proof: Certainly if f is continuous then H is closed. For the converse, we need only consider the case $x_o = 0$, since translations (i.e., vector additions) are homeomorphisms of V to itself.

For any v_o with $f(v_o) \neq 0$ and for any other $v \in V$ we have

$$f(v - f(v)f(v_o)^{-1}v_o) = f(v) - f(v)f(v_o)^{-1}f(v_o) = 0$$

Thus, V/H is one-dimensional. Let $\bar{f} : V/H \to k$ be the induced k-linear map on V/H so that $f = \bar{f} \circ q$:

$$\bar{f}(v + H) = f(v)$$

Then $\bar{f}$ is a homeomorphism to k, by the previous result, so f is continuous. ///

In the following theorem, the three assertions are related, and we prove them together by induction on dimension.

Theorem:
- A *finite-dimensional* vectorspace V over k has just one topology on it which makes it into a topological vectorspace over k.
- A finite-dimensional subspace V of an arbitrary topological vectorspace W over k is necessarily a *closed* subspace of W.
- A k-linear map $\phi : X \to V$ is continuous if and only if the kernel is closed.

Proof: To prove the uniqueness of the topology, it suffices to prove that for any k-basis $e_1, \ldots, e_n$ for V, the map

$$k \times \ldots \times k \to V$$

given by

$$(x_1, \ldots, x_n) \to x_1 e_1 + \ldots + x_n e_n$$

is a homeomorphism. We will prove this by induction on the dimension n, i.e., on the number of generators for V as a free k-module.

The case $n = 1$ was treated already. Granting this, we need only further note that, since k is complete, the lemma above asserting the closed-ness of complete subspaces shows that any one-dimensional subspace is necessarily closed.

Take $n > 1$. Let

$$H = ke_1 + \ldots + ke_{n-1}$$

By induction, H is closed in V, so we can form the quotient V/H. Let q be the quotient map. The space V/H is a one-dimensional topological vectorspace over k, with basis $q(e_n)$. By induction, the map

$$\phi : xq(e_n) = q(xe_n) \to x$$

is a homeomorphism to k.

Likewise, ke_n is a closed subspace and we have the quotient map

$$q' : V \to V/ke_n$$

We have a basis $q'(e_1), \ldots, q'(e_{n-1})$ for the image, and by induction the map

$$\phi' : x_1 q'(e_1) + \ldots + x_{n-1} q'(e_{n-1}) \to (x_1, \ldots, x_{n-1})$$

is a homeomorphism.

Then, invoking the induction hypothesis, the map

$$v \to (\phi \circ q)(v) \times (\phi' \circ q')(v)$$

is a continuous map to

$$k^{n-1} \times k \approx k^n$$

On the other hand, by the assumption that scalar multiplication and vector addition are continuous, the map

$$k^n \to V$$

by

$$(x_1, \ldots, x_n) \to x_1 e_1 + \ldots + x_n e_n$$

is continuous. These two maps are mutual inverses, proving that we have a homeomorphism.

Thus, a n-dimensional subspace is homeomorphic to k^n, so is complete, since (as follows readily) a finite product of complete spaces is complete. Thus, by the lemma asserting the closed-ness of complete subspaces, it is closed.

For a linear map $f : X \to k^n$ continuity certainly implies that the kernel N is closed. On the other hand, if N is closed, then X/N is a topological vectorspace of dimension at most n. Therefore, the induced map $\bar{f} : X/N \to V$ is unavoidably continuous. But then $f = \bar{f} \circ q$ is continuous, where q is the quotient map. This completes the induction step. ///

6. Baire Category Theorem

Let X be a complete metric space, with metric d. [**... iou ...**]

7. Seminorms and Minkowski functionals

As usual, when the scalar field k contains $\mathbf{R}$, we say that a subset E of a (not necessarily topological) vectorspace is **convex** if, for all $0 \leq t \leq 1$ and for all $x, y \in E$, the point $tx + (1-t)y$ lies in E. *For the rest of this section we suppose that k contains* $\mathbf{R}$.

For a set E in a not-necessarily topological vectorspace V we can define the **Minkowski functional** μ_E on V, as follows. For $v \in V$, let $\mu_E(v)$ be the *infimum* of all positive real numbers t_o so that for all $t \in \mathbf{R}$ with $t \geq t_o$ we have $v \in tE$. If for a given v there is no such t_o, then put

$$\mu_E(v) = +\infty$$

As usual,

$$t + \infty = \infty$$

for any (finite) real number t. Thus, we need not assume that E is *absorbing* in order to define these functionals, if we allow infinite values.

Proposition: If E is convex, then

$$\mu_E(x+y) \leq \mu_E(x) + \mu_E(y)$$

and for $t \geq 0$

$$\mu_E(tx) = t\mu_E(x)$$

If E is balanced, then we have the stronger assertion that for all $\alpha \in k$

$$\mu_E(\alpha v) = |\alpha|\mu_E(v)$$

Conversely, if μ is a function on V assuming values which are non-negative real or $+\infty$, satisfying

$$\mu(x+y) \leq \mu(x) + \mu(y)$$

$$\mu(\alpha v) = |\alpha|\mu(v)$$

then the set

$$E = \{v \in V : \mu(v) \leq 1\}$$

is balanced and convex.

Proof: The property

$$\mu_E(\alpha v) = |\alpha|\mu_E(v)$$

follows simply from the fact that E is balanced.

Suppose $x \in sE$ for all $|s| \geq s_o$ and $y \in tE$ for all $|t| \geq t_o$. Then

$$|\,| + |\,| \leq 1$$

$$x + y \in sE + tE$$

for all $|s| \geq s_o$ and for all $|t| \geq t_o$.

