

(January 11, 2007)

Banach-Alaoglu, boundedness, weak-to-strong principles

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- Banach-Alaoglu theorem: compactness of polars
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1. Banach-Alaoglu theorem

[1.1] **Definition:** The **polar** U° of an open neighborhood U of 0 in V is

$$U^\circ = \{\lambda \in V^* : |\lambda(u)| \leq 1, \text{ for all } u \in U\}$$

[1.2] **Theorem:** (*Banach-Alaoglu*) In the weak star-topology on V^* the polar U° of an open neighborhood U of 0 in V is *compact*.

Proof: For every v in V there is t_v sufficiently large real such that $v \in t_v \cdot U$. Then $|\lambda v| \leq t_v$. Let

$$D_v = \{z \in \mathbb{C} : |z| \leq t_v\}$$

and give

$$P = \prod_{v \in V} D_v$$

the product topology. By the Tychonoff theorem, P is compact. Certainly

$$U^\circ \subset V^* \cup P$$

To prove the compactness of U° it suffices to show that the weak topology on U° is identical to the subspace topology inherited from P , and that U° is closed in P .

Regarding the topologies, the sub-basis sets

$$\{\lambda \in V^* : |\lambda v - \lambda_o v| < \delta\}$$

for V^* and

$$\{p \in P : |p_v - \lambda_o v| < \delta\}$$

for P , respectively, (for $v \in V$ and $\delta > 0$) have identical intersections with U° . Thus, the weak star-topology on U° is the same as the product topology restricted to U° .

To show that U° is closed, consider L in the closure of U° in P . Given $x, y \in V$, $a, b \in \mathbb{C}$, the sets

$$\{p \in P : |(p - L)_x| < \delta\}$$

$$\{p \in P : |(p - L)_y| < \delta\}$$

$$\{p \in P : |(p - L)_{ax+by}| < \delta\}$$

are open in P and contain L , so meet U° . Let $\lambda \in U^\circ$ lie in the simultaneous intersection of these three sets and U° . Then

$$\begin{aligned} |aL_x + bL_y - L_{ax+by}| &\leq |a| \cdot |L_x - \lambda x| + |b| \cdot |L_y - \lambda y| + |L_{ax+by} - \lambda(ax + by)| + |a\lambda x + b\lambda y - \lambda(ax + by)| \\ &\leq |a| \cdot \delta + |b| \cdot \delta + \delta + 0 \end{aligned}$$

This holds for every δ , so L is linear.

Given $\varepsilon > 0$, let N be a neighborhood of 0 in V such that $x - y \in N$ implies

$$\lambda x - \lambda y \in N$$

Then

$$|L_x - L_y| = |L_x - \lambda x| + |L_y - \lambda y| + |\lambda x - \lambda y|\delta + \delta + \varepsilon$$

Thus, L is continuous.

Also,

$$|L_x - \lambda x| < \delta$$

for all $x \in U$ and all $\delta > 0$, so $L \in U^\circ$. ///

2. A variant Banach-Steinhaus theorem

The following variant of the Banach-Steinhaus (uniform boundedness) theorem is used along with Banach-Alaoglu to show that weak boundedness implies (original) boundedness in a locally convex space, which is the starting point for *weak-to-strong principles*.

[2.1] Theorem: (*Variant of Banach-Steinhaus*) Let K be a compact convex set in an arbitrary topological vector space X , and \mathcal{T} a set of continuous linear maps $X \rightarrow Y$ from X to some other topological vector space Y . Suppose that for every individual $x \in K$ the collection of images

$$\mathcal{T}(x) = \{Tx : T \in \mathcal{T}\}$$

is a *bounded* subset of Y . Then there is a bounded set B in Y so that $\mathcal{T}(x) \subset B$ for every $x \in K$.

Proof: Let B be the union

$$B = \bigcup_{x \in K} \mathcal{T}(x)$$

Let U, V be balanced neighborhoods of 0 in Y so that $\bar{U} + \bar{U} \subset V$, and let

$$E = \bigcap_{T \in \mathcal{T}} T^{-1}(\bar{U})$$

By the boundedness of $\mathcal{T}(x)$, there is a positive integer n so that $\mathcal{T}(x) \subset nU$. Then $x \in nE$. For each $x \in K$ there is such n (depending upon x), so surely

$$K = \bigcup_n (K \cap nE)$$

Since E is closed, the version of the Baire Category Theorem applicable to locally compact Hausdorff spaces implies that at least one set $K \cap nE$ has non-empty interior in K .

