(*September 10, 2016*)

01. Natural topologies on function spaces

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[This document is http://www.math.umn.edu/~garrett/m/fun/notes_2016-17/fun-notes-01.pdf]

We review natural topological vectorspaces of functions on relatively simple geometric objects, such as $\mathbb{R}$ or the circle $\mathbb{T}$.

In all cases, we specify a natural topology, in which differentiation or other natural operators are *continuous*, and so that the space is suitably *complete*.

Many familiar and useful spaces of continuous or differentiable functions, such as $C^k[a,b]$, have natural metric structures, and are *complete*. Often, the metric $d(,)$ comes from a *norm* $|\cdot|$, on the functions, giving Banach spaces.

Other natural function spaces, such as $C^\infty[a,b]$, $C^o(\mathbb{R})$, are *not* Banach, but still do have a metric topology and are complete: these are *Fréchet spaces*, appearing as (projective) *limits* of Banach spaces, as below. These lack some of the conveniences of Banach spaces, but their expressions as *limits* of Banach spaces is often sufficient.

Other important spaces, such as compactly-supported continuous functions $C_c^o(\mathbb{R})$ on $\mathbb{R}$, or compactly-supported smooth functions (test functions) $C_c^\infty(\mathbb{R}) = C_c^\infty(\mathbb{R})$ on $\mathbb{R}$, are not metrizable so as to be *complete*. Nevertheless, some are expressible as *colimits* (sometimes called *inductive limits*) of Banach or Fréchet spaces, and such descriptions suffice for many applications. An *LF-space* is a countable ascending union of Fréchet spaces with each Fréchet subspace *closed* in the next. These are *strict colimits* or *strict inductive limits* of Fréchet spaces. These are generally *not* complete in the strongest sense, but, nevertheless, as demonstrated in [???], are *quasi-complete*, and this suffices for applications.

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1. Banach spaces $C^k[a,b]$

We give the vector space $C^k[a,b]$ of k-times continuously differentiable functions on an interval $[a,b]$ a metric which makes it *complete*. Mere *pointwise* limits of continuous functions easily fail to be continuous. First recall the standard

[1.0.1] Claim: The set $C^o(K)$ of complex-valued continuous functions on a compact set K is *complete* with

the metric $|f-g|_{C^o}$, with the C^o-*norm* $|f|_{C^o} = \sup_{x\in K} |f(x)|$.

Proof: This is a typical three-epsilon argument. To show that a Cauchy sequence $\{f_i\}$ of continuous functions has a *pointwise* limit which is a continuous function, first argue that f_i has a pointwise limit at every $x \in K$. Given $\varepsilon > 0$, choose N large enough such that $|f_i - f_j| < \varepsilon$ for all $i, j \geq N$. Then $|f_i(x) - f_j(x)| < \varepsilon$ for any x in K. Thus, the sequence of values $f_i(x)$ is a Cauchy sequence of complex numbers, so has a limit $f(x)$. Further, given $\varepsilon' > 0$ choose $j \geq N$ sufficiently large such that $|f_j(x) - f(x)| < \varepsilon'$. For $i \geq N$

$$|f_i(x) - f(x)| \;\leq\; |f_i(x) - f_j(x)| + |f_j(x) - f(x)| \;<\; \varepsilon + \varepsilon'$$

This is true for every positive ε', so $|f_i(x) - f(x)| \leq \varepsilon$ for every x in K. That is, the pointwise limit is approached uniformly in $x \in [a, b]$.

To prove that $f(x)$ is continuous, for $\varepsilon > 0$, take N be large enough so that $|f_i - f_j| < \varepsilon$ for all $i, j \geq N$. From the previous paragraph $|f_i(x) - f(x)| \leq \varepsilon$ for every x and for $i \geq N$. Fix $i \geq N$ and $x \in K$, and choose a small enough neigborhood U of x such that $|f_i(x) - f_i(y)| < \varepsilon$ for any y in U. Then

$$|f(x) - f(y)| \;\leq\; |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f(y) - f_i(y)| \;\leq\; \varepsilon + |f_i(x) - f_i(y)| + \varepsilon \;<\; \varepsilon + \varepsilon + \varepsilon$$

Thus, the pointwise limit f is continuous at every x in U. ///

Unsurprisingly, but significantly:

[1.0.2] Claim: For $x \in [a, b]$, the *evaluation* map $f \to f(x)$ is a continuous linear functional on $C^o[a, b]$.

Proof: For $|f - g|_{C^o} < \varepsilon$, we have

$$|f(x) - g(x)| \;\leq\; |f - g|_{C^o} \;<\; \varepsilon$$

proving the continuity. ///

As usual, a real-valued or complex-valued function f on a closed interval $[a, b] \subset \mathbb{R}$ is *continuously differentiable* when it has a derivative which is itself a continuous function. That is, the limit

$$f'(x) \;=\; \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists for all $x \in [a, b]$, and the function $f'(x)$ is in $C^o[a, b]$. Let $C^k[a, b]$ be the collection of k-times continuously differentiable functions on $[a, b]$, with the C^k-*norm*

$$|f|_{C^k} \;=\; \sum_{0\leq i\leq k} \sup_{x\in[a,b]} |f^{(i)}(x)| \;=\; \sum_{0\leq i\leq k} |f^{(i)}|_\infty$$

where $f^{(i)}$ is the ith derivative of f. The *associated metric* on $C^k[a, b]$ is $|f - g|_{C^k}$.

Similar to the assertion about evaluation on $C^o[a, b]$,

[1.0.3] Claim: For $x \in [a, b]$ and $0 \leq j \leq k$, the *evaluation* map $f \to f^{(j)}(x)$ is a continuous linear functional on $C^k[a, b]$.

Proof: For $|f - g|_{C^k} < \varepsilon$,

$$|f^{(j)}(x) - g^{(j)}(x)| \;\leq\; |f - g|_{C^k} \;<\; \varepsilon$$

proving the continuity. ///

We see that $C^k[a, b]$ is a Banach space:

[1.0.4] Theorem: The normed metric space $C^k[a,b]$ is complete.

Proof: For a Cauchy sequence $\{f_i\}$ in $C^k[a,b]$, all the pointwise limits $\lim_i f_i^{(j)}(x)$ of j-fold derivatives exist for $0 \le j \le k$, and are uniformly continuous. The issue is to show that $\lim_i f^{(j)}$ is differentiable, with derivative $\lim_i f^{(j+1)}$. It suffices to show that, for a Cauchy sequence f_n in $C^1[a,b]$, with pointwise limits $f(x) = \lim_n f_n(x)$ and $g(x) = \lim_n f_n'(x)$ we have $g = f'$. By the fundamental theorem of calculus, for any index i,

$$f_i(x) - f_i(a) \;=\; \int_a^x f_i'(t)\,dt$$

Since the f_i' *uniformly* approach g, given $\varepsilon > 0$ there is i_o such that $|f_i'(t) - g(t)| < \varepsilon$ for $i \ge i_o$ and for *all* t in the interval, so for such i

$$\Big|\int_a^x f_i'(t)\,dt - \int_a^x g(t)\,dt\Big| \;\le\; \int_a^x |f_i'(t) - g(t)|\,dt \;\le\; \varepsilon \cdot |x-a| \;\longrightarrow\; 0$$

Thus,

$$\lim_i f_i(x) - f_i(a) \;=\; \lim_i \int_a^x f_i'(t)\,dt \;=\; \int_a^x g(t)\,dt$$

from which $f' = g$. ///

By design, we have

[1.0.5] Theorem: The map $\frac{d}{dx} : C^k[a,b] \to C^{k-1}[a,b]$ is continuous.

Proof: As usual, for a linear map $T : V \to W$, by linearity $Tv - Tv' = T(v - v')$ it suffices to check continuity at 0. For Banach spaces the homogeneity $|\sigma \cdot v|_V = |\alpha| \cdot |v|_V$ shows that continuity is equivalent to existence of a constant B such that $|Tv|_W \le B \cdot |v|_V$ for $v \in V$. Then

$$|\frac{d}{dx} f|_{C^{k-1}} \;=\; \sum_{0 \le i \le k-1} \sup_{x \in [a,b]} |(\frac{df}{dx})^{(i)}(x)| \;=\; \sum_{1 \le i \le k} \sup_{x \in [a,b]} |f^{(i)}(x)| \;\le\; 1 \cdot |f|_{C^k}$$

as desired. ///

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2. *Non-Banach limit $C^\infty[a,b]$ of Banach spaces $C^k[a,b]$*

The space $C^\infty[a,b]$ of infinitely differentiable complex-valued functions on a (finite) interval $[a,b]$ in $\mathbb{R}$ is not a Banach space. [1] Nevertheless, the topology is *completely determined* by its relation to the Banach spaces $C^k[a,b]$. That is, there is a *unique* reasonable topology on $C^\infty[a,b]$. After explaining and proving this uniqueness, we also show that this topology is *complete metric*.

This function space can be presented as

$$C^\infty[a,b] \;=\; \bigcap_{k \ge 0} C^k[a,b]$$

and we reasonably require that whatever topology $C^\infty[a,b]$ should have, each inclusion $C^\infty[a,b] \longrightarrow C^k[a,b]$ is continuous.

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[1] It is not essential to prove that there is no reasonable Banach space structure on $C^\infty[a,b]$, but this can be readily proven in a suitable context [???].

At the same time, given a family of *continuous linear* maps $Z \to C^k[a,b]$ from a vector space Z in some reasonable class (specified in the next section), with the *compatibility* condition of giving commutative diagrams

$$\begin{tikzcd} C^k[a,b] \arrow[r, "\subset"] & C^{k-1}[a,b] \\ & Z \arrow[ul] \arrow[u] \end{tikzcd}$$

the image of Z actually lies in the intersection $C^\infty[a,b]$. Thus, diagrammatically, for every family of compatible maps $Z \to C^k[a,b]$, there is a *unique* $Z \to C^\infty[a,b]$ fitting into a commutative diagram

$$\begin{tikzcd} C^\infty[a,b] \arrow[r, bend left] \arrow[rr, bend left] & \dots \arrow[r] & C^1[a,b] \arrow[r] & C^o[a,b] \\ & Z \arrow[ul, dotted, "\exists!"] \arrow[ur, dashed] \arrow[urr, dashed, "\forall"'] & & \end{tikzcd}$$

We require that this induced map $Z \to C^\infty[a,b]$ is *continuous*.

When we know that these conditions are met, we would say that $C^\infty[a,b]$ is the (projective) *limit* of the spaces $C^k[a,b]$, written

$$C^\infty[a,b] \;=\; \lim_k C^k[a,b]$$

with implicit reference to the inclusions $C^{k+1}[a,b] \to C^k[a,b]$ and $C^\infty[a,b] \to C^k[a,b]$.

[2.0.1] Claim: Up to unique isomorphism, there exists at most one topology on $C^\infty[a,b]$ such that to every compatible family of continuous linear maps $Z \to C^k[a,b]$ from a topological vector space Z there is a unique continuous linear $Z \to C^\infty[a,b]$ fitting into a commutative diagram as just above.

Proof: Let X, Y be $C^\infty[a,b]$ with two topologies fitting into such diagrams, and show $X \approx Y$, and for a unique isomorphism. First, claim that the identity map $\mathrm{id}_X : X \to X$ is the only map $\varphi : X \to X$ fitting into a commutative diagram

$$\begin{tikzcd} X \arrow[r, bend left] \arrow[rr, bend left] & \dots \arrow[r] & C^1[a,b] \arrow[r] \arrow[d, equal] & C^o[a,b] \arrow[d, equal] \\ X \arrow[u, "\varphi"] \arrow[rr, bend right] & \dots \arrow[r] & C^1[a,b] \arrow[r] & C^o[a,b] \end{tikzcd}$$

Indeed, given a compatible family of maps $X \to C^k[a,b]$, there is *unique* φ fitting into

$$\begin{tikzcd} X \arrow[r, bend left] \arrow[rr, bend left] & \dots \arrow[r] & C^1[a,b] \arrow[r] & C^o[a,b] \\ & X \arrow[ul, dotted, "\varphi"] \arrow[ur, dashed] \arrow[urr, dashed, "\forall"'] & & \end{tikzcd}$$

Since the identity map id_X fits, necessarily $\varphi = \mathrm{id}_X$. Similarly, given the compatible family of inclusions $Y \to C^k[a,b]$, there is unique $f : Y \to X$ fitting into

$$\begin{tikzcd} X \arrow[r, bend left] \arrow[rr, bend left] & \dots \arrow[r] & C^1[a,b] \arrow[r] & C^o[a,b] \\ & Y \arrow[ul, dotted, "f"] \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

Similarly, given the compatible family of inclusions $X \to C^k[a,b]$, there is unique $g : X \to Y$ fitting into

$$\begin{tikzcd}
Y \arrow[rr, bend left] \arrow[rrr, bend left] & \cdots \arrow[r] & C^1[a,b] \arrow[r] & C^o[a,b] \\
& X \arrow[ul, dotted, "g"] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

Then $f \circ g : X \to X$ fits into a diagram

$$\begin{tikzcd}
X \arrow[rr, bend left] \arrow[rrr, bend left] & \cdots \arrow[r] & C^1[a,b] \arrow[r] & C^o[a,b] \\
& X \arrow[ul, dotted, "f\circ g"] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

Therefore, $f \circ g = \mathrm{id}_X$. Similarly, $g \circ f = \mathrm{id}_Y$. That is, f, g are mutual inverses, so are isomorphisms of topological vector spaces. ///

Existence of a topology on $C^\infty[a,b]$ satisfying the condition above will be proven by identifying $C^\infty[a,b]$ as the obvious diagonal *closed subspace* of the *topological product* of the *limitands* $C^k[a,b]$:

$$C^\infty[a,b] \;=\; \{\{f_k : f_k \in C^k[a,b]\} : f_k = f_{k+1} \text{ for all } k\}$$

An arbitrary *product* of topological spaces X_α for α in an index set A is a topological space X with (*projections*) $p_\alpha : X \to X_\alpha$, such that every family $f_\alpha : Z \to X_\alpha$ of maps from any other topological space Z *factors through* the p_α *uniquely*, in the sense that there is a unique $f : Z \to X$ such that $f_\alpha = p_\alpha \circ f$ for all α. Pictorially, *all triangles commute* in the diagram

$$\begin{tikzcd}
Z \arrow[rr, dotted, "f"] \arrow[dr, dashed, "f_\alpha"] \arrow[drr, dashed, "f_\beta"] & & X \arrow[dl, "p_\alpha"] \arrow[dr, "p_\beta"] & & \\
\cdots & X_\alpha & \cdots & X_\beta & \cdots
\end{tikzcd}$$

A similar argument to that for uniqueness of limits proves *uniqueness* of products up to unique isomorphism. *Construction* of products is by putting the usual product topology with basis consisting of products $\prod_\alpha Y_\alpha$ with $Y_\alpha = X_\alpha$ for all but finitely-many indices, on the Cartesian product of the *sets* X_α, whose existence we grant ourselves. Proof that this usual is *a* product amounts to unwinding the definitions. By uniqueness, in particular, despite the plausibility of the *box topology* on the product, it cannot function as a product topology since it differs from the standard product topology in general.

[2.0.2] Claim: Giving the diagonal copy of $C^\infty[a,b]$ inside $\prod_k C^k[a,b]$ the subspace topology yields a (projective) limit topology.

Proof: The projection maps $p_k : \prod_j C^j[a,b] \to C^k[a,b]$ from the whole product to the factors $C^k[a,b]$ are continuous, so their restrictions to the diagonally imbedded $C^\infty[a,b]$ are continuous. Further, letting $i_k : C^k[a,b] \to C^{k-1}[a,b]$ be the inclusion, on that diagonal copy of $C^\infty[a,b]$ we have $i_k \circ p_k = p_{k-1}$ as required.

On the other hand, *any* family of maps $\varphi_k : Z \to C^k[a,b]$ induces a map $\widetilde{\varphi} : Z \to \prod C^k[a,b]$ such that $p_k \circ \widetilde{\varphi} = \varphi_k$, by the property of the product. *Compatibility* $i_k \circ \varphi_k = \varphi_{k-1}$ implies that the image of $\widetilde{\varphi}$ is inside the diagonal, that is, inside the copy of $C^\infty[a,b]$. ///

A *countable* product of *metric* spaces X_k with metrics d_k has no canonical single metric, but is *metrizable*. One of many topologically equivalent metrics is the usual

$$d(\{x_k\},\{y_k\}) \;=\; \sum_{k=0}^{\infty} 2^{-k} \frac{d_k(x_k - y_k)}{d_k(x_k - y_k) + 1}$$

When the metric spaces X_k are *complete*, the product is complete. A closed subspace of a complete metrizable space is complete metrizable, so we have

[2.0.3] Corollary: $C^\infty[a,b]$ is complete metrizable. ///

Abstracting the above, for a (not necessarily countable) family

$$\ldots \xrightarrow{\varphi_2} B_1 \xrightarrow{\varphi_1} B_o$$

of Banach spaces with continuous linear transition maps as indicated, *not* recessarily requiring the continuous linear maps to be injective (or surjective), a *(projective) limit* $\lim_i B_i$ is a topological vector space with continuous linear maps $\lim_i B_i \to B_j$ such that, for every compatible family of continuous linear maps $Z \to B_i$ there is unique continuous linear $Z \to \lim_i B_i$ fitting into

$$\begin{tikzcd}
\lim_i B_i \arrow[rr, bend left] \arrow[rrr, bend left] & \ldots \arrow[r, "\varphi_2"] & B_1 \arrow[r, "\varphi_1"] & B_o \\
& Z \arrow[ul, dotted] \arrow[ur, dashed] \arrow[urr, dashed] & &
\end{tikzcd}$$

The same *uniqueness* proof as above shows that there is at most one topological vector space $\lim_i B_i$. For *existence* by *construction*, the earlier argument needs only minor adjustment. The conclusion of complete metrizability would hold when the family is countable.

Before declaring $C^\infty[a,b]$ to be a *Fréchet* space, we must certify that it is *locally convex*, in the sense that every point has a local basis of *convex* opens. Normed spaces are immediately locally convex, because open balls are convex: for $0 \le t \le 1$ and x, y in the ε-ball at 0 in a normed space,

$$|tx + (1-t)y| \;\le\; |tx| + |(1-t)y| \;\le\; t|x| + (1-t)|y| \;<\; t\cdot\varepsilon + (1-t)\cdot\varepsilon \;=\; \varepsilon$$

Product topologies of locally convex vectorspaces are locally convex, from the *construction* of the product. The construction of the limit as the diagonal in the product, with the subspace topology, shows that it is locally convex. In particular, *countable limits of Banach spaces are locally convex, hence, are Fréchet.* All spaces of practical interest are locally convex for simple reasons, so demonstrating local convexity is rarely interesting.

