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Harish-Chandra's homomorphism, Verma modules

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Representations of semi-simple Lie algebras for the impatient.

The Harish-Chandra homomorphism is due to [Harish-Chandra 1951]. Attention to universal modules with highest weights is in]Harish-Chandra 1951], [Cartier 1955], as well as [Verma 1968], [Bernstein-Gelfand-Gelfand 1971a], [Bernstein-Gelfand-Gelfand 1971b], [Bernstein-Gelfand-Gelfand 1975]. See also [Jantzen 1979].

[1] We treat $\mathfrak{sl}(2)$ in as simple a style as possible, to highlight ideas. Then $\mathfrak{sl}(3)$ to illustrate that certain technical complications are harmless. One should be aware that some properties hold for $\mathfrak{sl}(3)$ that become more complicated or fail completely for $\mathfrak{sl}(4)$ and larger algebras. See the *Supplementary Remarks* to chapter 7 of [Dixmier 1977].

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1. *Highest weights for* $\mathfrak{sl}(2)$

In the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ of two-by-two real matrices with trace 0, let, as usual,

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

In this Lie algebra, the Lie bracket is

$$[a,b] = ab - ba$$

We have

$$[H,X] = 2X \quad [H,Y] = -2Y \quad [X,Y] = H$$

Sometimes X is called a **raising operator** and Y a **lowering operator**. These are also called **creation** and **annihilation** operators.

A **representation** or $\mathfrak{g}$**-module** (π, V) is a complex vectorspace V and π is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$$

where $\mathrm{End}_{\mathbb{C}}(V)$ has the Lie bracket

$$[A,B] = AB - BA$$

That is, π is $\mathbb{R}$-linear, and

$$[\pi(a), \pi(b)] = \pi([a,b])$$

[1] The terminology *Verma modules* was apparently promulgated by Dixmier and Kostant in the 1960's. For exposition with further historical notes, see, for example, [Knapp 1986], [Knapp 1996], [Wallach 1988]. Even though one might think that naming these modules after Harish-Chandra would have been more apt, one could argue that it would have been ill-advised to name yet another thing a *Harish-Chandra module*, since this would fail to distinguish the thing. By contrast, the terminology *Verma module* is unambiguous.

When convenient the π is suppressed in notation, and parentheses are minimized. Thus, the action of $x \in \mathfrak{g}$ on $v \in V$ is denoted

$$\pi(x)(v) = \pi(x)v = x \cdot v = xv$$

For a finite-dimensional (as $\mathbb{C}$-vectorspace) $\mathfrak{g}$-module V, for some complex number λ the operator H necessarily has a non-zero λ-eigenspace $V(\lambda)$ in V. Since $[H, X] = 2X$, X has a predictable effect on $V(\lambda)$: for v in $V(\lambda)$,

$$H(Xv) = (HX - XH)v + XHv = 2Xv + XHv = 2Xv + X\lambda v = (2 + \lambda)Xv$$

That is, Xv is in the $2+\lambda$ eigenspace $V(\lambda+2)$ for H. By finite-dimensionality, in the sequence of eigenspaces

$$V(\lambda),\ V(\lambda + 2),\ V(\lambda + 4),\ V(\lambda + 6),\ \ldots$$

only finitely-many can be non-zero. That is, there is some $\lambda \in \mathbb{C}$ such that X annihilates $V(\lambda)$. Call λ a **highest weight** (even though in this simple setting it's just a complex number), and any non-zero vector in $V(\lambda)$ (for highest weight λ) is a **highest weight vector** in V.

Similarly, the lowering operator Y maps an eigenspace $V(\mu)$ to $V(\mu-2)$. The easy half of Poincaré-Birkhoff-Witt asserts that the monomials

$$Y^a\, H^b\, X^c \quad \text{(non-negative integers } a, b, c\text{)}$$

in $U(\mathfrak{g})$ span $U(\mathfrak{g})$. Thus, since X annihilates v, and since H acts by a scalar on v, the submodule $U(\mathfrak{g}) \cdot v$ generated by a highest weight vector v is

$$U(\mathfrak{g}) \cdot v = \mathbb{C}[Y] \cdot \mathbb{C}[H] \cdot \mathbb{C}[X] \cdot v = \mathbb{C}[Y] \cdot \mathbb{C}[H] \cdot v = \mathbb{C}[Y] \cdot v$$

Thus, the submodule generated by v has a $\mathbb{C}$-basis consisting of vectors of the form

$$Y^n \cdot v$$

for integer n in some finite set.