8. Local convexity: Hahn-Banach Theorems

Now we suppose that the scalars are $\mathbf{R}$ or $\mathbf{C}$, since we will exploit *convexity*. Note that any complex vectorspace may be considered as a real vectorspace, simply by 'forgetting' the possibility of scalar multiplication by any but *scalars*. Thus, much of the following discussion will address *real* vectorspaces, with assertions for complex vectorspaces appearing as trivial consequences.

As usual, a subset E of a (not necessarily topological) vectorspace V over $\mathbf{R}$ is **convex** if $tx + (1-t)y \in E$ for all $x, y \in E$ and for all $0 \leq t \leq 1$.

A *topological* vectorspace V is **locally convex** if there is a local basis at 0 consisting of open *convex* neighborhoods of 0.

[... iou ...]

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9. Countably normed, countably Hilbert spaces

In practice, most Fréchet spaces have more structure than just the Fréchet structure: they are *projective limits of Hilbert spaces*, and even that in a rather special way. This type of additional information is exactly what is needed for several types of stronger results, concerning spectral theory, regularity results for differential operators, Schwartz-type kernel theorems, and so on.

The ideas here, although of considerable utility, are not made explicit as often as they merit. The present account is inspired by, and is partly an adaptation of, parts of the Gelfand-Shilov-Vilenkin-Graev monographs *Generalized Functions*. This material is meant to be a utilitarian substitute (following Gelfand *et alia*) for Grothendieck's somewhat more general concepts related to *nuclear spaces*.

Let V be a real or complex vectorspace with a collection of norms $|\,|_i$ for $i \in \mathbf{Z}$. We suppose that we have

$$\ldots \geq |v|_{-2} \geq |v|_{-1} \geq |v|_0 \geq |v|_1 \geq |v|_2 \geq \ldots$$

for all $v \in V$. Let V_i be the *Banach space* obtained by taking the *completion* of V with respect to the i^{th} norm $|\,|_i$. The inequalities relating the various norms assure that for $i \leq j$ the identity map of V to itself induces (extending by continuity) *continuous inclusions*

$$\phi_{ij} : V_i \to V_j$$

Then it makes sense to take the *intersection* of all the spaces V_i: this is more properly described as an example of a *projective limit of Banach spaces*

$$\bigcap_i V_i = \operatorname{proj}\lim_i V_i$$

It is clear that V is contained in this intersection (certainly in the sense that there is a natural injection, and so on). If the intersection is *exactly* V then V is a **countably normed space** or **countably Banach space**.

This situation can also arise when we have positive-definite hermitian inner products $\langle,\rangle_i$ with $i \in \mathbf{Z}$. Let $|\,|_i$ be the norm associated to $\langle,\rangle_i$. Again suppose that

$$|v|_i \geq |v|_{i+1}$$

for all $v \in V$ and for all indices i. If the intersection

$$\bigcap_i V_i = \operatorname{proj}\lim_i V_i \supset V$$

is *exactly* V then we say that V is a **countably Hilbert space**.

Remark: The notion of countably Hilbert space is worthwhile only for real or complex scalars, while the countably Banach concept has significant content over more general scalar fields.

Remark: We can certainly take projective limits over more complicated indexing sets. And we can take $|\,|_i = |\,|_{i+1}$ for $i \geq 0$ if we want to focus our attention only on the 'negatively indexed' norms or inner products.

10. Local countability

For any algebraic subspace Y of the dual space V^* of continuous linear functionals on V, *if Y separates points on V* we can form the Y-**(weak-)topology** on V by taking seminorms

$$\nu_\lambda(v) := |\lambda(v)|$$

for $\lambda \in Y$.

The assumption that Y separates points is necessary to assure that the topology attached to this collection of semi-norms is such that *points are closed*. For example, if V is locally convex and Y is all of V^*, then this separation property is assured by the Hahn-Banach theorem.

If Y separates points on V and if V is not a countable union of finite-dimensional subspaces, then the Y-topology on V cannot have a countable local basis.

For example, if V is an infinite-dimensional Frechet space, then (from Baire's theorem) its dual is not locally countable.

Proof: Given $y \in Y$ and $\varepsilon > 0$, suppose that there are $y_1, \ldots, y_n \in V$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ so that for $v \in V$ we have

$$|y_i(v)| < \varepsilon_i \ \forall i \ \Rightarrow |y(v)| < \varepsilon$$

If so, then certainly $y_i(v) = 0$ for all i would imply that $|y(v)| < \varepsilon$. Let H denote the closed subspace of $v \in V$ where $y_i(v) = 0$ for all i. Then $|y(v)| < \varepsilon$ on H implies that $y(v) = 0$ on H.

We claim that then y is a linear combination of the y_i. Without loss of generality we may suppose that the y_i are linearly independent. Consider the quotient map $q : V^{\rightarrow} V/H$. From elementary linear algebra, without any topological consideration, that the quotient V/H is n-dimensional, and has dual space spanned by the functionals

$$\bar{y}_i(v + H) = y_i(v)$$

The functional $y(v)$ induces a continuous functional

$$\bar{y} : V/H \to \mathbf{C}$$

since y vanishes on H. Thus, $\bar{y}$ is a linear combination of the $\bar{y}_i$.

The fact that $\bar{y}$ is a linear combination of the $\bar{y}_i$ implies that v is the corresponding linear combination of the v_i.

This shows that, if it were the case that V had a countable basis in the Y-topology, then there would be countably-many y_i so that every vector in Y would be a *finite* linear combination of the y_i. ///