[2.0.4] Theorem: $\frac{d}{dx} : C^\infty[a,b] \to C^\infty[a,b]$ is continuous.

Proof: In fact, the differentiation operator is characterized via the expression of $C^\infty[a,b]$ as a limit. We already know that differentiation d/dx gives a continuous map $C^k[a,b] \to C^{k-1}[a,b]$. Differentiation is compatible with the inclusions among the $C^k[a,b]$. Thus, we have a commutative diagram

$$\begin{tikzcd}
C^\infty[a,b] \arrow[rr, bend left] \arrow[rrr, bend left] & \ldots & C^k[a,b] \arrow[r] & C^{k-1}[a,b] \arrow[r] & \ldots \\
C^\infty[a,b] \arrow[rr, bend right] \arrow[rrr, bend right] & \ldots & C^k[a,b] \arrow[r] \arrow[ur, "\frac{d}{dx}"] & C^{k-1}[a,b] \arrow[r] \arrow[ur, "\frac{d}{dx}"] & \ldots
\end{tikzcd}$$

Composing the projections with d/dx gives (dashed) induced maps from $C^\infty[a,b]$ to the limitands, inducing a unique (dotted) continuous linear map to the limit, as in

$$\begin{tikzcd}
C^\infty[a,b] \arrow[rr, bend left] \arrow[rrr, bend left] & \ldots & C^k[a,b] \arrow[r] & C^{k-1}[a,b] \arrow[r] & \ldots \\
C^\infty[a,b] \arrow[u, dotted, "\frac{d}{dx}"] \arrow[rr, bend right] \arrow[rrr, bend right] \arrow[urr, dashed] \arrow[urrr, dashed] \arrow[urrrr, dashed] & \ldots & C^k[a,b] \arrow[r] \arrow[ur] & C^{k-1}[a,b] \arrow[r] \arrow[ur] & \ldots
\end{tikzcd}$$

This proves the continuity of differentiation in the limit topology. ///

In a slightly different vein, we have

[2.0.5] Claim: For all $x \in [a,b]$ and for all non-negative integers k, the evaluation map $f \to f^{(k)}(x)$ is a continuous linear map $C^\infty[a,b] \to \mathbb{C}$.

Proof: The inclusion $C^\infty[a,b] \to C^k[a,b]$ is continuous, and the evaluation of the kth derivative is continuous [???]. ///

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3. *Sufficient notion of topological vector space*

To describe a (projective) limit by characterizing its behavior in relation to *all* topological vectorspaces requires specification of what a topological vectorspace *should be*.

A *topological vector space* V (over $\mathbb{C}$) is a $\mathbb{C}$-vectorspace V with a topology on V in which *points are closed*, and so that *scalar multiplication*

$$x \times v \;\longrightarrow\; xv \qquad\qquad (\text{for } x \in k \text{ and } v \in V)$$

and *vector addition*

$$v \times w \;\longrightarrow\; v + w \qquad\qquad (\text{for } v, w \in V)$$

are *continuous*. For subsets X, Y of V, let

$$X + Y \;=\; \{x + y : x \in X, y \in Y\}$$

and

$$-X \;=\; \{-x : x \in X\}$$

The following trick is elementary, but indispensable. Given an open neighborhood U of 0 in a topological vectorspace V, continuity of vector addition yields an open neighborhood U' of 0 such that

$$U' + U' \;\subset\; U$$

Since $0 \in U'$, necessarily $U' \subset U$. This can be repeated to give, for any positive integer n, an open neighborhood U_n of 0 such that

$$\underbrace{U_n + \ldots + U_n}_{n} \;\subset\; U$$

In a similar vein, for fixed $v \in V$ the map $V \to V$ by $x \to x + v$ is a *homeomorphism*, being invertible by the obvious $x \to x - v$. Thus, *the open neighborhoods of v are of the form $v + U$ for open neighborhoods U of* 0. In particular, *a local basis at* 0 *gives the topology on a topological vectorspace.*

[3.0.1] Lemma: Given a compact subset K of a topological vectorspace V and a closed subset C of V not meeting K, there is an open neighborhood U of 0 in V such that

$$\text{closure}(K + U) \;\cap\; (C + U) \quad = \quad \phi$$

Proof: Since C is closed, for $x \in K$ there is a neighborhood U_x of 0 such that the neighborhood $x + U_x$ of x does not meet C. By continuity of vector addition

$$V \times V \times V \to V \quad \text{by} \quad v_1 \times v_2 \times v_3 \to v_1 + v_2 + v_3$$

there is a smaller open neighborhood N_x of 0 so that

$$N_x + N_x + N_x \;\subset\; U_x$$

By replacing N_x by $N_x \cap -N_x$, which is still an open neighborhood of 0, suppose that N_x is *symmetric* in the sense that $N_x = -N_x$.

Using this symmetry,

$$(x + N_x + N_x) \cap (C + N_x) \;=\; \phi$$

Since K is compact, there are finitely-many $x_1, \ldots, x_n$ such that

$$K \;\subset\; (x_1 + N_{x_1}) \cup \ldots \cup (x_n + N_{x_n})$$

Let $U = \bigcap_i \; N_{x_i}$. Since the intersection is finite, U is open. Then

$$K + U \;\subset\; \bigcup_{i=1,\ldots,n} (x_i + N_{x_i} + U) \;\subset\; \bigcup_{i=1,\ldots,n} (x_i + N_{x_i} + N_{x_i})$$

These sets do not meet $C + U$, by construction, since $U \subset N_{x_i}$ for all i. Finally, since $C + U$ is a union of opens $y + U$ for $y \in C$, it is open, so even the *closure* of $K + U$ does not meet $C + U$. ///

Conveniently, Hausdorff-ness of topological vectorspaces follows from the weaker assumption that points are closed:

[3.0.2] Corollary: A topological vectorspace is *Hausdorff*.

Proof: Take $K = \{x\}$ and $C = \{y\}$ in the lemma. ///

[3.0.3] Corollary: The topological closure $\bar{E}$ of a subset E of a topological vectorspace V can be expressed as

$$\bar{E} \;=\; \bigcap_U (E + U) \qquad\qquad \text{(where } U \text{ ranges over a local basis at 0)}$$

Proof: In the lemma, take $K = \{x\}$ and $C = \bar{E}$ for a point x of V not in C. Then we obtain an open neighborhood U of 0 so that $x + U$ does not meet $\bar{E} + U$. The latter contains $E + U$, so certainly $x \notin E + U$. That is, for x not in the closure, there is an open U containing 0 so that $x \notin E + U$. ///

As usual, for two topological vectorspaces V, W over $\mathbb{C}$, a function $f : V \longrightarrow W$ is *(k-)linear* when $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $\alpha, \beta \in k$ and $x, y \in V$. Almost without exception we care about *continuous* linear maps, meaning linear maps continuous for the topologies on V, W. As expected, the *kernel* $\ker f$ of a linear map is

$$\ker f \;=\; \{v \in V : f(v) = 0\}$$

Being the inverse image of a closed set by a continuous map, the kernel is a *closed* subspace of V.

For a *closed* subspace H of a topological vectorspace V, the *quotient* V/H is *characterized* as topological vectorspace with linear quotient map $q : V \to V/H$ through which any continuous $f : V \to W$ with $\ker f \supset H$ *factors*, in the sense that there is a unique continuous linear $\overline{f} : V/H \to W$ giving a commutative diagram

$$\begin{tikzcd} V/H \arrow[dr, dashed, "\overline{f}"] & \\ V \arrow[u, "q"] \arrow[r, "f"'] & W \end{tikzcd}$$

Uniqueness of the quotient $q : V \to V/H$, up to unique isomorphism, follows by the usual categorical arguments, as with limits and products above. The *existence* of the quotient is proven by the usual construction of V/H as the collection of cosets $v + H$, with q given as usual by $q : v \longrightarrow v + H$. We verify that this construction succeeds in the proposition below.

The *quotient topology* on V/H is the *finest* topology such that the quotient map $q : V \to V/H$ is continuous, namely, a subset E of V/H is open if and only if $q^{-1}(E)$ is open.

For *non*-closed subspaces H, the quotient topology on the collection of cosets $\{v+H\}$ would *not* be Hausdorff. Thus, the proper categorical notion of topological vectorspace quotient, by non-closed subspace, would produce the collection of cosets $v + \overline{H}$ for the *closure* $\overline{H}$ of H.

[3.0.4] Claim: For a closed subspace W of a topological vectorspace V, the collection $Q = \{v+W : v \in V\}$ of cosets by W with map $q(v) = v + W$ is a topological vectorspace and q is a quotient map.

Proof: The *algebraic* quotient $Q = V/W$ of cosets $v+W$ and $q(v) = v+W$ constructs a vectorspace quotient without any topological hypotheses on W. Since W is closed, and since vector addition is a homeomorphism, $v+W$ is closed as well. Thus, its complement $V - (v+W)$ is open, so $q(V - (v+W))$ is open, by definition of the quotient topology. Thus, the complement

$$q(v) \;=\; v + W \;=\; q(v+W) \;=\; V/W - q(V - (v+W))$$

of the open set $q(V - (v+W))$ is closed. ///

Unlike general topological quotient maps,

[3.0.5] Claim: For a closed subspace H of a topological vector space V, the quotient map $q : V \to V/H$ is *open*, that is, carries open sets to open sets.

Proof: For U open in V,

$$q^{-1}(q(U)) \;=\; q^{-1}(U+H) \;=\; U + H \;=\; \bigcup_{h \in H} h + U$$

This is a union of opens. ///

[3.0.6] Corollary: For $f : V \to X$ a linear map with a closed subspace W of V contained in $\ker f$, and $\bar{f}$ the induced map $\bar{f} : V/W \to X$ defined by $\bar{f}(v+W) = f(v)$, f is continuous if and only if $\bar{f}$ is continuous.

Proof: Certainly if $\bar{f}$ is continuous then $f = \bar{f} \circ q$ is continuous. The converse follows from the fact that q is *open*. ///

This proves that the *construction* of the quotient by cosets succeeds in producing a quotient: a continuous linear map $f : V \to X$ *factors through* any quotient V/W for W a closed subspace contained in the kernel of f.

The notions of *balanced subset, absorbing subset*, *directed set, Cauchy net*, and *completeness* are necessary:

A subset E of V is *balanced* when $xE \subset E$ for every $x \in k$ with $|x| \le 1$.

[3.0.7] Lemma: Every neighborhood u of 0 in a topological vectorspace V over k contains a *balanced* neighborhood N of 0.

Proof: By continuity of scalar multiplication, there is $\varepsilon > 0$ and a neighborhood U' of $0 \in V$ so that if $|x| < \varepsilon$ and $v \in U'$ then $xv \in U$. Since $\mathbb{C}$ is not discrete, there is $x_o \in k$ with $0 < |x_o| < \varepsilon$. Since scalar multiplication by a non-zero element is a homeomorphism, $x_o U'$ is a neighborhood of 0 and $x_o U' \subset U$. Put

$$N \;=\; \bigcup_{|y| \le 1} y x_o U'$$

For $|x| \le 1$, $|xy| \le |y| \le 1$, so

$$xN \;=\; \bigcup_{|y|\le 1} x(yx_oU') \subset \bigcup_{|y|\le 1} yx_oU' \;=\; N$$

producing the desired N. ///

A subset E of vectorspace V over k is *absorbing* when for every $v \in V$ there is $t_o \in R$ so that $v \in \alpha E$ for every $\alpha \in k$ so that $|\alpha| \ge t_o$.

[3.0.8] Lemma: Every neighborhood U of 0 in a topological vectorspace is *absorbing.*

Proof: We may *shrink* U to assume U is *balanced.* By continuity of the map $k \to V$ given by $\alpha \to \alpha v$, there is $\varepsilon > 0$ so that $|\alpha| < \varepsilon$ implies $\alpha v \in U$. By the *non-discreteness* of k, there is non-zero $\alpha \in k$ satisfying any such inequality. Then $v \in \alpha^{-1}U$, as desired. ///

A *poset* $S, \le$ is a partially ordered set. A *directed set* is a poset S such that, for any two elements $s, t \in S$, there is $z \in S$ so that $z \ge s$ and $z \ge t$.

A *net* in V is a subset $\{x_s : s \in S\}$ of V indexed by a directed set S. A net $\{x_s : s \in S\}$ in a topological vectorspace V is a *Cauchy net* if, for every neighborhood U of 0 in V, there is an index s_o so that for $s, t \ge s_o$ we have $x_s - x_t \in U$. A net $\{x_s : s \in S\}$ is *convergent* if there is $x \in V$ so that, for every neighborhood U of 0 in V there is an index s_o so that for $s \ge s_o$ we have $x - x_s \in U$. Since points are closed, there can be *at most* one point to which a net converges. Thus, *a convergent net is Cauchy.* Oppositely, a topological vectorspace is *complete* if every Cauchy net is convergent.

[3.0.9] Lemma: Let Y be a vector subspace of a topological vector space X, *complete* when given the subspace topology from X. Then Y is a *closed* subset of X.

Proof: Let $x \in X$ be in the closure of Y. Let S be a local basis of opens at 0, where we take the partial ordering so that $U \ge U'$ if and only if $U \subset U'$. For each $U \in S$ choose $y_U \in (x + U) \cap Y$. The net $\{y_U : U \in S\}$ converges to x, so is Cauchy. It must converge to a point in Y, so by uniqueness of limits of nets it must be that $x \in Y$. Thus, Y is closed. ///

Unfortunately, *completeness* as above is too strong a condition for general topological vectorspaces, beyond Fréchet spaces. A slightly weaker version of completeness, *quasi-completeness* or *local* completeness, *does* hold for most important natural spaces, as discussed in [???]

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4. *Unique vectorspace topology on* $\mathbb{C}^n$

Finite-dimensional topological vectorspaces, and their interactions with other topological vectorspaces, are especially simple:

[4.0.1] Theorem: A *finite-dimensional* complex vectorspace V has just one topological vectorspace topology, that of the product topology on $\mathbb{C}^n$ for $n = \dim V$. A finite-dimensional subspace V of a topological vectorspace W is *closed.* A $\mathbb{C}$-linear map $X \to V$ to a finite-dimensional space V is continuous if and only if the kernel is closed.

Proof: The argument is by induction. First treat the one-dimensional situation:

[4.0.2] Claim: For a one-dimensional topological vectorspace V with basis e the map $\mathbb{C} \to V$ by $x \to xe$ is a *homeomorphism.*

Proof: Since scalar multiplication is continuous, we need only show that the map is *open.* We need only do this at 0, since translation addresses other points. Given $\varepsilon > 0$, by the non-discreteness of $\mathbb{C}$ there is x_o in

$\mathbb{C}$ so that $0 < |x_o| < \varepsilon$. Since V is Hausdorff, there is a neighborhood U of 0 so that $x_o e \notin U$. Shrink U so it is *balanced*. Take $x \in k$ so that $xe \in U$. For $|x| \geq |x_o|$, $|x_o x^{-1}| \leq 1$, so

$$x_o e \;=\; (x_o x^{-1})(xe) \in U$$

by balanced-ness of U, contradiction. Thus, $xe \in U$ implies that $|x| < |x_o| < \varepsilon$. ///

[4.0.3] Corollary: For fixed $x_o \in \mathbb{C}$, a not-identically-zero $\mathbb{C}$-linear $\mathbb{C}$-valued function f on V is *continuous* if and only if the *affine hyperplane* $H = \{v \in V : f(v) = x_o\}$ is *closed* in V.

Proof: Certainly if f is continuous then H is closed. For the converse, consider only the case $x_o = 0$, since translations (vector additions) are homeomorphisms of V to itself.

For v_o with $f(v_o) \neq 0$ and for any other $v \in V$

$$f(v - f(v)f(v_o)^{-1}v_o) \;=\; f(v) - f(v)f(v_o)^{-1}f(v_o) \;=\; 0$$

Thus, V/H is one-dimensional. The induced $\mathbb{C}$-linear map $\bar{f} : V/H \to k$ so that $f = \bar{f} \circ q$, that is, $\bar{f}(v + H) = f(v)$, is a homeomorphism to $\mathbb{C}$, by the previous result, so f is continuous. ///

For the theorem, for uniqueness of the topology it suffices to prove that for any $\mathbb{C}$-basis $e_1, \ldots, e_n$ for V, the map $\mathbb{C} \times \ldots \times \mathbb{C} \;\longrightarrow\; V$ by

$$(x_1, \ldots, x_n) \;\longrightarrow\; x_1 e_1 + \ldots + x_n e_n$$

is a homeomorphism. Prove this by induction on the dimension n, that is, on the number of generators for V as a free $\mathbb{C}$-module. The case $n = 1$ was treated. Since $\mathbb{C}$ is complete, the lemma above asserting the closed-ness of complete subspaces shows that any one-dimensional subspace is closed. For $n > 1$, let $H = \mathbb{C}e_1 + \ldots + \mathbb{C}e_{n-1}$. By induction, H is closed in V, so the quotient $q : V \to V/H$ is constructed as expected, as the set of cosets $v + H$. The space V/H is a one-dimensional topological vectorspace over $\mathbb{C}$, with basis $q(e_n)$. By induction, $\phi : xq(e_n) = q(xe_n) \longrightarrow x$ is a homeomorphism $V/H \to \mathbb{C}$.

Likewise, $\mathbb{C}e_n$ is a closed subspace and we have the quotient map

$$q' : V \;\longrightarrow\; V/\mathbb{C}e_n$$

The image has basis $q'(e_1), \ldots, q'(e_{n-1})$, and by induction

$$\phi' : x_1 q'(e_1) + \ldots + x_{n-1} q'(e_{n-1}) \to (x_1, \ldots, x_{n-1})$$

is a homeomorphism. By the induction hypothesis,

$$v \;\longrightarrow\; (\phi \circ q)(v) \times (\phi' \circ q')(v)$$

is continuous to $\mathbb{C}^{n-1} \times \mathbb{C} \;\approx\; \mathbb{C}^n$. On the other hand, by the continuity of scalar multiplication and vector addition,

$$\mathbb{C}^n \;\longrightarrow\; V \quad \text{by} \quad x_1 \times \ldots \times x_n \;\longrightarrow\; x_1 e_1 + \ldots + x_n e_n$$

is continuous. These two maps are mutual inverses, certifying the homeomorphism.

Thus, a n-dimensional subspace is homeomorphic to $\mathbb{C}^n$ with its *product* topology, so is complete, since a finite product of complete spaces is complete. By the closed-ness of complete subspaces, it is closed.