Since $Y^n \cdot v$ is either 0 or is a basis for the $(\lambda - 2n)$-eigenspace of H in $U(\mathfrak{g}) \cdot v$, for each n there is some constant $c(n)$ such that (with v a highest weight vector with fixed weight λ) either $Y^n \cdot v = 0$ or

$$X \cdot Y \cdot Y^n v = c(n) \cdot Y^n v$$

For example,

$$X \cdot Y \cdot v = ([X, Y] + YX)v = Hv + YXv = \lambda v + Y \cdot 0 = \lambda v$$

and, using again the shifting of weights by Y and X,

$$X \cdot Y \cdot Yv = ([X, Y] + YX)Yv = HYv + YXYv = (\lambda - 2)Yv + Y(\lambda v) = (2\lambda - 2) \cdot Yv$$

$$X \cdot Y \cdot Y^2 v = ([X, Y] + YX)Y^2 v = HYv + YXY^2 v = (\lambda - 4)Y^2 v + Y((2\lambda - 2) \cdot v) = (3\lambda - 6) \cdot Y^2 v$$

Generally, by induction,

$$X \cdot Y \cdot Y^n v = ([X, Y] + YX)Y^n v = HY^n v + Y(XY^n v) = (\lambda - 2n)Y^n v + Y((n\lambda - n(n-1)) \cdot v) = (n+1)(\lambda - n) \cdot Y^n v$$

In particular, for finite-dimensional V, let v be a highest weight vector in V with weight (eigenvalue) λ, and let n be the smallest positive integer such that $Y^n v = 0$. Then, on one hand,

$$Y \cdot Y^{n-1} v = 0$$

and, on the other hand, by the computation of the previous paragraph,

$$XY \cdot Y^{n-1}v = n(\lambda - (n-1)) \cdot Y^{n-1}v$$

Thus, for finite-dimensional V with highest weight λ and $\mathbb{C}$-basis

$$v,\ Yv,\ Y^2v,\ \ldots, Y^{n-2}v,\ Y^{n-1}v$$

(i.e., the dimension of V is n) the highest weight is

$$\lambda = n - 1 = \dim_{\mathbb{C}} V - 1$$

That is, the dimension n determines the highest weight $\lambda = n-1$. Further, the complete collection of weights occurring is

$$\lambda,\ \lambda - 2,\ \lambda - 4,\ \ldots,\ 2 - \lambda,\ -\lambda$$

or, in other words, for $U(\mathfrak{g}) \cdot v$ of dimension n, the H-eigenvalues are

$$n-1,\ n-3,\ n-5,\ \ldots, 5-n,\ 3-n,\ -n$$

[1.0.1] Corollary: For an irreducible finite-dimensional representation of $\mathfrak{sl}(2)$ the highest weight is a non-negative integer. ///

[1.0.2] Remark: The preceding analysis does not *construct* any finite-dimensional irreducibles. On the other hand, in fact it did not *assume* the irreducibility of the $U(\mathfrak{g})$-module generated by a highest weight vector v. Rather, the argument shows that a *cyclic* $U(\mathfrak{g})$-module generated by a highest-weight vector is of the form indicated.

[1.0.3] Remark: Since we are looking at $\mathbb{R}$-linear (Lie algebra) maps of $\mathfrak{sl}(2,\mathbb{R})$ to endomorphism algebras of complex vector spaces, we may as well consider complex-linear representations of the complexification

$$\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

And the special unitary group $SU(2)$ has the *same* complexified Lie algebra, $\mathfrak{sl}(2,\mathbb{C})$. Thus, at the level of $\mathfrak{sl}(2)$ representations we are unable to distinguish $SU(2)$ and $SL(2,\mathbb{R})$. While on one hand this evidently loses information, on the other hand it *allows* (by ignoring some awkward points) a discussion which simultaneously addresses *some* common points of the two cases.

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2. *The Casimir element*

For $\mathfrak{g} = \mathfrak{sl}(2)$, with X, Y, H as above, we can define the **Casimir operator** [2]

$$\text{Casimir operator} = \Omega = \frac{1}{2}H^2 + XY + YX \in U(\mathfrak{g})$$

We will see that the Casimir operator is in the center of the universal enveloping algebra quite generally. Schur's Lemma (see the proof of the first corollary just below) assures that Ω acts by a *scalar* on an irreducible representation of $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{sl}(2)$ we determine this scalar. Therefore, the eigenvalues of Ω are *invariants* of the isomorphism class of the irreducible. We will make essential use of this in proving *complete reducibility* of finite-dimensional representations, for example.

[2] One might roughly imagine that this is a sort of *Laplacian*.

[2.0.1] Proposition: The Casimir operator $\Omega = \frac{1}{2} H^2 + XY + YX$ is in the center $Z(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{sl}(2)$.