For such n , let x_o be an interior point of $K \cap nE$. Pick a balanced neighborhood W of 0 in X so that

$$K \cap (x_o + W) \subset nE$$

Since K is compact, the set K is bounded, so $K - x_o$ is bounded, for large enough positive real t

$$K \subset x_o + tW$$

For any $x \in K$, for $t \geq 1$, since K is convex the point

$$(1 - t^{-1})x + t^{-1}x$$

also is in K . At the same time,

$$z - x_o = t^{-1}(x - x_o) \in W$$

for large enough t , from the boundedness of K , so $z \in x_o + W$. Thus,

$$z \in K \cap (x_o + V) \subset nE$$

with the latter inclusion following from above.

From the definition of E , $T(E) \subset \bar{U}$, so

$$T(nE) = nT(E) \subset n\bar{U}$$

And $x = tz - (t - 1)x_o$ yields

$$Tx \in tn\bar{U} - (t - 1)n\bar{U} \subset tn(\bar{U} + \bar{U})$$

by the balanced-ness of U . Since $\bar{U} + \bar{U} \subset V$, we have

$$B \subset tnV$$

Since V was arbitrary, this proves the boundedness of the set B . ///

3. Bipolars

The **bipolar** N^{oo} of an open neighborhood N of 0 in a topological vector space V is

$$N^{oo} = \{v \in V : |\lambda v| \leq 1 \text{ for all } \lambda \in N^o\}$$

where N^o is the polar of N .

[3.1] Proposition: (*On bipolars*) Let V be a locally convex topological vectorspace. Let N be a convex and balanced neighborhood N of 0. Then the bipolar N^{oo} of N is the closure \bar{N} of N .

Proof: Certainly N is contained in N^{oo} , and in fact \bar{N} is contained in N^{oo} since N^{oo} is closed. Now use the local convexity of V . The Hahn-Banach theorems imply that for $v \in V$ but $v \notin \bar{N}$ there exists $\lambda \in V^*$ so that $\lambda v > 1$ and $|\lambda v'| \leq 1$ for all $v' \in \bar{N}$. Thus, this λ is in N^o , and we have proven that every element $v \in N^{oo}$ is in \bar{N} , so $N^{oo} = \bar{N}$ as claimed. ///

4. Weak boundedness implies strong boundedness

[4.1] Theorem: Let V be a locally convex topological vectorspace. A subset E of V is bounded if and only if it is weakly bounded.

Proof: That boundedness implies weak boundedness is trivial. On the other hand, suppose E is weakly bounded, and let U be a neighborhood of 0 in V , in the original topology. By the local convexity, there is a convex (and balanced) neighborhood N of 0 so that the closure \bar{N} is contained in U .

By the weak boundedness of E , for each $\lambda \in V^*$ a bound b_λ so that $|\lambda x| \leq b_\lambda$ for $x \in E$. By the Banach-Alaoglu theorem the polar N° of N is compact in V^* . The functions $\lambda \rightarrow \lambda x$ are continuous, so by the variant Banach-Steinhaus theorem above there is a uniform constant $b < \infty$ so that $|\lambda x| \leq b$ for $x \in E$ and $\lambda \in N^\circ$. Thus, $b^{-1}x$ is in the bipolar $N^{\circ\circ}$ of N , which the previous proposition showed to be equal to the closure \bar{N} of N . That is, $b^{-1}x \in \bar{N}$. Thus, by the balanced-ness of N , for any $t > b$

$$E \subset t\bar{N} \subset tU$$

This shows that E is bounded. ///

5. Weak-to-strong differentiability

Here is a useful application of the fact that weak boundedness implies boundedness. It seems that the result below is well-known, at least folklorically, for the case of Banach-space-valued functions is well-known, but the simple general case is generally treated as though it were apocryphal. In fact, weak-versus-strong differentiability and holomorphy were treated definitively in

[Grothendieck 1954] A. Grothendieck, *Espaces vectoriels topologiques*, mimeographed notes, Univ. Sao Paulo, Sao Paulo, 1954.

[Grothendieck 1953] A. Grothendieck, *Sur certains espaces de fonctions holomorphes, I, II*, J. Reine Angew. Math. **192** (1953), 35–64 and 77–95.

The first-mentioned of these is cited, for example, in

[Barros-Neto 1964] J. Barros-Neto, *Spaces of vector-valued real analytic functions*,

[5.1] Definition: Let $f : [a, c] \rightarrow V$ be a V -valued function on an interval $[a, c] \subset \mathbb{R}$. The function f is *differentiable* if for each $x_o \in [a, c]$

$$f'(x_o) = \lim_{x \rightarrow x_o} (x - x_o)^{-1} (f(x) - f(x_o))$$

exists. The function f is *continuously differentiable* if it is differentiable and if f' is continuous. A k -times continuously differentiable function is said to be C^k , and a continuous function is said to be C^0 .