Continuity of a linear map $f : X \to \mathbb{C}^n$ implies that the kernel $N = \ker f$ is closed. On the other hand, for N closed, the set of cosets $x + N$ constructs a quotient, and is a topological vectorspace of dimension at most n. Therefore, the induced map $\bar{f} : X/N \to V$ is unavoidably continuous. Then $f = \bar{f} \circ q$ is continuous, where q is the quotient map. This completes the induction step. ///

5. *Non-Banach limits* $C^k(\mathbb{R})$, $C^\infty(\mathbb{R})$ *of Banach spaces* $C^k[a,b]$

For a *non-compact* topological space such as $\mathbb{R}$, the space $C^o(\mathbb{R})$ of continuous functions is *not* a Banach space with sup norm, because the sup of the absolute value of a continuous function may be $+\infty$.

But, $C^o(\mathbb{R})$ has a Fréchet-space structure: express $\mathbb{R}$ as a *countable union of compact subsets* $K_n = [-n, n]$. Despite the likely non-injectivity of the map $C^o(\mathbb{R}) \to C^o(K_i)$, giving $C^o(\mathbb{R})$ the (projective) limit topology $\lim_i C^o(K_i)$ is reasonable: certainly the restriction map $C^o(\mathbb{R}) \to C^o(K_i)$ should be continuous, as should all the restrictions $C^o(K_i) \to C^o(K_{i-1})$, whether or not these are *surjective*.

The argument in favor of giving $C^o(\mathbb{R})$ the limit topology is that a *compatible* family of maps $f_i : Z \to C^o(K_i)$ gives *compatible fragments* of functions F on $\mathbb{R}$. That is, for $z \in Z$, given $x \in \mathbb{R}$ take K_i such that x is in the interior of K_i. Then for all $j \geq i$ the function $x \to f_j(z)(x)$ is continuous near x, and the compatibility assures that all these functions are the same.

That is, the compatibility of these fragments is exactly the assertion that they fit together to make a function $x \to F_z(x)$ on the whole space X. Since continuity is a *local* property, $x \to F_z(x)$ is in $C^o(X)$. Further, there is *just one* way to piece the fragments together. Thus, diagrammatically,

$$\begin{tikzcd}
C^o(\mathbb{R}) \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & C^o(K_2) \arrow[r] & C^o(K_1) \\
& & Z \arrow[ul, dotted, "z \to F_z"] \arrow[u, dashed, "f_2"] \arrow[ur, dashed, "f_1"'] &
\end{tikzcd}$$

Thus, $C^o(X) = \lim_n C^o(K_n)$ is a Fréchet space. Similarly, $C^k(\mathbb{R}) = \lim_n C^k(K_n)$ is a Fréchet space.

[5.0.1] Remark: The question of whether the restriction maps $C^o(K_n) \to C^o(K_{n-1})$ or $C^o(\mathbb{R}) \to C^o(K_n)$ are *surjective* need not be addressed.

Unsurprisingly, we have

[5.0.2] Theorem: $\frac{d}{dx} : C^k(\mathbb{R}) \to C^{k-1}(\mathbb{R})$ is continuous.

Proof: The argument is structurally similar to the argument for $\frac{d}{dx} : C^\infty[a,b] \to C^\infty[a,b]$. The differentiations $\frac{d}{dx} : C^k(K_n) \to C^{k-1}(K_n)$ are a compatible family, fitting into a commutative diagram

$$\begin{tikzcd}
C^{k-1}(\mathbb{R}) \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & C^{k-1}(K_{n+1}) \arrow[r] & C^{k-1}(K_n) \arrow[r] & \dots \\
C^k(\mathbb{R}) \arrow[rr, bend right] \arrow[rrr, bend right] & \dots \arrow[r] & C^k(K_{n+1}) \arrow[r] \arrow[u, "\frac{d}{dx}"] & C^k(K_n) \arrow[r] \arrow[u, "\frac{d}{dx}"] & \dots
\end{tikzcd}$$

Composing the projections with d/dx gives (dashed) induced maps from $C^k(\mathbb{R})$ to the limitands, inducing a unique (dotted) continuous linear map to the limit, as in

$$\begin{tikzcd}
C^{k-1}(\mathbb{R}) \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & C^{k-1}(K_{n+1}) \arrow[r] & C^{k-1}(K_n) \arrow[r] & \dots \\
C^k(\mathbb{R}) \arrow[u, dotted, "\frac{d}{dx}"] \arrow[urr, dashed] \arrow[urrr, dashed] \arrow[rr, bend right] \arrow[rrr, bend right] & \dots \arrow[r] & C^k(K_{n+1}) \arrow[r] \arrow[u] & C^{k-1}(K_n) \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

That is, there is a unique continuous linear map $\frac{d}{dx} : C^k(\mathbb{R}) \to C^{k-1}(\mathbb{R})$ compatible with the differentiations on finite intervals. ///

Similarly,

[5.0.3] Theorem: $C^\infty(\mathbb{R}) = \lim_k C^k(\mathbb{R})$, also $C^\infty(\mathbb{R}) = \lim_n C^\infty(K_n)$, and $\frac{d}{dx} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is continuous.

Proof: From $C^\infty(\mathbb{R}) = \lim_k C^k(\mathbb{R})$ we can obtain the induced map d/dx, as follows. Starting with the commutative diagram

$$\begin{tikzcd}
C^\infty(\mathbb{R}) \arrow[rrr, bend left] \arrow[rrrr, bend left] & \dots \arrow[r] & C^{k-1}(\mathbb{R}) \arrow[r] & C^{k-1}(\mathbb{R}) \arrow[r] & \dots \\
C^\infty(\mathbb{R}) \arrow[rr, bend right] \arrow[rrr, bend right] & \dots \arrow[r] & C^k(\mathbb{R}) \arrow[r] \arrow[ur, "\frac{d}{dx}"'] & C^k(\mathbb{R}) \arrow[r] \arrow[ur, "\frac{d}{dx}"'] & \dots
\end{tikzcd}$$

Composing the projections with d/dx gives (dashed) induced maps from $C^k(\mathbb{R})$ to the limitands, inducing a unique (dotted) continuous linear map to the limit, as in

$$\begin{tikzcd}
C^\infty(\mathbb{R}) \arrow[rrr, bend left] \arrow[rrrr, bend left] & \dots \arrow[r] & C^{k-1}(\mathbb{R}) \arrow[r] & C^{k-1}(\mathbb{R}) \arrow[r] & \dots \\
C^\infty(\mathbb{R}) \arrow[u, dotted, "\frac{d}{dx}"] \arrow[urr, dashed] \arrow[urrr, dashed] \arrow[rr, bend right] \arrow[rrr, bend right] & \dots \arrow[r] & C^k(\mathbb{R}) \arrow[r] \arrow[ur] & C^k(\mathbb{R}) \arrow[r] \arrow[ur] & \dots
\end{tikzcd}$$

A novelty is the assertion that (projective) limits *commute* with each other, so that the limits of $C^k(K_n)$ in k and in n can be taken in either order. Generally, in a situation

$$\begin{tikzcd}
\lim_j(\lim_i V_{ij}) \arrow[rrr, dashed, bend left] \arrow[rr, dashed, bend left] & \dots \arrow[r] & \lim_i V_{i2} \arrow[r] \arrow[d, two heads, bend left] \arrow[d, two heads, bend right] & \lim_i V_{i1} \arrow[d, two heads, bend left] \arrow[d, two heads, bend right] \\
\lim_i(\lim_j V_{ij}) \arrow[d, dashed, bend right] \arrow[d, dashed, bend left] & & & \\
\lim_j V_{2j} \arrow[d] \arrow[rrr, bend left] \arrow[rrr, bend right] & \dots \arrow[r] & V_{22} \arrow[r] \arrow[d] & V_{21} \arrow[d] \\
\lim_j V_{1j} \arrow[rrr, bend left] \arrow[rrr, bend right] & \dots \arrow[r] & V_{12} \arrow[r] & V_{11}
\end{tikzcd}$$

the maps $\lim_j(\lim_i V_{ij}) \to V_{k\ell}$ induce a map $\lim_j(\lim_i V_{ij}) \to \lim_\ell V_{k\ell}$, which induce a unique $\lim_j(\lim_i V_{ij}) \to \lim_k(\lim_\ell V_{k\ell})$. Similarly, a unique map is induced in the opposite direction, and, for the usual reason, these are mutual inverses. ///

[5.0.4] Claim: For fixed $x \in \mathbb{R}$ and fixed non-negative integer k, the *evaluation map* $f \to f^{(k)}(x)$ is *continuous*.

Proof: Take n large enough so that $x \in [-n, n]$. Evaluation $f \to f^{(k)}(x)$ was shown in [???] to be continuous on $C^k[-n, n]$. Composing with the continuous $C^\infty(\mathbb{R}) \to C^k(\mathbb{R}) \to C^k[-n, n]$ gives the continuity. ///

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6. *Banach completion* $C_o^k(\mathbb{R})$ *of* $C_c^k(\mathbb{R})$

It is reasonable to ask about the completion of the space $C_c^o(\mathbb{R})$ of compactly-supported continuous functions in the metric given by the sup-norm, and, more generally, about the completion of the space $C_c^k(\mathbb{R})$ of compactly-supported k-times continuously differentiable functions in the metric given by the sum of the sups of the k derivatives.

The spaces $C_c^k(\mathbb{R})$ are *not complete* with those norms, because supports can *leak* out to infinity: for example, in fix any u such that $u(x) = 1$ for $|x| \le 1$, $0 \le u(x) \le 1$ for $1 \le |x| \le 2$, and $u(x) = 0$ for $|x| \ge 2$. Then

$$f(x) \;=\; \sum_{n=0}^{\infty} \frac{u(x-n)}{n^2}$$

converges in sup-norm, the partial sums have compact support, but the whole does not have compact support.

[6.0.1] Claim: The completion of the space $C_c^o(\mathbb{R})$ of compactly-supported continuous functions in the metric given by the sup-norm $|f|_{C^o} = \sup_{x\in\mathbb{R}} |f(x)|$ is the space $C_o^o(\mathbb{R})$ of continuous functions f *vanishing at infinity*, in the sense that, given $\varepsilon > 0$, there is a compact interval $K = [-N, N] \subset X$ such that $|f(x)| < \varepsilon$ for $x \notin K$.

[6.0.2] Remark: Since we need to distinguish compactly-supported functions $C_c^o(\mathbb{R})$ from functions $C_o^o(\mathbb{R})$ going to 0 at infinity, we cannot use the latter notation for the former, unlike some sources.

Proof: This is almost a tautology. Given $f \in C_o^o(\mathbb{R})$, given $\varepsilon > 0$, let $K = [-N, N] \subset X$ be compact such that $|f(x)| < \varepsilon$ for $x \notin K$. It is easy to make an auxiliary function φ that is continuous, *compactly-supported*, real-valued function such that $\varphi = 1$ on K and $0 \le \varphi \le 1$ on X. Then $f - \varphi \cdot f$ is 0 on K, and of absolute value $|\varphi(x) \cdot f(x)| \le |f(x)| < \varepsilon$ off K. That is, $\sup_{\mathbb{R}} |f - \varphi \cdot f| < \varepsilon$, so $C_c^o(\mathbb{R})$ is dense in $C_o^o(\mathbb{R})$.

On the other hand, a sequence f_i in $C_c^o(\mathbb{R})$ that is a Cauchy sequence with respect to sup norm gives a Cauchy sequence in each $C^o[a, b]$, and converges uniformly pointwise to a continuous function on $[a, b]$ for every $[a, b]$. Let f be the pointwise limit. Given $\varepsilon > 0$ take i_o such that $\sup_x |f_i(x) - f_j(x)| < \varepsilon$ for all $i, j \ge i_o$. With K the support of f_{i_o},

$$\sup_{x\notin K} |f(x)| \;\le\; \sup_{x\notin K} |f(x) - f_{i_o}(x)| + \sup_{x\notin K} |f_{i_o}(x)| \;=\; \sup_{x\notin K} |f(x) - f_{i_o}(x)| + 0 \;\le\; \varepsilon \;<\; 2\varepsilon$$

showing that f goes to 0 at infinity. ///

[6.0.3] Corollary: Continuous functions vanishing at infinity are *uniformly* continuous.

Proof: For $f \in C_o^o(\mathbb{R})$, given $\varepsilon > 0$, let $g \in C_c^o(\mathbb{R})$ be such that $\sup |f - g| < \varepsilon$. By the uniform continuity of g, there is $\delta > 0$ such that $|x - y| < \delta$ implies $|g(x) - g(y)| < \varepsilon$, and

$$|f(x) - f(y)| \;\le\; |f(x) - g(x)| + |f(y) - g(y)| + |g(x) - g(y)| \;<\; 3\varepsilon$$

as desired. ///

The arguments for $C^k(\mathbb{R})$ are completely parallel: the completion of the space $C_c^k(\mathbb{R})$ of compactly supported k-times continuously differentiable functions is the space $C_c^k(\mathbb{R})$ of k-times continuously differentiable functions whose k derivatives go to zero at infinity. Similarly,

[6.0.4] Corollary: The space of C^k functions whose k derivatives all vanish at infinity have *uniformly* continuous derivatives. ///

[6.0.5] Claim: The limit $\lim_k C_o^k(\mathbb{R})$ is the space $C_o^\infty(\mathbb{R})$ of smooth functions all whose derivatives go to 0 at infinity. All those derivatives are *uniformly* continuous.

Proof: As with $C^\infty[a,b] = \bigcap_k C^k[a,b] = \lim_k C^k[a,b]$, by its very definition $C_o^\infty(\mathbb{R})$ is the intersection of the Banach spaces $C_o^k(\mathbb{R})$. For any compatible family $Z \to C_o^k(\mathbb{R})$, the compatibility implies that the image of Z is in that intersection. ///

[6.0.6] Corollary: The space $C_o^\infty(\mathbb{R})$ is a Fréchet space, so is *complete*.

Proof: As earlier, countable limits of Banach spaces are Fréchet. ///

[6.0.7] Remark: In contrast, the space of merely *bounded* continuous functions does not behave so well. Functions such as $f(x) = \sin(x^2)$ are not *uniformly* continuous. This has the bad side effect that $\sup_x |f(x+h) - f(x)| = 1$ for all $h \neq 0$, which means that the *translation action* of $\mathbb{R}$ on that space of functions is *not continuous*.

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7. *Rapid-decay functions, Schwartz functions*

A continuous function f on $\mathbb{R}$ is *of rapid decay* when

$$\sup_{x\in\mathbb{R}} (1+x^2)^n \cdot |f(x)| \;<\; +\infty \qquad\qquad \text{(for every } n = 1, 2, \ldots)$$

With norm $\nu_n(f) = \sup_{x\in\mathbb{R}}(1+x^2)^n \cdot |f(x)|$, let the Banach space B_n be the completion of $C_c^o(\mathbb{R})$ with respect to the metric $\nu_n(f-g)$ associated to ν_n.

[7.0.1] Lemma: The Banach space B_n is isomorphic to $C_o^o(\mathbb{R})$ by the map $T : f \to (1+x^2)^n \cdot f$. Thus, B_n is the space of continuous functions f such that $(1+x^2)^n \cdot f(x)$ goes to 0 at infinity.

Proof: By design, $\nu_n(f)$ is the sup-norm of Tf. Thus, the result [???] for $C_o^o(\mathbb{R})$ under sup-norm gives this lemma. ///

[7.0.2] Remark: Just as we want the completion $C_o^o(\mathbb{R})$ of $C_c^o(\mathbb{R})$, rather than the space of all *bounded* continuous functions, we want B_n rather than the space of all continuous functions f with $\sup_x (1+x^2) \cdot |f(x)| < \infty$. This distinction disappears in the limit, but it is only via the density of $C_c^o(\mathbb{R})$ in every B_n that it follows that $C_c^o(\mathbb{R})$ is dense in the space of continuous functions of rapid decay, in the corollary below.

[7.0.3] Claim: The space of continuous functions of rapid decay on $\mathbb{R}$ is the nested *intersection*, thereby the *limit*, of the Banach spaces B_n, so is Fréchet.

Proof: The key issue is to show that rapid-decay f is a ν_n-limit of *compactly-supported* continuous functions for every n. For each fixed n the function $f_n = (1+x^2)^n f$ is continuous and goes to 0 at infinity. By [???], f_n is the sup-norm limit of *compactly supported* continuous functions F_{nj}. Then $(1+x^2)^{-n} F_{nj} \to f$ in the topology on B_n, and $f \in B_n$. Thus, the space of rapid-decay functions lies *inside* the intersection.

On the other hand, a function $f \in \bigcap_k B_k$ is continuous. For each n, since $(1+x^2)^{n+1}|f(x)|$ is continuous and goes to 0 at infinity, it has a finite sup σ, and

$$\sup_x (1+x^2)^n \cdot |f(x)| \;=\; \sup_x (1+x^2)^{-1} \cdot (1+x^2)^{n+1}|f(x)| \;\le\; \sup_x (1+x^2)^{-1} \cdot \sigma \;<\; +\infty$$

This holds for all n, so f is of rapid decay. ///

[7.0.4] Corollary: The space $C_c^o(\mathbb{R})$ is *dense* in the space of continuous functions of rapid decay.

Proof: That every B_n is a completion of $C_c^o(\mathbb{R})$ is essential for this argument.

Use the model of the limit $X = \lim_n B_n$ as the diagonal in $\prod_n B_n$, with the product topology restricted to X. Let $p_n : \prod_k B_k \to B_n$ be the projection. Thus, given $x \in X$, there is a basis of neighborhood N of x in X of the form $N = X \cap U$ for an open U in the product of the form $U = \prod_n U_n$ with all but finitely-many $U_n = B_n$. Thus, for $y \in C_c^o(\mathbb{R})$ such that $p_n(y) \in p_n(N) = p_n(U)$ for the finitely-many indices such that $U_n \neq B_n$, we have $y \in N$. That is, approximating x in only *finitely-many* of the limitands B_n suffices to approximate x in the limit. Thus, density in the limitands B_n implies density in the limit. ///

[7.0.5] Remark: The previous argument applies generally, showing that a common subspace dense in all limitands is dense in the limit.

Certainly the operator of multiplication by $1 + x^2$ preserves $C_c^o(\mathbb{R})$, and is a continuous map $B_n \to B_{n-1}$. Much as d/dx was treated earlier,

[7.0.6] Claim: Multiplication by $1 + x^2$ is a continuous map of the space of continuous rapidly-decreasing functions to itself.