[2.0.2] Corollary: On an irreducible finite-dimensional representation V of $\mathfrak{g} = \mathfrak{sl}(2)$ with highest weight $\lambda = \dim_{\mathbb{C}} V - 1$, the Casimir operator acts by the scalar

$$\frac{1}{2}(\dim_{\mathbb{C}} V)(\dim_{\mathbb{C}} V - 1)$$

Proof: First, we prove that anything in the center of $U(\mathfrak{g})$ must act by a scalar on a finite-dimensional irreducible V. Indeed, by irreducibility, any $\mathfrak{g}$-endomorphism is either 0 or is an isomorphism, since the kernel and image, being $\mathfrak{g}$-subspaces, can only be V or 0. Thus, the $\mathfrak{g}$-endomorphism algebra E is a division ring. And E is finite-dimensional over $\mathbb{C}$, since V is finite-dimensional. The finite-dimensionality implies that any element of E is algebraic over $\mathbb{C}$. The algebraic closure of $\mathbb{C}$ implies that $E = \mathbb{C}$. Thus, any central element in $U(\mathfrak{g})$ acts by a scalar on an irreducible.

In the case of $\mathfrak{g} = \mathfrak{sl}(2)$, we saw above that a finite-dimensional irreducible has a highest weight vector v with eigenvalue λ equal to $\dim V - 1$. Since Ω acts by a scalar, we can compute this scalar on the highest weight vector.

$$\Omega v = (\frac{1}{2}H^2 + XY + YX)\, v = (\frac{1}{2}H^2 + [X,Y] + YX + YX)\, v = (\frac{1}{2}H^2 + H + YX + YX)\, v = (\frac{1}{2}\lambda^2 + \lambda + 0 + 0)\, v$$

as claimed. ///

Conceivably it is possible to prove the proposition by direct computation for the little algebra $\mathfrak{sl}(2)$, since we need only show that it commutes (in $U(\mathfrak{g})$ with H, X, and Y, but such an argument would fail to reveal how the thing was found in the first place. So we may as well prove a version for $\mathfrak{sl}(n)$. Indeed, the proof of the following theorem applies to arbitrary semi-simple Lie algebras.

[2.0.3] Theorem: Let E_{ij} be the n-by-n matrix with a 1 at the ij^{th} position (i^{th} row and j^{th} column) and 0 at all other positions. Let $H_i = E_{ii} - E_{i+1,i+1}$ for $1 \le i < n$, and

$$H_i^* = \frac{1}{n}\left((n-i)E_{11} + (n-i)E_{22} + \ldots + (n-i)E_{ii} - iE_{i+1,i+1} - \ldots - iE_{nn}\right)$$

The **Casimir operator** Ω in $\mathfrak{g} = \mathfrak{sl}(n)$ defined by

$$\sum_{1 \le i < n} H_i H_i^* + \sum_{i \ne j} E_{ij}\, E_{ji}$$

is in the center $Z(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

[2.0.4] Remark: The particular form of the expression for the Casimir operator is best explained by a yet more general claim made in the course of the proof below. Note that in the case of $\mathfrak{sl}(2)$, there is just a single element $H = H_1$, and

$$H^* = H_1^* = \frac{1}{2}H$$

giving the simple version for $\mathfrak{sl}(2)$.

[2.0.5] Remark: The annoying constants arising in the expression for H_i^* can be dispatched entirely if we tolerate[3] considering the Lie algebra $\mathfrak{gl}(n)$ of *all* n-by-n complex matrices, rather than just trace zero ones. In that case, the diagonal matrices have a simpler basis

$$H_i = E_{ii}$$

[3] The Lie algebra $\mathfrak{g} = \mathfrak{gl}(n)$ is not *semi-simple*, but only *reductive*, meaning, among other things, that the set of brackets $[x,y] = xy - yx$ for $x, y \in \mathfrak{g}$ is *not* all of $\mathfrak{g}$. The pairing $\langle x, y\rangle = \mathrm{tr}(xy)$ is no longer a constant multiple of the Killing form. The Lie algebra is *not* spanned by images of $\mathfrak{sl}(2)$. But it is still true that this pairing is invariant under conjugation by $GL(n,\mathbb{C})$, and behaves well with respect to the operation $[x,y]$ in $\mathfrak{gl}(n)$, so the argument that a Casimiar operator is central would work in this case, too.

and the dual basis is

$$H_i^* = H_i$$

Proof: Let $\{x_i\}$ be a $\mathbb{C}$-basis for $\mathfrak{g} = \mathfrak{sl}(n)$, with *dual basis* $\{x_i^*\}$ with respect to the pairing

$$\langle x, y\rangle = \mathrm{tr}(xy)$$

We claim that for *any* such choice the Casimir operator is

$$\Omega = \sum_i x_i\, x_i^*$$

Thus, in $\mathfrak{sl}(2)$, with basis H, X, Y, the dual basis element H^* for H is $H^* = \frac{1}{2}H$, for X it is $X^* = Y$, and for Y it is $Y^* = X$.