[5.2] Definition: A V -valued function f is **weakly** C^k if for every $\lambda \in V^*$ the scalar-valued function $\lambda \circ f$ is C^k .

If there were any doubt, the present sense of *weak differentiability* of a function f does not refer to distributional derivatives, but rather to differentiability of every scalar-valued function $\lambda \circ f$ where f is vector-valued and λ ranges over suitable continuous linear functionals.

[5.3] Theorem: Let V be a quasi-complete locally convex topological vector space. Let f be a V -valued function defined on an interval $[a, c]$. Suppose that f is weakly C^k . Then the V -valued function f is (strongly) C^{k-1} .

First we need

[5.4] **Lemma:** Let V be a quasi-complete locally convex topological vector space. Fix real numbers $a \leq b \leq c$. Let f be a V -valued function defined on $[a, b] \cup (b, c]$. Suppose that for each $\lambda \in V^*$ the scalar-valued function $\lambda \circ f$ has an extension to a function F_λ on the whole interval $[a, c]$ which is C^1 . Then $f(b)$ can be chosen so that the extended $f(x)$ is (strongly) continuous on $[a, c]$.

Proof: For each $\lambda \in V^*$, let F_λ be the extension of $\lambda \circ f$ to a C^1 function on $[a, c]$. For each λ , the differentiability of F_λ implies that

$$\Phi_\lambda(x, y) = \frac{F_\lambda(x) - F_\lambda(y)}{x - y}$$

has a continuous extension $\tilde{\Phi}_\lambda$ to the compact set $[a, c] \times [a, c]$. Thus, the image C_λ of $[a, c] \times [a, c]$ under this continuous map is a compact subset of \mathbb{R} , so bounded. Thus, the subset

$$\left\{ \frac{\lambda f(x) - \lambda f(y)}{x - y} : x \neq y \right\} \subset C_\lambda$$

is also bounded in \mathbb{R} . Therefore, the set

$$E = \left\{ \frac{f(x) - f(y)}{x - y} : x \neq y \right\} \subset V$$

is weakly bounded. It is a standard fact (from Banach-Steinhaus, Hahn-Banach, and Baire category arguments) that weak boundedness implies (strong) boundedness in a locally convex topological vectorspace, so E is (strongly) bounded. Thus, for a (strong, balanced, convex) neighborhood N of 0 in V , there is t_o so that $(f(x) - f(y))/(x - y) \in tN$ for any $x \neq y$ in $[a, c]$ and any $t \geq t_o$. That is,

$$f(x) - f(y) \in (x - y)tN$$

Thus, given N and the t_o determined as just above, for $|x - y| < \frac{1}{t_o}$ we have

$$f(x) - f(y) \in N$$

That is, as $x \rightarrow y$ the collection $f(x)$ is a bounded Cauchy net. Thus, by the quasi-completeness, we can define $f(b) \in V$ as the limit of the values $f(x)$. And in fact we see that for $x \rightarrow y$ the values $f(x)$ approach $f(y)$, so this extended version of f is continuous on $[a, c]$. ///

Proof: (of theorem) Fix $b \in (a, c)$, and consider the function

$$g(x) = \frac{f(x) - f(b)}{x - b}$$

for $x \neq b$. The assumed weak C^2 -ness implies that every $\lambda \circ g$ extends to a C^1 function on $[a, c]$. Thus, by the lemma, g itself has a continuous extension to $[a, c]$. In particular,

$$\lim_{x \rightarrow b} g(x)$$

exists, which exactly implies that f is differentiable at b . Thus, f is differentiable throughout $[a, c]$.

To prove the continuity of f' , consider again the function of two variables (initially for $x \neq y$)

$$g(x, y) = \frac{f(x) - f(y)}{x - y}$$

The weak C^2 -ness of f assures that g extends to a weakly C^1 function on $[a, c] \times [a, c]$. In particular, the function $x \rightarrow g(x, x)$ of (the extended) g is weakly C^1 . This function is $f'(x)$. Thus, f' is weakly C^1 , so is (strongly) C^0 .

Suppose that we already know that f is C^ℓ , for $\ell < k - 1$. Then consider the ℓ^{th} derivative $g = f^{(\ell)}$ of f . This function g is at least weakly C^2 , so is (strongly) C^1 by the first part of the argument. That is, f is at least $C^{\ell+1}$. ///