Proof: Let T denote the multiplication by $1+x^2$, and let $B = \lim_n B_n$ be the space of rapid-decay continuous functions. From the commutative diagram

$$\begin{tikzcd}
B \arrow[rr, bend left] \arrow[rrr, bend left] & \cdots \arrow[r] & B_n \arrow[r] & B_{n-1} \arrow[r] & \cdots \\
B \arrow[rrr, bend right] \arrow[rr, bend right] & \cdots \arrow[r] & B_n \arrow[r] \arrow[ur, "T"] & B_{n-1} \arrow[r] \arrow[ur, "T"] & \cdots
\end{tikzcd}$$

composing the projections with T giving (dashed) induced maps from B to the limitands, inducing a unique (dotted) continuous linear map to the limit, as in

$$\begin{tikzcd}
B \arrow[rr, bend left] \arrow[rrr, bend left] & \cdots \arrow[r] & B_n \arrow[r] & B_{n-1} \arrow[r] & \cdots \\
B \arrow[u, dotted, "T"] \arrow[rrr, bend right] \arrow[rr, bend right] \arrow[urr, dashed] \arrow[urrr, dashed] & \cdots \arrow[r] & B_n \arrow[r] \arrow[urr] & B_{n-1} \arrow[r] \arrow[urr] & \cdots
\end{tikzcd}$$

giving the *continuous* multiplication map on the space of rapid-decay continuous functions. ///

Similarly, adding differentiability conditions, the space of *rapidly decreasing* C^k functions is the space of k-times continuously differentiable functions f such that, for every $\ell = 0, 1, 2, \ldots, k$ and for every $n = 1, 2, \ldots$,

$$\sup_{x \in \mathbb{R}} (1 + x^2)^n \cdot |f^{(\ell)}(x)| \;<\; +\infty$$

Let B_n^k be the completion of $C_c^k(\mathbb{R})$ with respect to the metric from the norm

$$\nu_n^k(f) \;=\; \sum_{0 \le \ell \le k} \sup_{x \in \mathbb{R}} (1 + x^2)^n |f^{(\ell)}(x)|$$

Essentially identical arguments give

[7.0.7] Claim: The space of C^k functions of rapid decay on $\mathbb{R}$ is the *nested intersection*, thereby the *limit*, of the Banach spaces B_n^k, so is Fréchet. ///

[7.0.8] Corollary: The space $C_c^k(\mathbb{R})$ is *dense* in the space of C^k functions of rapid decay. ///

Identifying B_n^k as a space of C^k functions with additional decay properties at infinity gives the obvious map $\frac{d}{dx} : B_n^k \to B_n^{k-1}$.

[7.0.9] Claim: $\frac{d}{dx} : B_n^k \to B_n^{k-1}$ is *continuous.*

Proof: Since B_n^k is the closure of $C_c^k(\mathbb{R})$, it suffices to check the continuity of $\frac{d}{dx} : C_c^k(\mathbb{R}) \to C_c^{k-1}(\mathbb{R})$ for the B_n^k and B_n^{k-1} topologies. As usual, that continuity was designed into the situation. ///

The space of *Schwartz functions* is

$$\mathscr{S}(\mathbb{R}) \;=\; \{\text{smooth functions } f \text{ all whose derivatives are of rapid decay}\}$$

One reasonable topology on $\mathscr{S}(\mathbb{R})$ is as a limit

$$\mathscr{S}(\mathbb{R}) \;=\; \bigcap_k \{C^k \text{ functions of rapid decay}\} \;=\; \lim_k \{C^k \text{ functions of rapid decay}\}$$

As a countable limit of Fréchet spaces, this makes $\mathscr{S}(\mathbb{R})$ Fréchet.

[7.0.10] Corollary: $\frac{d}{dx} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ is *continuous.*

Proof: This is structurally the same as before: from the commutative diagram

$$\begin{tikzcd}
\mathscr{S}(\mathbb{R}) \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & B_n^{k-1} \arrow[r] & B_n^{k-1} \arrow[r] & \dots \\
\mathscr{S}(\mathbb{R}) \arrow[rrr, bend right] \arrow[rr, bend right] & \dots \arrow[r] & B_n^{k} \arrow[r] \arrow[u, "\frac{d}{dx}"] & B_n^{k} \arrow[r] \arrow[u, "\frac{d}{dx}"] & \dots
\end{tikzcd}$$

composing the projections with d/dx to give (dashed) induced maps from $\mathscr{S}(\mathbb{R})$ to the limitands, inducing a unique (dotted) continuous linear map to the limit:

$$\begin{tikzcd}
\mathscr{S}(\mathbb{R}) \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & B_{n+1}^{k-1} \arrow[r] & B_n^{k-1} \arrow[r] & \dots \\
\mathscr{S}(\mathbb{R}) \arrow[u, dotted, "\frac{d}{dx}"] \arrow[urr, dashed] \arrow[urrr, dashed] \arrow[rrr, bend right] \arrow[rr, bend right] & \dots \arrow[r] & B_{n+1}^{k} \arrow[r] \arrow[u] & B_n^{k-1} \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

as desired. ///

Finally, to induce a canonical continuous map $T : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ by multiplication by $1 + x^2$, examine the behavior of this multiplication map on the auxiliary spaces B_n^k and its interaction with $\frac{d}{dx}$:

[7.0.11] Claim: $T : B_n^k \to B_{n-1}^{k-1}$ is continuous.

Proof: Of course,

$$\Big|\frac{d}{dx}\Big((1+x^2)\cdot f(x)\Big)\Big| \;=\; \Big|2x\cdot f(x) + (1+x^2)\cdot f'(x)\Big| \;\le\; 2\cdot(1+x^2)\cdot|f(x)| + (1+x^2)\cdot|f'(x)|$$

Thus, $T : C_c^k(\mathbb{R}) \to C_c^{k-1}(\mathbb{R})$ is continuous with the B_n^k and B_{n-1}^{k-1} topologies. As noted earlier in [???], cofinal limits are isomorphic, so the same argument gives a unique continuous linear map $\mathscr{S}(\mathbb{R})$. ///

It is worth noting

[7.0.12] Claim: Compactly-supported smooth functions are *dense* in $\mathscr{S}$.

Proof: At least up to rearranging the order of limit-taking, the description of $\mathscr{S}$ above is as a limit of spaces in each of which compactly-supported smooth functions are dense. Thus, we claim a general result: for a limit $X = \lim_i X_i$ and compatible maps $f_i : V \to X_i$ with dense image, the induced map $f : V \to X$ has dense image. As in [???], the limit is the diagonal

$$D \;=\; \{\{x_i\} \in \prod_i X_i : x_i \to x_{i-1}, \text{ for all } i\} \;\subset\; \prod_i X_i$$

with the subspace topology from the product. Suppose we are given a finite collection of neighborhoods $x_{i_1} \in U_{i_1} \subset X_{i_1}, \ldots, x_{i_n} \in U_{i_n} \subset X_{i_n}$, with $x_{i_j} \to x_{i_k}$ if $i_j \geq i_k$. Take $i = \max_j i_j$, and U a neighborhood of x_i such that the image of U is inside every U_{i_j}, by continuity. Since the image of V is dense in X_i, there is $v \in V$ such that $f_i(v) \in U$. By compatibility, $f_{i_j}(v) \in U_{i_j}$ for all j. Thus, the image of V is dense in the limit. ///

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8. *Non-Fréchet colimit $\mathbb{C}^\infty$ of $\mathbb{C}^n$, quasi-completeness*

Toward topologies in which $C_c^o(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$ could be *complete*, we consider first

$$\mathbb{C}^\infty \;=\; \bigcup_n \mathbb{C}^n$$

where $i_n : \mathbb{C}^n \subset C^{n+1}$ by $i_n : (x_1, \ldots, x_n) \to (x_1, \ldots, x_n, 0)$. We want to topologize $\mathbb{C}^\infty$ so that it is *complete*, in a suitable sense. Above, we saw that finite-dimensional complex vectorspaces have *unique* vectorspace topologies, so the only question is how to fit them together.

A countable ascending union of complete metric topological vector spaces, each a proper *closed* subspace of the next, such as $C^\infty = \bigcup \mathbb{C}^n$, *cannot* be a complete *metric* space, because it is exactly *presented* as a countable union of nowhere-dense closed subsets, contradicting the conclusion of the Baire Category Theorem. The function spaces $C_c^o(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$ are also of this type, being the ascending unions of spaces C_K^o or C_K^∞, continuous or smooth functions with supports inside compact $K \subset \mathbb{R}$.

Thus, we cannot hope to give such space *metric* topologies for which they are *complete*.

Nevertheless, ascending unions are a type of *colimit*, just as descending intersections are a type of *limit*. That is, the topology on $\mathbb{C}^\infty$ is characterized by a universal property: for every collection of maps $f_n : \mathbb{C}^n \to Z$ with the compatibility $i_n \circ f_n = f_{n+1}$, there is a unique $f : \mathbb{C}^\infty \to Z$ through which all f_n's *factor*. That is, given a commutative diagram

$$\begin{tikzcd}
\mathbb{C}^1 \arrow[r] \arrow[rrr, bend left] \arrow[drr, dashed] & \mathbb{C}^2 \arrow[r] \arrow[rr, bend left] \arrow[dr, dashed] & \ldots & \mathbb{C}^\infty \arrow[dl, dashed] \\
& & Z &
\end{tikzcd}$$

there is a *unique* (dotted) map $\mathbb{C}^\infty \to Z$ giving a commutative diagram

$$\begin{tikzcd}
\mathbb{C}^1 \arrow[r] \arrow[rrr, bend left] \arrow[drr, dashed] & \mathbb{C}^2 \arrow[r] \arrow[rr, bend left] \arrow[dr, dashed] & \ldots & \mathbb{C}^\infty \arrow[dl, dotted] \\
& & Z &
\end{tikzcd}$$

To argue that an ascending union $X = \bigcup_n X_n$ with $X_1 \subset X_2 \subset \ldots$ is an example of a colimit, observe that every $x \in X$ lies in some X_n, so all values $f(x)$ for a map $f : X \to Z$ are completely determined by the restrictions of f to the limitands X_n. Thus, on one hand, given a compatible family $f_n : X_n \to Z$, there is *at most one* compatible $f : X \to Z$. On the other hand, a compatible family $f_n : X_n \to Z$ *defines* a map $X \to Z$: given $x \in X$, take n sufficiently large so that $x \in X_n$, and define $f(x) = f_n(x)$. The compatibility assures that it doesn't matter which sufficiently large n we use.

For the topology of $\mathbb{C}^\infty$, the colimit characterization has a possibly-counterintuitive consequence:

[8.0.1] Claim: *Every* linear map from the space $\mathbb{C}^\infty = \mathrm{colim}_n \mathbb{C}^n$ with the colimit topology to *any* topological vectorspace is *continuous*.

Proof: Given arbitrary linear $f : \mathbb{C}^\infty \to Z$, composition with inclusion gives a compatible family of linear maps $f_n : \mathbb{C}^n \to Z$. From [???], *every* linear map from a finite-dimensional space is *continuous*. The collection $\{f_n\}$ induces a unique *continuous* map $F : \mathbb{C}^\infty \to Z$ such that $F \circ i_n : \mathbb{C}^n \to Z$ is the same as $f \circ i_n$. In general, this might not be force $f = F$. However, because X is an ascending union, the values of both F and f are completely determined by their values on the limitands, and these are the same. Thus, $f = F$. ///

The *uniqueness* argument for locally convex colimits of locally convex topological vectorspaces, that there is *at most one* such topology, is identical to the uniqueness argument for *limits* in [???] with arrows reversed.

[8.0.2] Remark: The fact that a colimit of finite-dimensional spaces has a unique canonical topology, from which *every* linear map from such a colimit is *continuous*, is often misunderstood and misrepresented as suggesting that there is *no* topology on that colimit. Again, there is a *unique canonical* topology, from which every *linear* map is *continuous*.

To prove *existence* of colimits, just as limits are *subobjects* of products, colimits are *quotients* of *coproducts*, as follows. A *locally convex colimit* of topological vector spaces X_α with *transition maps* $j^\alpha_\beta : X_\alpha \to X_\beta$ is the *quotient* of the *locally convex coproduct* X of the X_α by the *closure* of the subspace Z spanned by vectors

$$j_\alpha(x_\alpha) - (j_\beta \circ j^\alpha_\beta)(x_\alpha) \qquad \text{(for all } \alpha < \beta \text{ and } x_\alpha \in X_\alpha)$$

Annihilation of these differences in the quotient forces the desired compatibility relations. Obviously, quotients of locally convex spaces are locally convex.

Locally convex coproducts X of topological vector spaces X_α are coproducts (also called *direct sums*) of the *vector spaces* X_α topologized by the *diamond topology*, described as follows. [2] For a collection U_α of convex neighborhoods of 0 in the X_α, let

$$U \;=\; \text{convex hull in } X \text{ of the union of } j_\alpha(U_\alpha) \qquad \text{(with } j_\alpha : X_\alpha \to X \text{ the } \alpha^{th} \text{ canonical map)}$$

The diamond topology has local basis at 0 consisting of such U. Thus, it is locally convex by construction. *Closedness of points* follows from the corresponding property of the X_α. Thus, *existence* of a *locally convex* coproduct of locally convex spaces is assured by the *construction*.

A *countable* colimit of a family $V_1 \to V_2 \to \ldots$ of topological vectorspaces is a *strict* colimit, or *strict inductive limit*, when each $V_i \to V_{i+1}$ is an isomorphism to its image, and each image is closed. A strict colimit of *Fréchet* spaces is called an *LF-space*.

[2] The *product* topology of locally convex topological vector spaces is locally convex, whether in the category of locally convex topological vector spaces or in the larger category of not-necessarily-locally-convex topological vector spaces. However, *coproducts* behave differently: the locally convex coproduct of *uncountably many* locally convex spaces is *not* a coproduct in the larger category of not-necessarily-locally-convex spaces. This already occurs with an uncountable coproduct of *lines*.

Just to be sure:

[8.0.3] Claim: In a colimit indexed by positive integers $V = \text{colim} V_i$, if every transition $V_i \to V_{i+1}$ is *injective*, then every limitand V_i *injects* to the colimit V. Further, the colimit is the ascending union of the limitands V_i, suitably topologized.

Proof: In effect, the argument presents the colimit corresponding to an ascending union more directly, not as a quotient of the coproduct, although it is convenient to already have *existence* of the colimit. Certainly each V_i injects to $W = \bigcup_n V_n$. We will give W a locally convex topology so that every inclusion $V_i \to W$ is continuous. The universal property of the colimit produces a unique compatible map $V \to W$, so every V_i must inject to V itself.

Since the maps j_i of V_i to the colimit V are injections, the ascending union W injects to V by $j(w) = j_i(w)$ for any index i large enough so that $w \in V_i$. The compatibility of the maps among the V_i assures that j is well-defined. We claim that $j(W)$ with the subspace topology from V, and the inclusions $V_i \to j_i(V_i) \subset j(W)$, give a colimit of the V_i. Indeed for any compatible, family $f_i : V_i \to Z$ and induced $f : V \to Z$, the restriction of f to $j(W)$ gives a map $j(W) \to Z$ through which the f_i factor. Thus, in fact, such a colimit is the ascending union with a suitable topology.

Now we describe a topology on the ascending union W so that all inclusions $V_i \to W$ are continuous. Give W a local basis $\{U\}$ at 0, by taking arbitrary convex opens $U_i \subset V_i$ containing 0, and letting U be the convex hull of $\bigcup_i U_i$. Every injection $V_i \to W$ is continuous, because the inverse image of such $U \cap V_i$ contains U_i, giving continuity at 0.

To be sure that *points are closed* in W, given $0 \neq x \in W$, we find a neighborhood of 0 in W not containing x. Let i_o be the first index such that $x \in V_{i_o}$. By Hahn-Banach, there is a continuous linear functional λ_{i_o} on V_{i_o} such that $\lambda_{i_o}(x) \neq 0$. Without loss of generality, $\lambda_{i_o}(x) = 1$ and $|\lambda_{i_o}| = 1$. Use Hahn-Banach to extend λ_{i_o} to a continuous linear functional λ_i on V_i for every $i \geq i_o$, with $|\lambda_i| \leq 1$. λ_{i_o} gives a continuous linear functional on V_i for $i < i_o$ by composition with the injection $V_i \to V_{i_o}$. Then $U_i = \{y \in V_i : |\lambda_i(y)| < 1\}$ is open in V_i and does not contain x, for all i. The convex hull of the ascending union $\bigcup_i U_i$ is just $\bigcup_i U_i$ itself, so does not contain x.

We did not quite prove that this topology is exactly the colimit topology, but we will never need that fact. ///

Typical *colimit* topologies are *not* complete in the strongest possible sense (see below), but are *quasi-complete*, a property sufficient for all applications. To describe quasi-completeness, we need a notion of *boundedness* in general topological vectorspaces, not merely metrizable ones. A subset B of a topological vector space V is *bounded* when, for every open neighborhood N of 0 there is $t_o > 0$ such that $B \subset tN$ for every $t \geq t_o$. A space is *quasi-complete* when every *bounded* Cauchy *net* is *convergent*.

Nothing new for metric spaces:

[8.0.4] Lemma: Complete metric spaces are quasi-complete. In particular, Cauchy nets converge, and contain cofinal *sequences* converging to the same limit.

Proof: Let $\{s_i : i \in I\}$ be a Cauchy net in X. Given a natural number n, let $i_n \in I$ be an index such that $d(x_i, x_j) < \frac{1}{n}$ for $i, j \geq i_n$. Then $\{x_{i_n} : n = 1, 2, \ldots\}$ is a Cauchy sequence, with limit x. Given $\varepsilon > 0$, let $j \geq i_n$ be also large enough such that $d(x, x_j) < \varepsilon$. Then

$$d(x, x_{i_n}) \leq d(x, x_j) + d(x_j, x_{i_n}) < \varepsilon + \frac{1}{n} \qquad \text{(for every } \varepsilon > 0)$$

Thus, $d(x, x_{i_n}) \leq \frac{1}{n}$. The original Cauchy net also converges to x: given $\varepsilon > 0$, for n large enough so that $\varepsilon > \frac{1}{n}$,

$$d(x_i, x) \;\leq\; d(x_i, x_{i_n}) + d(x_{i_n}, x) \;<\; \varepsilon + \varepsilon \qquad \text{(for } i \geq i_n)$$

with the strict inequality coming from $d(x_{i_n}, x) < \varepsilon$. ///

[8.0.5] Theorem: A bounded subset of an LF-space $X = \mathrm{colim}_n X_n$ lies in some limitand X_n. An LF-space is *quasi-complete*.

Proof: Let B be a bounded subset of X. Suppose B does *not* lie in any X_i. Then there is a sequence $i_1, i_2, \ldots$ of positive integers and x_{i_ℓ} in $X_{i_\ell} \cap B$ with x_{i_ℓ} *not* lying in $X_{i_\ell - 1}$. Using $X = \bigcup_j X_{i_\ell}$, without loss of generality, suppose that $i_\ell = \ell$.

By the Hahn-Banach theorem and induction, using the closedness of X_{i-1} in X_i, there are continuous linear functionals λ_i on X_i's such that $\lambda_i(x_i) = i$ and the restriction of λ_i to X_{i-1} is λ_{i-1}, for example. Since X is the colimit of the X_i, this collection of functionals exactly describes a unique compatible continuous linear functional λ on X.

But $\lambda(B)$ is *bounded* since B is bounded and λ is continuous, precluding the possibility that λ takes on all positive integer values at the points x_i of B. Thus, it could *not* have been that B failed to lie inside some single X_i. The strictness of the colimit implies that B is bounded as a subset of X_i, proving one direction of the equivalence. The other direction of the equivalence is less interesting.

Thus a *bounded* Cauchy net lies in some limitand Fréchet space X_n, so is convergent there, since Fréchet spaces are complete. ///

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9. *Non-Fréchet colimit $C_c^\infty(\mathbb{R})$ of Fréchet spaces*

The space of compactly-supported continuous functions

$$C_c^o(\mathbb{R}) \;=\; \text{compactly-supported continuous functions on } \mathbb{R}$$

is an *ascending union* of the subspaces

$$C^o_{[-n,n]} \;=\; \{f \in C^o(\mathbb{R}) \;:\; \mathrm{spt} f \subset [-n,n]\}$$

Each space $C^o_{[-n,n]}$ is a Banach space, being a closed subspace of the Banach space $C^o[-n,n]$, further requiring vanishing of the functions on the boundary of $[-n,n]$. A closed subspace of a Banach space is a Banach space. Thus, $C_c^o(\mathbb{R})$ is an LF-space, and is *quasi-complete*.

Similarly,

$$C_c^k(\mathbb{R}) \;=\; \text{compactly-supported } C^k \text{ functions on } \mathbb{R}$$

is an *ascending union* of the subspaces

$$C^k_{[-n,n]} \;=\; \{f \in C^k(\mathbb{R}) \;:\; \mathrm{spt} f \subset [-n,n]\}$$

Each space $C^k_{[-n,n]}$ is a Banach space, being a closed subspace of the Banach space $C^k[-n,n]$, further requiring vanishing of the functions and derivatives on the boundary of $[-n,n]$. A closed subspace of a Banach space is a Banach space. Thus, $C_c^k(\mathbb{R})$ is an LF-space, and is *quasi-complete*.

The space of *test functions* is

$$C_c^\infty(\mathbb{R}) \;=\; C_c^\infty(\mathbb{R}) \;=\; \text{compactly-supported } C^\infty \text{ functions on } \mathbb{R}$$

is an *ascending union* of the subspaces

$$C_c^\infty{}_{[-n,n]} \;=\; C^\infty_{[-n,n]} \;=\; \{f \in C^\infty(\mathbb{R}) \;:\; \mathrm{spt} f \subset [-n,n]\}$$

Each space ${C_c^\infty}_{[-n,n]}$ is a Fréchet space, being a closed subspace of the Fréchet space $C^\infty[-n,n]$, by further requiring vanishing of the functions and derivatives on the boundary of $[-n,n]$. A closed subspace of a Fréchet space is a Fréchet space. Thus, $C_c^\infty(\mathbb{R}) = C_c^\infty(\mathbb{R})$ is an LF-space, and is *quasi-complete*.

The operator $\frac{d}{dx} : C^k[-n,n] \to C^{k-1}[-n,n]$ is continuous, and preserves the vanishing conditions at the endpoints, so restricts to a continuous map $\frac{d}{dx} : C^k_{[-n,n]} \to C^{k-1}_{[-n,n]}$ on the Banach sub-spaces of functions vanishing suitably at the endpoints. Composing with the inclusions $C^{k-1}_{[-n,n]} \to C_c^{k-1}(\mathbb{R})$ gives a compatible family of continuous maps $\frac{d}{dx} : C^k_{[-n,n]} \to C_c^{k-1}(\mathbb{R})$. This induces a unique continuous map on the colimit: $\frac{d}{dx} : C_c^k(\mathbb{R}) \to C_c^{k-1}(\mathbb{R})$.

Similarly, $\frac{d}{dx} : C^\infty[-n,n] \to C^\infty[-n,n]$ is continuous, and preserves the vanishing conditions at the endpoints, so restricts to a continuous map $\frac{d}{dx} : {C_c^\infty}_{[-n,n]} \to {C_c^\infty}_{[-n,n]}$ on the Frechet sub-spaces of functions vanishing to all orders at the endpoints. Composing with the inclusions ${C_c^\infty}_{[-n,n]} \to C_c^\infty(\mathbb{R})$ gives a compatible family of continuous maps $\frac{d}{dx} : {C_c^\infty}_{[-n,n]} \to C_c^\infty(\mathbb{R})$. This induces a unique continuous map on the colimit: $\frac{d}{dx} : C_c^\infty(R) \to C_c^\infty(\mathbb{R})$. Diagrammatically,

$$\begin{tikzcd}
\cdots \arrow[r] & C^\infty_{[-n,n]} \arrow[r] \arrow[d, "\frac{d}{dx}"] \arrow[dr, dashed, "\frac{d}{dx}"] & \cdots \arrow[r, bend left] & C_c^\infty(R) \arrow[d, dotted, "\frac{d}{dx}"] \\
\cdots \arrow[r] & C^\infty_{[-n,n]} \arrow[r] & \cdots \arrow[r, bend right] & C_c^\infty(\mathbb{R})
\end{tikzcd}$$

That is, $\frac{d}{dx}$ is continuous in the LF-space topology on test functions $C_c^\infty(\mathbb{R}) = C_c^\infty(\mathbb{R})$.

[9.0.1] Claim: For fixed $x \in \mathbb{R}$ and non-negative integer k, the evaluation map $f \to f^{(k)}(x)$ on $C_c^\infty(\mathbb{R}) = C_c^\infty(\mathbb{R})$ is *continuous*.

Proof: This evaluation map is continuous on $C^\infty[-n,n]$ for every large-enough n so that $x \in [-n,n]$, so is continuous on the closed subspace ${C_c^\infty}_{[-n,n]}$ of $C^\infty[-n,n]$. The inclusions among these spaces are extend-by-0, so the evaluation map is the 0 map on ${C_c^\infty}_{[-n,n]}$ if $|x| \geq n$. These maps to $\mathbb{C}$ fit together into a compatible family, so extend uniquely to a continuous linear map of the colimit $C_c^\infty(\mathbb{R})$ to $\mathbb{C}$. ///

[9.0.2] Claim: For $F \in C^\infty(\mathbb{R})$, the map $f \to F \cdot f$ is a continuous map of $C_c^\infty(\mathbb{R})$ to itself.

Proof: By the colimit characterization, it suffices to show that such a map is continuous on $C^\infty_{[-n,n]}$, or on the larger Fréchet space $C^\infty[-n,n]$ without vanishing conditions on the boundary. This is the limit of $C^k[-n,n]$, so it suffices to show that $f \to F \cdot f$ is a continuous map $C^k[-n,n] \to C^k[-n,n]$ for every k. The sum of sups of derivatives is

$$\sum_{0 \leq i \leq k} \sup_{|x| \leq n} \Big| \big(\frac{d}{dx}\big)^i (Ff)(x) \Big| \;\leq\; 2^k \Big(\sum_{0 \leq i \leq k} \sup_{|x| \leq n} |F^{(i)}(x)| \Big) \cdot \Big(\sum_{0 \leq i \leq k} \sup_{|x| \leq n} |f^{(i)}(x)| \Big)$$

Although F and its derivatives need not be bounded, this estimate only uses their boundedness on $[-n,n]$. This is a bad estimate, but sufficient for continuity. ///

[9.0.3] Claim: The inclusion $C_c^\infty(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ is *continuous*, and the image is *dense*.

Proof: At least after changing order of limits, as in [???], we have described $\mathscr{S}(\mathbb{R})$ as a limit of spaces in which $C_c^\infty(\mathbb{R})$ is dense. By [???], $C_c^\infty(\mathbb{R})$ is dense in that limit.

The slightly more serious issue is that $C_c^\infty(\mathbb{R})$ with its LF-space topology maps continuously to $\mathscr{S}(\mathbb{R})$. Since $C_c^\infty(\mathbb{R})$ is a colimit, we need only check that the limitands (compatibly) map continuously. On a limitand

$C^\infty_{[-n,n]}$, the norms

$$\nu_{N,k}(f) \;=\; \sup_x (1+x^2)^N \cdot |f^{(k)}(x)|$$

differ from the norms $\sup_x |f^{(k)}(x)|$ defining the topology on $C^\infty_{[-n,n]}$ merely by *constants*, namely, the sups of $(1+x^2)^N$ on $[-n,n]$. Thus, we have the desired continuity on the limitands. ///

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10. *LF-spaces of moderate-growth functions*

The space $C^o_{\text{mod}}(\mathbb{R})$ of continuous functions of *moderate growth* on $\mathbb{R}$ is

$$C^o_{\text{mod}}(\mathbb{R}) \;=\; \{f \in C^o(\mathbb{R}) \;:\; \sup_{x\in\mathbb{R}} (1+x^2)^{-N} \cdot |f(x)| \;<\; +\infty \text{ for some } N\}$$

Literally, it is an ascending union

$$C^o_{\text{mod}}(\mathbb{R}) \;=\; \bigcup_N \Big\{f \in C^o(\mathbb{R}) \;:\; \sup_{x\in\mathbb{R}} (1+x^2)^{-N} \cdot |f(x)| \;<\; +\infty\Big\}$$

However, it is ill-advised to use the individual spaces

$$B_N \;=\; \Big\{f \in C^o(\mathbb{R}) \;:\; \sup_{x\in\mathbb{R}} (1+x^2)^{-N} \cdot |f(x)| \;<\; +\infty\Big\}$$

with norms $\nu_N(f) = \sup_{x\in\mathbb{R}} (1+x^2)^{-N} \cdot |f(x)|$ because $C^o_c(\mathbb{R})$ is not *dense* in these spaces B_N. Indeed, in the simple case $N=0$, the norm ν_0 is the sup-norm, and the sup-norm closure of $C^o_c(\mathbb{R})$ is continuous functions *going to* 0 *at infinity* [???], which excludes many *bounded* continuous functions.

In particular, there are many bounded continuous functions f which are not *uniformly* continuous, and the translation action of $\mathbb{R}$ on such functions cannot be continuous: no matter how small $\delta > 0$ is, $\sup_{x\in\mathbb{R}} |f(x+\delta) - f(x)|$ may be large. For example, $f(x) = \sin(x^2)$ has this feature.

This difficulty does not mean that the characterization of the whole set of moderate-growth functions is incorrect, nor that the norms ν_N are inappropriate, but only that the Banach spaces B_N are too large, and that the topology of the whole should *not* be the strict colimit of the Banach spaces B_N. Instead, take the smaller

$$V_N \;=\; \text{completion of } C^o_c(\mathbb{R}) \text{ with respect to } \nu_N$$

As in the case of completion of $C^o_c(\mathbb{R})$ with respect to sup-norm ν_0,

[10.0.1] Claim: $V_N = \{\text{continuous } f \text{ such that } (1+x^2)^{-N} f(x) \text{ goes to 0 at infinity}\}$. ///

Of course, if $(1+x^2)^{-N} f(x)$ is merely *bounded*, then $(1+x^2)^{-(N+1)} f(x)$ then goes to 0 at infinity. Thus, as *sets*, $B_N \subset V_{N+1}$, but this inclusion *cannot be continuous*, since $C^o_c(\mathbb{R})$ is dense in V_{N+1}, but not in B_N. That is, there is a non-trivial effect on the topology in setting

$$C^o_{\text{mod}} \;=\; \text{colim}_N V_N$$

instead of the colimit of the too-large spaces B_N.

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11. *Seminorms and locally convex topologies*

So far, the vectorspace topologies have been described as Banach spaces, limits of Banach spaces, and colimits of limits of Banach spaces. By design, these descriptions facilitate proof of (quasi-) *completeness*. *Weaker*

topologies are not usually described in this fashion. For example, for a topological vectorspace V, with (continuous) *dual*

$$V^* = \{\text{continuous linear maps } V \to \mathbb{C}\}$$

the *weak dual topology*[3] on V^* has a local sub-basis at 0 consisting of sets

$$U \;=\; U_{v,\varepsilon} \;=\; \{\lambda \in V^* : |\lambda(v)| < \varepsilon\} \qquad\qquad (\text{for fixed } v \in V \text{ and } \varepsilon > 0)$$

Unless V is finite-dimensional, this topology on V^* is much coarser than a Banach, Fréchet, or LF-topology. The map $\lambda \to |\lambda(v)|$ is a natural example of a *seminorm*. It is not a norm, because $\lambda(v) = 0$ can easily happen.

Seminorms are a general device to describe topologies on vectorspaces. These topologies are invariably *locally convex*, in the sense of having a local basis at 0 consisting of *convex* sets.

Description of a vectorspace topology by seminorms does *not* generally give direct information about *completeness*. Nevertheless, we can prove *quasi-completeness* for an important class of examples, just below.

A *seminorm* ν on a complex vectorspace V is a real-valued function on V so that $\nu(x) \geq 0$ for all $x \in V$ *(non-negativity)*, $\nu(\alpha x) = |\alpha| \cdot \nu(x)$ for all $\alpha \in \mathbb{C}$ and $x \in V$ *(homogeneity)*, and $\nu(x+y) \leq \nu(x) + \nu(y)$ for all $x, y \in V$ *(triangle inequality)*. This differs from the notion of *norm* only in the significant point that we allow $\nu(x) = 0$ for $x \neq 0$.

To compensate for the possibility that an individual seminorm can be 0 on a particular non-zero vector, since we want Hausdorff topologies, we mostly care about *separating* families $\{\nu_i : i \in I\}$ of semi-norms: for every $0 \neq x \in V$ there is ν_i so that $\nu_i(x) \neq 0$.

[11.0.1] Claim: The collection Φ of all *finite intersections* of sets

$$U_{i,\varepsilon} \;=\; \{x \in V : \nu_i(x) < \varepsilon\} \qquad\qquad (\text{for } \varepsilon > 0 \text{ and } i \in I)$$

is a *local basis* at 0 for a locally convex topology on V.

Proof: As expected, we intend to define a topological vector space topology on V by saying a set U is *open* if and only if for every $x \in U$ there is some $N \in \Phi$ so that $x + N \subset U$ This would be the *induced topology* associated to the family of seminorms.

That we have a *topology* does not use the hypothesis that the family of seminorms is *separating*, although points will not be closed without the separating property. Arbitrary unions of sets containing sets of the form $x + N$ containing each point x have the same property. The empty set and the whole space V are visibly in the collection. The least trivial issue is to check that finite intersections of such sets are again of the same form. Looking at each point x in a given finite intersection, this amounts to checking that finite intersections of sets in Φ are again in Φ. But Φ is *defined* to be the collection of all finite intersections of sets $U_{i,\varepsilon}$, so this succeeds: we have closure under finite intersections, and a topology on V.

To verify that this topology makes V a topological vectorspace is to verify the continuity of vector addition and continuity of scalar multiplication, and closed-ness of points. None of these verifications is difficult:

The *separating* property implies that for each $x \in V$ the intersection of *all* the sets $x + N$ with $N \in \Phi$ is just x. Given $y \in V$, for each $x \neq y$ let U_x be an open set containing x but not y. Then

$$U \;=\; \bigcup_{x \neq y} U_x$$

is *open* and has complement $\{y\}$, so the singleton $\{y\}$ is *closed*.

[3] The weak dual topology is traditionally called the *weak-*-topology*, but replacing * by *dual* is more explanatory.

For continuity of vector addition, it suffices to prove that, given $N \in \Phi$ and given $x, y \in V$ there are $U, U' \in \Phi$ so that

$$(x+U)+(y+U') \;\subset\; x+y+N$$

The triangle inequality implies that for a fixed index i and for $\varepsilon_1, \varepsilon_2 > 0$

$$U_{i,\varepsilon_1} + U_{i,\varepsilon_2} \;\subset\; U_{i,\varepsilon_1+\varepsilon_2}$$

Then

$$(x+U_{i,\varepsilon_1})+(y+U_{i,\varepsilon_2}) \;\subset\; (x+y)+U_{i,\varepsilon_1+\varepsilon_2}$$

Thus, given

$$N \;=\; U_{i_1,\varepsilon_1} \cap \ldots \cap U_{i_n,\varepsilon_n}$$

take

$$U \;=\; U' \;=\; U_{i_1,\varepsilon_1/2} \cap \ldots \cap U_{i_n,\varepsilon_n/2}$$

proving continuity of vector addition.

For continuity of scalar multiplication, prove that for given $\alpha \in k$, $x \in V$, and $N \in \Phi$ there are $\delta > 0$ and $U \in \Phi$ so that

$$(\alpha + B_\delta)\cdot(x+U) \;\subset\; \alpha x + N \qquad\qquad (\text{with } B_\delta = \{\beta \in k : |\alpha-\beta| < \delta\})$$

Since N is an intersection of the sub-basis sets $U_{i,\varepsilon}$, it suffices to consider the case that N is such a set. Given α and x, for $|\alpha' - \alpha| < \delta$ and for $x - x' \in U_{i,\delta}$,

$$\begin{aligned}\nu_i(\alpha x - \alpha' x') \;&=\; \nu_i((\alpha-\alpha')x + (\alpha'(x-x')) \;\le\; \nu_i((\alpha-\alpha')x) + \nu_i(\alpha'(x-x'))\\ &=\; |\alpha-\alpha'|\cdot\nu_i(x) + |\alpha'|\cdot\nu_i(x-x') \;\le\; |\alpha-\alpha'|\cdot\nu_i(x) + (|\alpha|+\delta)\cdot\nu_i(x-x')\\ &\le\; \delta\cdot(\nu_i(x)+|\alpha|+\delta)\end{aligned}$$

Thus, for the joint continuity, take $\delta > 0$ small enough so that

$$\delta\cdot(\delta + \nu_i(x) + |\alpha|) \;<\; \varepsilon$$

Taking finite intersections presents no further difficulty, taking the corresponding finite intersections of the sets B_δ and $U_{i,\delta}$, finishing the demonstration that separating families of seminorms give a structure of topological vectorspace.

Last, check that finite intersections of the sets $U_{i,\varepsilon}$ are *convex*. Since intersections of convex sets are convex, it suffices to check that the sets $U_{i,\varepsilon}$ themselves are convex, which follows from the homogeneity and the triangle inequality: with $0 \le t \le 1$ and $x, y \in U_{i,\varepsilon}$,

$$\nu_i(tx+(1-t)y) \;\le\; \nu_i(tx) + \nu_i((1-t)y) \;=\; t\nu_i(x) + (1-t)\nu_i(y) \;\le\; t\varepsilon + (1-t)\varepsilon \;=\; \varepsilon$$

Thus, the set $U_{i,\varepsilon}$ is convex. ///

The converse, that *every* locally convex topology is given by a family of seminorms, is more difficult:

Let U be a *convex* open set containing 0 in a topological vectorspace V. Every open neighborhood of 0 contains a *balanced* neighborhood of 0, so shrink U if necessary so it is balanced, that is, $\alpha v \in U$ for $v \in U$ and $|\alpha| \le 1$. The *Minkowski functional* ν_U associated to U is

$$\nu_U(v) \;=\; \inf\{t \ge 0 : v \in tU\}$$

[11.0.2] Claim: The Minkowski functional ν_U associated to a balanced convex open neighborhood U of 0 in a topological vectorspace V is a *seminorm* on V, and is *continuous* in the topology on V.

Proof: The argument is as expected:

By continuity of scalar multiplication, *every* neighborhood U of 0 is *absorbing*, in the sense that every $v \in V$ lies inside tU for large enough $|t|$. Thus, the set over which we take the infimum to define the Minkowski functional is *non-empty*, so the infimum exists.

Let α be a scalar, and let $\alpha = s\mu$ with $s = |\alpha|$ and $|\mu| = 1$. The balanced-ness of U implies the balanced-ness of tU for any $t \geq 0$, so for $v \in tU$ also

$$\alpha v \;\in\; \alpha tU \;=\; s\mu tU \;=\; stU$$

From this,

$$\{t \geq 0 : \alpha v \in \alpha U\} \;=\; |\alpha| \cdot \{t \geq 0 : \alpha v \in tU\}$$

from which follows the *homogeneity* property required of a seminorm:

$$\nu_U(\alpha v) \;=\; |\alpha| \cdot \nu_U(v) \qquad\qquad \text{(for scalar } \alpha)$$

For the triangle inequality use the convexity. For $v, w \in V$ and $s, t > 0$ such that $v \in sU$ and $w \in tU$,

$$v + w \;\in\; sU + tU \;=\; \{su + tu' \;:\; u, u' \in U\}$$

By convexity,

$$su + tu' \;=\; (s+t) \cdot \Big(\frac{s}{s+t} \cdot u + \frac{t}{s+t} \cdot u'\Big) \;\in\; (s+t) \cdot U$$

Thus,

$$\nu_U(v+w) \;=\; \inf\{r \geq 0 : v + w \in rU\} \;\leq\; \inf\{r \geq 0 : v \in rU\} + \inf\{r \geq 0 : w \in rU\} \;=\; \nu_U(v) + \nu_U(w)$$

Thus, the Minkowski functional ν_U attached to balanced, convex U is a continuous seminorm. ///

[11.0.3] Theorem: The topology of a *locally convex* topological vectorspace V is given by the collection of seminorms obtained as Minkowski functionals ν_U associated to a local basis at 0 consisting of convex, balanced opens.

Proof: The proof is straightforward, once we decide to tolerate an extravagantly large collection of seminorms. With or without local convexity, every neighborhood of 0 contains a *balanced* neighborhood of 0. Thus, a locally convex topological vectorspace has a local basis X at 0 of *balanced convex* open sets.

We claim that every open $U \in X$ can be recovered from the corresponding seminorm ν_U by

$$U \;=\; \{v \in V \;:\; \nu_U(v) < 1\}$$

Indeed, for $v \in U$, the continuity of scalar multiplication gives $\delta > 0$ and a neighborhood N of v such that $z \cdot v - 1 \cdot v \in U$ for $|1 - z| < \delta$. Thus, $v \in (1+\delta)^{-1} \cdot U$, so

$$\nu_U(v) \;=\; \inf\{t \geq 0 \;:\; v \in t \cdot U\} \;\leq\; \frac{1}{1+\delta} \;<\; 1$$

On the other hand, for $\nu_U(v) < 1$, there is $t < 1$ such that $v \in tU \subset U$, since U is convex and contains 0. Thus, the seminorm topology is at least as fine as the original.

Oppositely, the same argument shows that every seminorm local basis open

$$\{v \in V \;:\; \nu_U(v) < t\}$$

is simply tU. Thus, the original topology is at least as fine as the seminorm topology. ///

The comparison of descriptions of topologies is straightforward, as follows. For a seminorm ν on a topological vectorspace V, we can form a Banach space *completing* with respect to the *pseudo-metric* $\nu(x-y)$. In particular, unlike completions with respect to genuine metrics, there can be *collapsing*, so that the natural map of V to this completion need not be an injection.

[11.0.4] Claim: Let V be a topological vectorspace with topology given by a (separating) family of seminorms $S = \{\nu\}$. Order the set of finite subsets of S by inclusion, and

$$\nu_F \;=\; \sum_{\nu \in F} \nu \qquad\qquad \text{(for finite subset } F \text{ of } S)$$

Then V with its seminorm topology is a dense subspace of the limit $\lim_{F\in\Phi} V_F$ of the Banach-space completions V_F with respect to ν_F.

Proof: As earlier, the seminorm topology is literally the subspace topology on the diagonal copy of V in the product of the V_F.

Of course, the poset of finite subsets of S is more complicated than the poset of positive integers, so such a limit can be large. Certainly V has a natural map to every V_F. Indeed, by definition of the seminorm topology, the open sets in V are exactly the inverse images in V of open sets in the various V_F.

For $F \subset F'$, since $\nu_{F'} \geq \nu_F$, there is a natural continuous linear map $V_{F'} \to V_F$. The maps $V \to V_F$ are compatible, in the sense that the composite $V \to V_{F'} \to V_F$ is the same as $V \to V_F$, for $F \subset F'$. This induces a unique continuous linear map of V to the limit of the V_F.

As in [???], the limit is the diagonal

$$D \;=\; \{\{v_F\} \in \prod_F V_F : v_{F'} \to v_F, \text{ for all } F' \supset F\} \;\subset\; \prod_F V_F$$

with the subspace topology. Repeating part of the argument in [???], given a finite collection of finite subsets $F_1, \ldots, F_n$ of S, for $\{v_F\} \in D$, take neighborhoods $U_i \subset V_{F_i}$ containing v_{F_i}. Let $\Phi = \bigcup_i F_i$. The compatibility implies that there is $v_\Phi \in V_\Phi$ such that $v_\Phi \to v_{F_i}$ for all i. Also, there is a sufficiently small neighborhood U of v_Φ such that its image in every V_{F_i} is inside the neighborhood U_i of v_{F_i}. Since the image of V is dense in V_Φ, take $v \in V$ with image inside U. Then the image of v is inside U_i for all i. Thus, the image of V is dense in the limit. ///

Although it turns out that we only care about locally convex topological vectorspaces, there *do exist* complete-metric topological vectorspaces which *fail* to be locally convex. This underscores the need to explicitly specify that a Fréchet space should be locally convex. The usual example of a not-locally-convex complete-metric space is the sequence space

$$\ell^p \;=\; \{x = (x_1, x_2, \ldots) \;:\; \sum_i |x_i|^p < \infty\}$$

for $0 < p < 1$ with metric

$$d(x,y) \;=\; \sum_i |x_i - y_i|^p \qquad\qquad \text{(note: no } p^{th} \text{ root, unlike the } p \geq 1 \text{ case)}$$

This example's interest is mostly as a counterexample to a naive presumption that local convexity is automatic.

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12. *Quasi-completeness theorem*

We have already seen that LF-spaces such as the space of test functions $C_c^\infty(\mathbb{R}) = C_c^\infty(\mathbb{R})$, although *not* complete metrizable, are *quasi-complete*. It is fortunate that most important topological vector spaces are quasi-complete.

At the end of this section, we show that the fullest notion of completeness easily fails to hold, even for quasi-complete spaces.

It is clear that *closed subspaces* of quasi-complete spaces are quasi-complete. Products and finite sums of quasi-complete spaces are quasi-complete.

Let $\mathrm{Hom}(X,Y)$ be the space of continuous linear functions from a topological vectorspace X to another topological vectorspace Y. Give $\mathrm{Hom}(X,Y)$ the topology by seminorms $p_{x,U}$ where $x \in X$ and U is a convex, balanced neighborhood of 0 in Y, defined by

$$p_{x,U}(T) \;=\; \inf\{t > 0 : Tx \in tU\} \qquad\qquad (\text{for } T \in \mathrm{Hom}(X,Y))$$

For $Y = \mathbb{C}$, this gives the weak dual topology on X^*.

[12.0.1] Theorem: For X a Fréchet space or LF-space, and Y quasi-complete, the space $\mathrm{Hom}(X,Y)$, with the topology induced by the seminorms $p_{x,U}$, is *quasi-complete.*

Proof: Some preparation is required. A set E of continuous linear maps from one topological vectorspace X to another topological vectorspace Y is *equicontinuous* when, for every neighborhood U of 0 in Y, there is a neighborhood N of 0 in X so that $T(N) \subset U$ for every $T \in E$.

[12.0.2] Claim: Let V be a strict colimit of a well-ordered countable collection of locally convex closed subspaces V_i. Let Y be a locally convex topological vectorspace. Let E be a set of continuous linear maps from V to Y. Then E is *equicontinuous* if and only if for each index i the collection of continuous linear maps $\{T|_{V_i} : T \in E\}$ is equicontinuous.

Proof: Given a neighborhood U of 0 in Y, shrink U if necessary so that U is convex and balanced. For each index i, let N_i be a convex, balanced neighborhood of 0 in V_i so that $TN_i \subset U$ for all $T \in E$. Let N be the convex hull of the union of the N_i. By the convexity of N, still $TN \subset U$ for all $T \in E$. By the construction of the diamond topology, N is an open neighborhood of 0 in the coproduct, hence in the colimit, which is a quotient of the coproduct. This gives the equicontinuity of E. The other direction of the implication is easy. ///

[12.0.3] Claim: (*Banach-Steinhaus/uniform boundedness theorem*) Let X be a Fréchet space or LF-space and Y a locally convex topological vector space. A set E of linear maps $X \to Y$, such that every set $Ex = \{Tx : T \in E\}$ is *bounded* in Y, is *equicontinuous.*

Proof: First consider X Fréchet. Given a neighborhood U of 0 in Y, let $A = \bigcap_{T\in E} T^{-1}\overline{U}$. By assumption, $\bigcup_n nA = X$. By the Baire category theorem, the complete metric space X is not a countable union of nowhere dense subsets, so at least one of the closed sets nA has non-empty interior. Since (non-zero) scalar multiplication is a homeomorphism, A itself has non-empty interior, containing some $x + N$ for a neighborhood N of 0 and $x \in A$. For every $T \in E$,

$$TN \;\subset\; T\{a - x : a \in A\} \;\subset\; \{u_1 - u_2 \;:\; u_1, u_2 \in \overline{U}\} \;=\; \overline{U} - \overline{U}$$

By continuity of addition and scalar multiplication in Y, given an open neighborhood U_o of 0, there is U such that $\overline{U} - \overline{U} \subset U_o$. Thus, $TN \subset U_o$ for every $T \in E$, and E is equicontinuous.

For $X = \bigcup_i X_i$ an LF-space, this argument already shows that E restricted to each X_i is equicontinuous. As in the previous claim, this gives equicontinuity on the strict colimit. ///

For the proof of the theorem on quasi-completeness, let $E = \{T_i : i \in I\}$ be a bounded Cauchy net in $\mathrm{Hom}(X,Y)$, where I is a directed set. Of course, attempt to define the limit of the net by $Tx = \lim_i T_i x$. For $x \in X$ the evaluation map $S \to Sx$ from $\mathrm{Hom}(X,Y)$ to Y is continuous. In fact, the topology on $\mathrm{Hom}(X,Y)$ is the coarsest with this property. Therefore, by the quasi-completeness of Y, for each fixed $x \in X$ the net $T_i x$ in Y is bounded and Cauchy, so converges to an element of Y suggestively denoted Tx.

To prove *linearity* of T, fix x_1, x_2 in X, $a, b \in \mathbb{C}$ and fix a neighborhood U_o of 0 in Y. Since T is in the closure of E, for any open neighborhood N of 0 in $\mathrm{Hom}(X, Y)$, there exists

$$T_i \;\in\; E \cap (T + N)$$

In particular, for any neighborhood U of 0 in Y, take

$$N = \{S \in \mathrm{Hom}(X, Y) : S(ax_1 + bx_2) \in U,\; S(x_1) \in U,\; S(x_2) \in U\}$$

Then

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2)$$

$$= \; (T(ax_1 + bx_2) - aT(x_1) - bT(x_2)) - (T_i(ax_1 + bx_2) - aT_i(x_1) - bT_i(x_2))$$

since T_i is linear. The latter expression is

$$T(ax_1 + bx_2) - (ax_1 + bx_2) + a(T(x_1) - T_i(x_1) + b(T(x_2) - T_i(x_2)$$

$$\in \; U + aU + bU$$

By choosing U small enough so that

$$U + aU + bU \;\subset\; U_o$$

we find that

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2) \;\in\; U_o$$

Since this is true for every neighborhood U_o of 0 in Y,

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2) \;=\; 0$$

which proves linearity.

Continuity of the limit operator T exactly requires *equicontinuity* of $E = \{T_i x : i \in I\}$. Indeed, for each $x \in X$, $\{T_i x : i \in I\}$ is *bounded* in Y, so by Banach-Steinhaus, $\{T_i : i \in I\}$ is equicontinuous.

Fix a neighborhood U of 0 in Y. Invoking the equicontinuity of E, let N be a small enough neighborhood of 0 in X so that $T(N) \subset U$ for all $T \in E$. Let $x \in N$. Choose an index i sufficiently large so that $Tx - T_i x \in U$, vis the definition of the topology on $\mathrm{Hom}(X, Y)$. Then

$$Tx \in U + T_i x \subset U + U$$

The usual rewriting, replacing U by U' such that $U' + U' \subset U$, shows that T is continuous. ///

Finally, we demonstrate that weak duals of reasonable topological vector spaces, such as infinite-dimensional Hilbert, Banach, or Fréchet spaces, are definitely *not* complete in the strongest sense. That is, in these weak duals there are Cauchy *nets* which *do not converge*.

[12.0.4] Theorem: The weak dual of a locally-convex topological vector space V is *complete* if and only if every linear functional on V is *continuous*.

Proof: A vectorspace V can be (re-) topologized as the colimit V_{init} of all its finite-dimensional subspaces. Although the poset of finite-dimensional subspaces is much larger than the poset of positive integers, the argument in [???] still applies: this colimit really is the ascending union with a suitable topology.

[12.0.5] Claim: For a locally-convex topological vector space V the identity map $V_{\mathrm{init}} \to V$ is *continuous*. That is, V_{init} is the finest locally convex topological vector space topology on V.

Proof: Finite-dimensional topological vector spaces have unique topologies [???]. Thus, for any finite-dimensional vector subspace X of V the inclusion $X \to V$ is continuous with that unique topology on X. These inclusions form a compatible family of maps to V, so by the characterization of colimit there is a *unique* continuous map $V_{\text{init}} \to V$. This map is the identity on every finite-dimensional subspace, so is the identity on the underlying set V. ///

[12.0.6] Claim: Every linear functional $\lambda : V_{\text{init}} \to \mathbb{C}$ is *continuous*.

Proof: The restrictions of a given linear function λ on V to finite-dimensional subspaces are compatible with the inclusions among finite-dimensional subspaces. Every linear functional on a finite-dimensional space is continuous, so the characterizing property of the colimit implies that λ is continuous on V_{init}. ///

[12.0.7] Claim: The weak dual V^* of a locally-convex topological vector space V injects continuously to the limit of the finite-dimensional Banach spaces

$$V^*_\Phi \;=\; \text{completion of } V^* \text{ under seminorm } \; p_\Phi(\lambda) \;=\; \sum_{v\in\Phi} |\lambda(v)| \qquad\qquad (\text{finite } \Phi \subset V)$$

and the weak dual topology is the subspace topology.

Proof: The weak dual topology on the continuous dual V^* of a topological vector space V is given by the seminorms

$$p_v(\lambda) \;=\; |\lambda(v)| \qquad\qquad (\text{for } \lambda \in V^* \text{ and } v \in V)$$

The corresponding local basis is finite intersections

$$\{\lambda \in V^* \;:\; |\lambda(v)| < \varepsilon, \text{ for all } v \in \Phi\} \qquad\qquad (\text{for arbitrary finite sets } \Phi \subset V)$$

These sets contain, and are contained in, sets of the form

$$\{\lambda \in V^* \;:\; \sum_{v\in\Phi} |\lambda(v)| < \varepsilon\} \qquad\qquad (\text{for arbitrary finite sets } \Phi \subset V)$$

Therefore, the weak dual topology on V^* is also given by semi-norms

$$p_\Phi(\lambda) \;=\; \sum_{v\in\Phi} |\lambda(v)| \qquad\qquad (\text{finite } \Phi \subset V)$$

These have the convenient feature that they form a projective family, indexed by (reversed) inclusion. Let $V^*(\Phi)$ be V^* with the p_Φ-topology: this is not Hausdorff, so continuous linear maps $V^* \to V^*(\Phi)$ descend to maps $V^* \to V^*_\Phi$ to the *completion* V^*_Φ of V^* with respect to the pseudo-metric attached to p_Φ. The quotient map $V^*(\Phi) \to V^*_\Phi$ typically has a large kernel, since

$$\dim_{\mathbb{C}} V^*_\Phi \;=\; \text{card}\,\Phi \qquad\qquad (\text{for finite } \Phi \subset V)$$

The maps $V^* \to V^*_\Phi$ are compatible with respect to (reverse) inclusion $\Phi \supset Y$, so V^* has a natural induced map to the $\lim_\Phi V^*_\Phi$. Since V separates points in V^*, V^* *injects* to the limit. The weak topology on V^* is exactly the subspace topology from that limit. ///

[12.0.8] Claim: The weak dual V^*_{init} of V_{init} is the limit of the finite-dimensional Banach spaces

$$V^*_\Phi \;=\; \text{completion of } V^*_{\text{init}} \text{ under seminorm } \; p_\Phi(\lambda) \;=\; \sum_{v\in\Phi} |\lambda(v)| \qquad\qquad (\text{finite } \Phi \subset V)$$

Proof: The previous proposition shows that V^*_{init} *injects* to the limit, and that the subspace topology from the limit is the weak dual topology. On the other hand, the limit consists of linear functionals on V, without

regard to topology or continuity. Since *all* linear functionals are continuous on V_{init}, the limit is naturally a subspace of V^*_{init}. ///

Returning to the proof of the theorem, $\lim_\Phi V^*_\Phi$ is a closed subspace of the corresponding *product*, so is *complete* in the fullest sense. Any other locally convex topologization V_τ of V has weak dual $(V_\tau)^* \subset (V_{\text{init}})^*$ with the subspace topology, and the image is *dense* in $(V_{\text{init}})^*$. Thus, unless $(V_\tau)^* = (V_{\text{init}})^*$, the weak dual V^*_τ *is not complete.* ///

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13. *Strong operator topology*

For X and Y Hilbert spaces, the topology on $\text{Hom}(X,Y)$ given as in [???] by seminorms

$$p_{x,U}(T) \;=\; \inf\{t>0 : Tx \in tU\} \qquad\qquad (\text{for } T \in \text{Hom}(X,Y))$$

where $x \in X$ and U is a convex, balanced neighborhood of 0 in Y, is the *strong operator topology.* Indeed, every neighborhood of 0 in Y contains an open *ball*, so this topology can also be given by seminorms

$$q_x(T) \;=\; |Tx|_Y \qquad\qquad (\text{for } T \in \text{Hom}(X,Y))$$

where $x \in X$. The strong operator topology is weaker than the *uniform* topology given by the operator norm $|T| = \sup_{|x|\le 1} |Tx|_Y$.

The *uniform* operator-norm topology makes the space of operators a Banach space, certainly simpler than the strong operator topology, but the uniform topology is too strong for many purposes.

For example, *group actions* on Hilbert spaces are rarely continuous for the uniform topology: letting $\mathbb{R}$ act on $L^2(\mathbb{R})$ by $T_x f(y) = f(x+y)$, no matter how small $\delta > 0$ is, there is an L^2 function f with $|f|_{L^2} = 1$ such that $|T_\delta f - f|_{L^2} = \sqrt{2}$.

Despite the strong operator topology being less elementary than the uniform topology, the theorem [???] of the previous section shows that $\text{Hom}(X,Y)$ with the strong operator topology is *quasi-complete.*

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14. *Generalized functions (distributions) on* $\mathbb{R}$

The most immediate definition of the space of *distributions* or *generalized functions* on $\mathbb{R}$ is as the dual $C_c^{\infty *} = C_c^\infty(\mathbb{R})^* = C_c^\infty(\mathbb{R})^*$ to the space $C_c^\infty = C_c^\infty(\mathbb{R})$ of test functions, with the *weak dual topology* by seminorms $\nu_f(u) = |u(f)|$ for test functions f and distributions u.

Similarly, the *tempered* distributions are the weak dual $\mathscr{S}^* = \mathscr{S}(\mathbb{R})^*$, and the *compactly-supported* distributions are the weak dual $C^{\infty *} = C^\infty(\mathbb{R})^*$, in this context writing $C^\infty(\mathbb{R}) = C^\infty(\mathbb{R})$. Naming $C^{\infty *}$ *compactly-supported* will be justified below.

By dualizing, the continuous containments $C_c^\infty \subset \mathscr{S} \subset C^\infty$ give continuous maps $C^{\infty *} \to \mathscr{S}^* \to C_c^\infty$. When we know that C_c^∞ is *dense* in $\mathscr{S}$ and in C^∞, it will follow that these are *injections.* The most straightforward argument for density uses Gelfand-Pettis integrals, as in [???]. Thus, for the moment, we cannot claim that $C^{\infty *}$ and $\mathscr{S}^*$ *are* distributions, but only that they naturally *map to* distributions.

[???] shows that $C_c^{\infty *}$, $\mathscr{S}^*$, and $C^{\infty *}$ are *quasi-complete*, despite not being complete in the strongest possible sense [???].

The description of the space of distributions as the weak dual to the space of test functions falls far short of explaining its utility! There is a natural imbedding $C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})^*$ of test functions into distributions, by

$$f \to u_f \qquad \text{by} \qquad u_f(g) = \int_{\mathbb{R}} f(x)\, g(x)\, dx \qquad\qquad (\text{for } f,g \in C_c^\infty(\mathbb{R}))$$

That is, via this imbedding we consider distributions to be *generalized functions*. Indeed, [???] shows that test functions $C_c^\infty(\mathbb{R})$ are *dense* in $C_c^\infty(\mathbb{R})^*$.

The simplest example of a distribution not obtained by integration against a test function on $\mathbb{R}$ is the *Dirac delta*, the evaluation map $\delta(f) = f(0)$. By [???], this is *continuous* for the LF-space topology on test functions.

This imbedding, and integration by parts, explain how to define $\frac{d}{dx}$ on distributions in a form compatible with the imbedding $C_c^\infty \subset C_c^{\infty *}$: noting the sign, due to integration by parts,

$$\Big(\frac{d}{dx}u\Big)(f) \;=\; -\,u\Big(\frac{d}{dx}f\Big) \qquad\qquad (\text{for } u \in C_c^{\infty *} \text{ and } f \in C_c^\infty)$$

[14.0.1] Claim: $\frac{d}{dx} : C_c^{\infty *} \to C_c^{\infty *}$ is continuous.

Proof: By the nature of the weak dual topology, it suffices to show that for each $f \in C_c^\infty$ and $\varepsilon > 0$ there are $g \in C_c^\infty$ and $\delta > 0$ such that $|u(g)| < \delta$ implies $|(\frac{d}{dx}u)(f)| < \varepsilon$. Taking $g = \frac{d}{dx}f$ and $\delta = \varepsilon$ succeeds. ///

From [???], multiplications by $F \in C^\infty(\mathbb{R})$ give continuous maps C_c^∞ to itself. These multiplications are compatible with the imbedding $C_c^\infty \to C_c^{\infty *}$ in the sense that

$$\int_{\mathbb{R}} (F \cdot u)(x)\; f(x)\; dx \;=\; \int_{\mathbb{R}} u(x)\; (F \cdot f)(x)\; dx \qquad\qquad (\text{for } F \in C^\infty(\mathbb{R}) \text{ and } u, f \in C_c^\infty(\mathbb{R}))$$

Extend this to a map $C_c^{\infty *} \to C_c^{\infty *}$ by

$$(F \cdot u)(f) \;=\; u(F \cdot f) \qquad\qquad (\text{for } F \in C^\infty,\; u \in C_c^{\infty *}, \text{ and } f \in C_c^\infty)$$

[14.0.2] Claim: Multiplication operators $C_c^{\infty *} \to C_c^{\infty *}$ by $F \in C^\infty$ are continuous.

Proof: By the nature of the weak dual topology, it suffices to show that for each $f \in C_c^\infty$ and $\varepsilon > 0$ there are $g \in C_c^\infty$ and $\delta > 0$ such that $|u(g)| < \delta$ implies $|F \cdot u)(f)| < \varepsilon$. Taking $g = F \cdot f$ and $\delta = \varepsilon$ succeeds. ///

Since $\mathscr{S}$ is mapped to itself by Fourier transform [???], this gives a way to define Fourier transform on $\mathscr{S}^*$, as in [???].

Recall that the support of a *function* is the *closure* of the set on which it is non-zero, slightly complicating the notion of support for a *distribution* u: support of u is the *complement* of the *union* of all open sets U such that $u(f) = 0$ for all test functions f with support inside U.

[14.0.3] Theorem: A distribution with *support* $\{0\}$ is a finite linear combination of Dirac's δ and its derivatives.

Proof: Since C_c^∞ is a colimit of ${C_c^\infty}_K$ over $K = [-n, n]$, it suffices to classify u in ${C_c^{\infty *}}_K$ with support $\{0\}$. We claim that a continuous linear functional on ${C_c^\infty}_K = \lim_k C_K^k$ factors through some limitand

$$C_K^k \;=\; \{f \in C^k(K) :\; f^{(i)} \text{ vanishes on } \partial K \text{ for } 0 \le i \le k\}$$

This is a special case of

[14.0.4] Claim: Let $X = \lim_n B_n$ be a limit of Banach spaces, with the image of X *dense* in each B_n. A continuous linear map $T : \lim_n B_n \to Z$ from a, to a *normed* space Z factors through some limitand B_n. For $Z = \mathbb{C}$, the same conclusion holds without the density assumption.

Proof: Let $X = \lim_i B_i$ with projections $p_i : X \to B_i$. Each B_i is the closure of the image of X. By the continuity of T at 0, there is an open neighborhood U of 0 in X such that TU is inside the open unit ball at 0 in Z. By the description of the limit topology as the product topology restricted to the diagonal, there are finitely-many indices $i_1, \ldots, i_n$ and open neighborhoods V_{i_t} of 0 in B_{i_t} such that

$$\bigcap_{t=1}^{n} p_{i_t}^{-1}\big(p_{i_t} X \cap V_{i_t}\big) \;\subset\; U$$

We can make a *smaller* open in X by a condition involving a single limitand, as follows. Let j be *any* index with $j \geq i_t$ for all t, and

$$N \;=\; \bigcap_{t=1}^{n} p_{i_t,j}^{-1}\big(p_{i_t,j} B_j \cap V_{i_t}\big) \;\subset\; B_j$$

By the compatibility $p_{i_t}^{-1} = p_j^{-1} \circ p_{i_t,j}^{-1}$, we have $p_{i_t,j} N \subset V_{i_t}$ for $i_1, \ldots, i_n$, and $p_j^{-1}(p_j X \cap N) \subset U$. By the linearity of T, for any $\varepsilon > 0$,

$$T(\varepsilon \cdot p_j^{-1}\big(p_j X \cap N\big)) \;=\; \varepsilon \cdot T(p_j^{-1}\big(p_j X \cap N\big)) \;\subset\; \varepsilon\text{-ball in } Z$$

We claim that T factors through $p_j X$ with the subspace topology from B_j. One potential issue in general is that $p_j : X \to B_j$ can have a non-trivial kernel, and we must check that $\ker p_j \subset \ker T$. By the linearity of T,

$$T(\frac{1}{n} \cdot p_j^{-1}(p_j \cap N)) \;\subset\; \frac{1}{n}\text{-ball in } Z$$

so

$$T\left(\bigcap_n \frac{1}{n} \cdot p_j^{-1}(p_j X \cap N)\right) \;\subset\; \frac{1}{m}\text{-ball in } Z \qquad\qquad (\text{for all } m)$$

and then

$$T\left(\bigcap_n \frac{1}{n} \cdot p_j^{-1}(p_j \cap N)\right) \;\subset\; \bigcap_m \frac{1}{m}\text{-ball in } Z \;=\; \{0\}$$

Thus,

$$\bigcap_n p_j^{-1}(p_j X \cap \frac{1}{n} \cdot N) \;=\; \bigcap_n \frac{1}{n} \cdot p^{-1}(p_j X \cap N) \;\subset\; \ker T$$

Thus, for $x \in X$ with $p_j x = 0$, certainly $p_j x \in \frac{1}{n} N$ for all $n = 1, 2, \ldots$, and

$$x \;\in\; \bigcap_n p_j^{-1}(p_j X \cap \frac{1}{n} N) \;\subset\; \ker T$$

This proves the subordinate claim that T factors through $p_j : X \to B_j$ via a (not necessarily continuous) linear map $T' : p_j X \to Z$. The continuity follows from continuity at 0, which is

$$T(\varepsilon \cdot p_j^{-1}\big(p_j X \cap N\big)) \;=\; \varepsilon \cdot T(p_j^{-1}\big(p_j X \cap N\big)) \;\subset\; \varepsilon\text{-ball in } Z$$

Then $T' : p_j X \to Z$ extends to a map $B_j \to Z$ by continuity: given $\varepsilon > 0$, take symmetric convex neighborhood U of 0 in B_j such that $|T'y|_Z < \varepsilon$ for $y \in p_j X \cap U$. Let y_i be a Cauchy net in $p_j X$ approaching $b \in B_j$. For y_i and y_j inside $b + \frac{1}{2}U$, $|T'y_i - T'y_j| = |T'(y_i - y_j)| < \varepsilon$, since $y_i - y_j \in \frac{1}{2} \cdot 2U = U$. Then unambiguously define $T'b$ to be the Z-limit of the $T'y_i$. The closure of $p_j X$ in B_j is B_j, giving the desired map.

When u is a *functional*, that is, a map to $\mathbb{C}$, we can extend it by Hahn-Banach. ///

Returning to the proof of the theorem: thus, there is $k \geq 0$ such that u factors through a limitand C_K^k. In particular, u is continuous for the C^k topology on ${C_c^\infty}_K$.

We need an auxiliary gadget. Fix a test function ψ identically 1 on a neighborhood of 0, bounded between 0 and 1, and (necessarily) identically 0 outside some (larger) neighborhood of 0. For $\varepsilon > 0$ let

$$\psi_\varepsilon(x) \;=\; \psi(\varepsilon^{-1}x)$$

Since the support of u is just $\{0\}$, for all $\varepsilon > 0$ and for all $f \in C_c^\infty(\mathbb{R}^n)$ the support of $f - \psi_\varepsilon \cdot f$ does not include 0, so

$$u(\psi_\varepsilon \cdot f) \;=\; u(f)$$

Thus, for implied constant depending on k and K, but not on f,

$$|\psi_\varepsilon f|_k \;=\; \sup_{x\in K} \sum_{0\le i\le k} |(\psi_\varepsilon f)^{(i)}(x)| \;\ll\; \sum_{i\le k}\sum_{0\le j\le i} \sup_x \varepsilon^{-j} \left|\psi^{(j)}(\varepsilon^{-1}x)\, f^{(i-j)}(x)\right|$$

For test function f vanishing to order k at 0, that is, $f^{(i)}(0) = 0$ for all $0 \le i \le k$, on a fixed neighborhood of 0, by a Taylor-Maclaurin expansion, $|f(x)| \ll |x|^{k+1}$, and, generally, for i^{th} derivatives with $0 \le i \le k$, $|f^{(i)}(x)| \ll |x|^{k+1-i}$. By design, all derivatives $\psi', \psi'', \ldots$ are identically 0 in a neighborhood of 0, so, for suitable implied constants independent of ε,

$$|\psi_\varepsilon f|_k \;\ll\; \sum_{0\le i\le k}\sum_{0\le j\le i} \varepsilon^{-j} \cdot \left|\psi^{(j)}(\varepsilon^{-1}x)\, f^{(i-j)}(x)\right| \;\ll\; \sum_{0\le i\le k}\sum_{j=0} \varepsilon^{-j}\cdot 1 \cdot \varepsilon^{k+1-i}$$

$$=\; \sum_{0\le i\le k} \varepsilon^{k+1-i} \;\ll\; \varepsilon^{k+1-k} \;=\; \varepsilon$$

Thus, for sufficiently small $\varepsilon > 0$, for smooth f vanishing to order k at 0, $|u(f)| = |u(\psi_\varepsilon f)| \ll \varepsilon$, and $u(f) = 0$. That is,

$$\ker u \;\supset\; \bigcap_{0\le i\le k} \ker \delta^{(i)}$$

The conclusion, that u is a linear combination of the distributions $\delta, \delta', \delta^{(2)}, \ldots, \delta^{(k)}$, follows from

[14.0.5] Claim: A linear functional $\lambda \in V^*$ vanishing on the intersection $\bigcap_i \ker \lambda_i$ of kernels of a finite collection $\lambda_1, \ldots, \lambda_n \in V^*$ is a *linear combination* of the λ_i.

Proof: The linear map

$$q : V \;\longrightarrow\; \mathbb{C}^n \quad \text{by} \quad v \;\longrightarrow\; (\lambda_1 v,\, \ldots,\, \lambda_n v)$$

is *continuous* since each λ_i is continuous, and λ factors through q, as $\lambda = L \circ q$ for some linear functional L on $\mathbb{C}^n$. We know all the linear functionals on $\mathbb{C}^n$, namely, L is of the form

$$L(z_1, \ldots, z_n) \;=\; c_1 z_1 + \ldots + c_n z_n \qquad\qquad (\text{for some } c_i \in \mathbb{C})$$

Thus,

$$\lambda(v) \;=\; (L \circ q)(v) \;=\; L(\lambda_1 v,\, \ldots,\, \lambda_n v) \;=\; c_1\lambda_1(v) + \ldots + c_n\lambda_n(v)$$

expressing λ as a linear combination of the λ_i. ///

The *order* of a distribution $u : C_c^\infty \to \mathbb{C}$ is the integer k, if such exists, such that u is continuous when C_c^∞ is given the weaker topology from $\mathrm{colim}_K C_K^k$. Not every distribution has finite order, but the claim [???] has a useful technical application:

[14.0.6] Corollary: A distribution $u \in C_c^{\infty *}$ with *compact support* has *finite order*.

Proof: Let ψ be a test function that is identically 1 on an open containing the support of u. Then

$$u(f) \;=\; u((1-\psi)\cdot f) + u(\psi \cdot f) \;=\; 0 + u(\psi \cdot f)$$

since $(1-\psi)\cdot f$ is a test function with support not meeting the support of u. With $K = \operatorname{spt}\psi$, this suggests that u factors through a subspace of ${C_c^\infty}_K$ via $f \to \psi\cdot f \to u(\psi\cdot f)$, but there is the issue of continuity. Distinguishing things a little more carefully, the compatibility embodied in the commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & {C_c^\infty}_K \arrow[r] \arrow[dr, "u_K"'] & \dots \arrow[r, bend left, "i"] & C_c^\infty \arrow[dl, "u"] \\
& & \mathbb{C} &
\end{tikzcd}$$

gives

$$u(f) \;=\; u(\psi\cdot f) \;=\; u\big(i(\psi f)\big) \;=\; u_K(\psi f)$$

The map u_K is continuous, as is the multiplication $f \to \psi f$. The map u_K is from the limit ${C_c^\infty}_K$ of Banach spaces C_K^k to the normed space $\mathbb{C}$, so factors through some limitand C_K^k, by [???]. As in the proofs that multiplication is continuous in the C^∞ topology, by Leibniz' rule, the C^k norm of ψf is

$$|\psi f|_k \;=\; \sum_{0\le i\le k} \sup_{x\in K} |(\psi f)^{(i)}(x)| \;\ll\; \sum_{i\le k}\sum_{0\le j\le i} \sup_x \Big|\psi^{(j)}(x)\, f^{(i-j)}(x)\Big|$$

$$\ll \sum_{0\le i\le k} \sup_{x\in K} |f^{(i)}(x)| \cdot \sum_{j\le k} \sup_x |\psi^{(j)}(x)| \;=\; |f|_{C^k}\cdot|\psi|_{C^k}$$

Since ψ is fixed, this gives continuity in f in the C^k topology. ///

[14.0.7] Claim: In the inclusion $C^{\infty *} \subset \mathscr{S}^* \subset C_c^{\infty *}$, the image of $C^{\infty *}$ really is the collection of distributions with compact support.

Proof: On one hand the previous shows that $u \in C_c^{\infty *}$ with compact support can be composed as $u(f) = u_K(\psi f)$ for suitable $\psi \in C_c^\infty$. The map $f \to \psi\cdot f$ is also continuous as a map $C^\infty \to C_c^\infty$, so the same expression $f \to \psi f \to u_K(\psi f)$ *extends* $u \in C_c^{\infty *}$ to a continuous linear functional on C^∞.

On the other hand, let $u \in C^{\infty *}$. Composition of u with $C_c^\infty \to C^\infty$ gives an element of $C_c^{\infty *}$, which we must check has compact support. By [???], C^∞ is a limit of the Banach spaces $C^k(K)$ with K running over compacts $[-n,n]$, *without* claiming that the image of C^∞ is necessarily dense in any of these. By [???], u factors through some limitand $C^k(K)$. The map $C_c^\infty \to C^\infty$ is compatible with the *restriction* maps $\mathrm{Res}_K : C_c^\infty \to C^k(K)$: the diagram

$$\begin{tikzcd}
C^\infty \arrow[rr, bend left] & \dots \arrow[r] & C^k(K) \arrow[r] \arrow[dr, "u"] & \dots \\
& C_c^\infty \arrow[ul] \arrow[ur, "\mathrm{Res}_K"'] & & \mathbb{C}
\end{tikzcd}$$

commutes. For $f \in C_c^\infty$ with support disjoint from K, $\mathrm{Res}_K(f) = 0$, and $u(f) = 0$. This proves that the support of the (induced) distribution is contained in K, so is compact. ///

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15. *Tempered distributions and Fourier transforms on* $\mathbb{R}$

One normalization of the *Fourier transform* integral is

$$\widehat{f}(\xi) \;=\; \mathbb{F}f(\xi) \;=\; \int_{\mathbb{R}} \overline{\psi}_\xi(x)\, f(x)\, dx \qquad\qquad (\text{with } \psi_\xi(x) = e^{2\pi i\xi x})$$

converges nicely for f in the space $\mathscr{S}(\mathbb{R})$ of Schwartz functions.

[15.0.1] Theorem: Fourier transform is a topological isomorphism of $\mathscr{S}(\mathbb{R})$ to itself, with Fourier inversion map $\varphi \to \check{\varphi}$ given by

$$\check{\varphi}(x) \;=\; \int_{\mathbb{R}} \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi$$

Proof: Using the idea [14.3] that Schwartz functions extend to smooth functions on a suitable one-point compactification of $\mathbb{R}$ vanishing to infinite order at the point at infinity, Gelfand-Pettis integrals justify moving a differentiation under the integral,

$$\frac{d}{d\xi}\widehat{f}(\xi) \;=\; \frac{d}{d\xi}\int_{\mathbb{R}} \overline{\psi}_\xi(x)\, f(x)\, dx \;=\; \int_{\mathbb{R}} \frac{\partial}{\partial\xi}\overline{\psi}_\xi(x)\, f(x)\, dx$$

$$=\; \int_{\mathbb{R}} (-2\pi i x)\, \overline{\psi}_\xi(x)\, f(x)\, dx \;=\; (-2\pi i)\int_{\mathbb{R}} \overline{\psi}_\xi(x)\, x f(x)\, dx \;=\; (-2\pi i)\widehat{xf}(\xi)$$

Similarly, with an integration by parts,

$$-2\pi i\xi \cdot \widehat{f}(\xi) \;=\; \int_{\mathbb{R}} \frac{\partial}{\partial x}\overline{\psi}_\xi(x) \cdot f(x)\, dx \;=\; -\mathbb{F}\frac{df}{dx}(\xi)$$

Thus, $\mathbb{F}$ maps $\mathscr{S}(\mathbb{R})$ to itself.

The natural idea to prove Fourier inversion for $\mathscr{S}(\mathbb{R})$, that unfortunately begs the question, is the obvious:

$$\int_{\mathbb{R}} \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi \;=\; \int_{\mathbb{R}} \psi_\xi(x)\Big(\int_{\mathbb{R}} \overline{\psi}_\xi(t)\, f(t)\, dt\Big)\, d\xi \;=\; \int_{\mathbb{R}} f(t)\Big(\int_{\mathbb{R}} \psi_\xi(x-t)\, dt\Big)\, dt$$

If we could *justify* asserting that the inner integral is $\delta_x(t)$, which it *is*, then Fourier inversion follows. However, Fourier inversion for $\mathscr{S}(\mathbb{R})$ is used to make sense of that inner integral in the first place.

Despite that issue, a dummy *convergence factor* will legitimize the idea. For example, let $g(x) = e^{-\pi x^2}$ be the usual Gaussian. Various computations show that it is its own Fourier transform. For $\varepsilon > 0$, as $\varepsilon \to 0^+$, the dilated Gaussian $g_\varepsilon(x) = g(\varepsilon \cdot x)$ approaches 1 uniformly on compacts. Thus,

$$\int_{\mathbb{R}} \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi \;=\; \int_{\mathbb{R}} \lim_{\varepsilon\to 0^+} g(\varepsilon\xi)\, \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi \;=\; \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} g(\varepsilon\xi)\, \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi$$

by *monotone convergence* or more elementary reasons. Then the iterated integral is legitimately rearranged:

$$\int_{\mathbb{R}} g(\varepsilon\xi)\, \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} g(\varepsilon\xi)\, \psi_\xi(x)\, \overline{\psi}_\xi(t)\, f(t)\, dt\, d\xi \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} g(\varepsilon\xi)\, \psi_\xi(x-t)\, f(t)\, d\xi\, dt$$

By changing variables in the definition of Fourier transform, $\widehat{g_\varepsilon} = \frac{1}{\varepsilon} g_{1/\varepsilon}$. Thus,

$$\int_{\mathbb{R}} \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi \;=\; \int_{\mathbb{R}} \frac{1}{\varepsilon}\, g\big(\frac{x-t}{\varepsilon}\big)\, f(t)\, dt \;=\; \int_{\mathbb{R}} \frac{1}{\varepsilon}\, g\big(\frac{t}{\varepsilon}\big) \cdot f(x+t)\, dt$$

The sequence of function $g_{1/\varepsilon}/\varepsilon$ is not an *approximate identity* in the strictest sense, since the supports are the entire line. Nevertheless, the integral of each is 1, and as $\varepsilon \to 0^+$, the mass is concentrated on smaller and smaller neighborhoods of $0 \in \mathbb{R}$. Thus, for $f \in \mathscr{S}(\mathbb{R})$,

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} \frac{1}{\varepsilon}\, g\big(\frac{t}{\varepsilon}\big) \cdot f(x+t)\, dt \;=\; f(x)$$

This proves Fourier inversion. In particular, this proves that Fourier transform *bijects* the Schwartz space to itself. ///

With Fourier inversion in hand, we can prove the Plancherel identity for Schwartz functions:

[15.0.2] Corollary: For $f, g \in \mathscr{S}$, the Fourier transform is an isometry in the $L^2(\mathbb{R})$ topology, that is, $\langle \widehat{f}, \widehat{g} \rangle = \langle f, g \rangle$.

Proof: There is an immediate preliminary identity:

$$\int_{\mathbb{R}} \widehat{f}(\xi)\, h(\xi)\, d\xi \;=\; \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\pi i \xi x}\, f(x)\, h(\xi)\, d\xi\, dx \;=\; \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\pi i \xi x}\, f(x)\, h(\xi)\, dx\, d\xi \;=\; \int_{\mathbb{R}} f(x)\, \widehat{h}(x)\, dx$$

To get from this identity to Plancherel requires, given $g \in \mathscr{S}$, existence of $h \in \mathscr{S}$ such that $\widehat{h} = \overline{g}$, with complex conjugation. By Fourier inversion on Schwartz functions, $h = (\overline{g})\check{}$ succeeds. ///

[15.0.3] Corollary: Fourier transform extends by continuity to an isometry $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

Proof: Schwartz functions are dense in in $L^2(\mathbb{R})$. ///

[15.0.4] Corollary: Fourier transform extends to give a bijection of the space tempered distributions $\mathscr{S}^*$ to itself, by

$$\widehat{u}(\varphi) \;=\; u(\widehat{\varphi}) \qquad\qquad (\text{for all } \varphi \in \mathscr{S})$$

Proof: Fourier transform is a topological isomorphism of $\mathscr{S}$ to itself. ///

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16. *Test functions and Paley-Wiener spaces*

Of course, the original [Paley-Wiener 1934] referred to L^2 functions, not distributions. The distributional aspect is from [Schwartz 1952]. An interesting point is that rate-of-growth of the Fourier transforms in the imaginary part determines the support of the inverse Fourier transforms.

The class PW of entire functions appearing in the following theorem is the *Paley-Wiener space* in one complex variable. The assertion is that, in contrast to the fact that Fourier transform maps the Schwartz space to itself, on test functions the Fourier transform has less symmetrical behavior, bijecting to the Paley-Wiener space.

[16.0.1] Theorem: A test function f supported on $[-r, r] \subset \mathbb{R}$ has Fourier transform $\widehat{f}$ extending to an entire function on $\mathbb{C}$, with

$$|\widehat{f}(z)| \;\ll_N\; (1 + |z|)^{-N}\, e^{r \cdot |y|} \qquad\qquad (\text{for } z = x + iy \in \mathbb{C}, \text{ for every } N)$$

Conversely, an entire function satisfying such an estimate has (inverse) Fourier transform which is a test function supported in $[-r, r]$.

Proof: First, the integral for $\widehat{f}(z)$ is the integral of the compactly-supported, continuous, entire-function-valued function,

$$\xi \;\longrightarrow\; \Big(z \to f(\xi) \cdot e^{-i\xi z} \Big)$$

where the space of entire functions is given the sups-on-compacts semi-norms $\sup_{z \in K} |f(z)|$. Since $\mathbb{C}$ can be covered by countably-many compacts, this topology is metrizable. Cauchy's integral formula proves

completeness, so this space is Fréchet. Thus, the Gelfand-Pettis integral exists, and is entire. Multiplication by z is converted to differentiation inside the integral,

$$(-iz)^N \cdot \widehat{f}(z) \;=\; \int_{|\xi|\le r} \frac{\partial^N}{\partial \xi^N} e^{-iz\cdot\xi} \cdot f(\xi)\, d\xi \;=\; (-1)^N \int_{|\xi|\le r} e^{-iz\cdot\xi} \cdot \frac{\partial^N}{\partial \xi^N} f(\xi)\, d\xi$$

by integration by parts. Differentiation does not enlarge support, so

$$|\widehat{f}(z)| \;\ll_N\; (1+|z|)^{-N} \cdot \Big| \int_{|\xi|\le r} e^{-iz\cdot\xi}\, f^{(N)}(\xi)\, d\xi \Big| \;\le\; (1+|z|)^{-N} \cdot e^{r\cdot|y|} \cdot \Big| \int_{|\xi|\le r} e^{-ix\cdot\xi}\, f^{(N)}(\xi)\, d\xi \Big|$$

$$\le\; (1+|z|)^{-N} \cdot e^{r\cdot|y|} \cdot \int_{|\xi|\le r} |f^{(N)}(\xi)|\, d\xi \;\ll_{f,N}\; (1+|z|)^{-N} \cdot e^{r\cdot|y|}$$

Conversely, for an entire function F with the indicated growth and decay property, we show that

$$\varphi(\xi) \;=\; \int_{\mathbb{R}} e^{ix\xi}\, F(x)\, dx$$

is a test function with support inside $[-r,r]$. The assumptions on F do *not* directly include any assertion that F is Schwartz, so we cannot directly conclude that φ is smooth. Nevertheless, a similar obvious computation would give

$$\int_{\mathbb{R}} (ix)^N \cdot e^{ix\xi}\, F(x)\, dx \;=\; \int_{\mathbb{R}} \frac{\partial^N}{\partial \xi^N} e^{ix\xi}\, F(x)\, dx \;=\; \frac{\partial^N}{\partial \xi^N} \int_{\mathbb{R}} e^{ix\xi}\, F(x)\, dx$$

Moving the differentiation outside the integral is *necessary*, justified via Gelfand-Pettis integrals by a compactification device, as in [14.3], as follows. Since F strongly vanishes at ∞, the integrand extends continuously to the stereographic-projection one-point compactification of $\mathbb{R}$, giving a compactly-supported smooth-function-valued function on this compactification. The measure on the compactification can be adjusted to be finite, taking advantage of the rapid decay of F:

$$\varphi(\xi) \;=\; \int_{\mathbb{R}} e^{ix\xi}\, F(x)\, dx \;=\; \int_{\mathbb{R}} e^{ix\xi}\, F(x)\, (1+x^2)^N\, \frac{dx}{(1+x^2)^N}$$

Thus, the Gelfand-Pettis integral exists, and φ is smooth. Thus, in fact, the justification proves that such an integral of smooth functions is smooth without necessarily producing a formula for derivatives.

To see that φ is supported inside $[-r,r]$, observe that, taking y of the same sign as ξ,

$$\Big| F(x+iy) \cdot e^{i\xi(x+iy)} \Big| \;\ll_N\; (1+|z|)^{-N} \cdot e^{(r-|\xi|)\cdot|y|}$$

Thus,

$$|\varphi(\xi)| \;\ll_N\; \int_{\mathbb{R}} (1+|z|)^{-N} \cdot e^{(r-|\xi|)\cdot|y|}\, dx \;\le\; e^{(r-|\xi|)\cdot|y|} \cdot \int_{\mathbb{R}} \frac{dx}{(1+|x|)^{-N}}$$

For $|\xi| > r$, letting $|y| \to +\infty$ shows that $\varphi(\xi) = 0$. ///

[16.0.2] Corollary: We can topologize PW by requiring that the linear bijection $C_c^\infty \to PW$ be a topological vector space isomorphism. ///

[16.0.3] Remark: The latter topology on PW is finer than the sups-on-compacts topology on all entire functions, since the latter cannot detect growth properties.

Let $\check{\varphi}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t}\, \varphi(\xi)\, d\xi$ be the inverse Fourier transform, mapping $PW \to C_c^\infty$.

[16.0.4] Corollary: Fourier transform can be defined on *all* distributions $u \in C_c^{\infty *}$ by $\widehat{u}(\varphi) = u(\check{\varphi})$ for $\varphi \in PW$, giving an isomorphism $C_c^{\infty *} \to PW^*$ to the dual of the Paley-Wiener space. ///

For example, the exponential $t \to e^{iz\cdot t}$ with $z \in \mathbb{C}$ but $z \notin \mathbb{R}$ is *not* a tempered distribution, but is a distributions, and its Fourier transform is the Dirac delta $\delta_z \in PW'$.

Compactly-supported distributions have a similar characterization:

[16.0.5] Theorem: The Fourier transform $\widehat{u}$ of a distribution u supported in $[-r, r]$, of order N, is (integration against) the function $x \to u(\xi \to e^{-ix\xi})$, which is *smooth*, and extends to an *entire* function satisfying

$$|\widehat{u}(z)| \ll (1+|z|)^N \cdot e^{r\cdot|y|}$$

Conversely, an entire function meeting such a bound is the Fourier transform of a distribution of order N supported inside $[-r, r]$.

Proof: The Fourier transform $\widehat{u}$ is the tempered distribution defined for Schwartz functions φ by

$$\widehat{u}(\varphi) \;=\; u(\widehat{\varphi}) \;=\; u\Big(\xi \to \int_{\mathbb{R}} e^{-ix\xi}\,\varphi(x)\,dx\Big) \;=\; \int_{\mathbb{R}} u(\xi \to e^{-ix\xi})\,\varphi(x)\,dx$$

since $x \to (\xi \to e^{-ix\xi}\varphi(\xi)$ extends to a continuous smooth-function-valued function on the one-point compactification of $\mathbb{R}$, and Gelfand-Pettis applies. Thus, as expected, $\widehat{u}$ is integration against $x \to u(\xi \to e^{-ix\xi})$.

The smooth-function-valued function $z \to (\xi \to e^{-iz\xi})$ is holomorphic in z. Compactly-supported distributions constitute the dual of $C^\infty(\mathbb{R})$. Application of u gives a holomorphic *scalar*-valued function $z \to u(\xi \to e^{-iz\xi})$.

Let ν_N be the Nth-derivative seminorm on $C^\infty[-r, r]$, so

$$|u(\varphi)| \ll_\varepsilon \nu_N(\varphi)$$

Then

$$|\widehat{u}(z)| \;=\; |u(\xi \to e^{-iz\xi})| \ll_\varepsilon \nu_N(\xi \to e^{-iz\xi}) \ll \sup_{[-r,r]} \Big|(1+|z|)^N\, e^{-iz\xi}\Big| \;\le\; (1+|z|)^N e^{r\cdot|y|}$$

Conversely, let F be an entire function with $|F(z)| \ll (1+|z|)^N e^{r\cdot|y|}$. Certainly F is a tempered distribution, so $F = \widehat{u}$ for a tempered distribution. We show that u is of order at most N and has support in $[-r, r]$.

With η supported on $[-1, 1]$ with $\eta \ge 0$ and $\int \eta = 1$, make an *approximate identity* $\eta_\varepsilon(x) = \eta(x/\varepsilon)/\varepsilon$ for $\varepsilon \to 0^+$. By the easy half of Paley-Wiener for test functions, $\widehat{\eta}_\varepsilon$ is entire and satisfies

$$|\widehat{\eta}_\varepsilon(z)| \ll_{\varepsilon,N} (1+|z|)^{-N} \cdot e^{\varepsilon\cdot|y|} \qquad \text{(for all } N\text{)}$$

Note that $\widehat{\eta}_\varepsilon(x) = \widehat{\eta}(\varepsilon \cdot x)$ goes to 1 as tempered distribution

By the more difficult half of Paley-Wiener for test functions, $F \cdot \widehat{\eta}_\varepsilon$ is $\widehat{\varphi}_\varepsilon$ for some test function φ_ε supported in $[-(r+\varepsilon), r+\varepsilon]$. Note that $F \cdot \widehat{\eta}_\varepsilon \to F$.

For Schwartz function g with the support of $\widehat{g}$ not meeting $[-r, r]$, $\widehat{g} \cdot \varphi_\varepsilon$ for sufficiently small $\varepsilon > 0$. Since $F \cdot \widehat{\eta}_\varepsilon$ is a Cauchy net as tempered distributions,

$$u(\widehat{g}) \;=\; \widehat{u}(g) \;=\; \int F \cdot g \;=\; \int \lim_\varepsilon (F \cdot \widehat{\eta}_\varepsilon)\; g \;=\; \lim_\varepsilon \int (F \cdot \widehat{\eta}_\varepsilon)\; g \;=\; \lim_\varepsilon \int \widehat{\varphi}_\varepsilon\; g \;=\; \lim_\varepsilon \int \varphi_\varepsilon\; \widehat{g} \;=\; 0$$

This shows that the support of u is inside $[-r, r]$. ///