To approach the proof of this claim, first let V be any finite-dimensional $\mathbb{C}$-vectorspace, with $\mathbb{C}$-linear dual V^*. Let G be any group acting on V, denoted

$$g \times v \to gv$$

and let G act on the dual V^* as usual by [4]

$$(g\lambda)(v) = \lambda(g^{-1}v)$$

Certainly G acts on the tensor product $V \otimes V^*$ by the $\mathbb{C}$-linear extension of

$$g(v \otimes \lambda) = gv \otimes g\lambda$$

For a $\mathbb{C}$-basis $\{v_i\}$ of V, let $\{\lambda_i\}$ be the corresponding dual basis of V^*. This has the property that for any $v \in V$

$$v = \sum_i \lambda_i(v) \cdot v_i$$

We claim that the expression $\sum_i v_i \otimes \lambda_i \in V \otimes V^*$ is independent of the choice of basis, and that, for any $g \in G$,

$$\sum_i v_i \otimes \lambda_i = \sum_i gv_i \otimes g\lambda_i$$

To see this, recall the natural isomorphism

$$V \otimes_{\mathbb{C}} V^* \approx \mathrm{End}_{\mathbb{C}}(V)$$

by

$$(v \otimes \lambda)(v') = \lambda(v') \cdot v$$

By the very definition of *dual basis*, the endomorphism

$$\sum_i v_i \otimes \lambda_i$$

of V is the *identity map* on V, regardless of the choice of basis. And, in particular, it does commute with the action of G on V. Thus, for $v \in V$ and $g \in G$

$$\left(\sum_i v_i \otimes \lambda_i\right)(v) = v = (g \circ \mathrm{id}_V \circ g^{-1})(v) = \sum_i g \cdot (v_i \otimes \lambda_i)(g^{-1}v)$$

[4] *Contragredient* or *adjoint* action.

$$= \sum_i g \cdot \lambda_i(g^{-1}v) \cdot v_i = \Big(\sum_i (gv_i) \otimes (g\lambda_i\Big))(v)$$

proving the claim.

Next, a non-degenerate bilinear form $\langle , \rangle$ on V gives a natural isomorphism $v \to \lambda_v$ of V with its dual V^*, by

$$\lambda_v(v') = \langle v', v\rangle$$

If also the action of the group G is by isometries of $\langle , \rangle$, that is, has the property that

$$\langle gv', gv\rangle = \langle v, v'\rangle$$

for all $v, v' \in V$, then the previous paragraph shows that

$$\sum_i v_i \otimes v_i^* \in V \otimes V$$

is independent of the choice of basis, and is G-invariant, where $\{v_i^*\}$ is the dual basis corresponding via $\langle , \rangle$ to a given basis $\{v_i\}$ for V.

The conjugation invariance of trace

$$\mathrm{tr}(gxg^{-1}) = \mathrm{tr}(x)$$

for $g \in GL(n, \mathbb{C})$ shows that conjugation action of $GL(nm, \mathbb{C})$ gives isometries of the pairing $\langle x, y\rangle = \mathrm{tr}(xy)$: for $g \in GL(n, \mathbb{C})$ and $x, y \in \mathfrak{g} = \mathfrak{sl}(n)$

$$\langle gxg^{-1}, gyg^{-1}\rangle = \mathrm{tr}(gxg^{-1} \cdot gyg^{-1}) = \mathrm{tr}(gxyg^{-1}) = \mathrm{tr}(xy)$$

Thus, for any basis $\{v_i\}$ and dual basis $\{v_i^*\}$ with respect to trace,

$$\sum_i v_i \otimes v_i^* \in V \otimes V$$

is independent of the choice of basis and is $GL(n, \mathbb{C})$-invariant.

The tensor product $V \otimes V$ sits inside the tensor algebra

$$T(V) = \mathbb{C} \oplus \bigoplus_{n\geq 1} \underbrace{V \otimes \ldots \otimes V}_{n}$$

Taking $V = \mathfrak{g}$, the enveloping algebra $U(\mathfrak{g})$ is a quotient of $T(\mathfrak{g})$ by the ideal generated by all expressions $[x, y] - (xy - yx)$. This ideal is stable under the action [5]

$$g \times x \to gxg^{-1}$$

so this action descends to [6] the quotient $U(\mathfrak{g})$. Use the bilinear form $\langle x, y\rangle = \mathrm{tr}(xy)$ on $\mathfrak{g}$ to create a $GL(n, \mathbb{C})$-invariant element

$$\sum_i x_i \otimes x_i^* \in T(\mathfrak{g})$$

in $\mathfrak{g} \otimes \mathfrak{g}$, where x_i is a basis for $\mathfrak{g}$ and the x_i^* are the dual basis. And, from above, this element does not depend on the choice of basis. Under the quotient map to $U(\mathfrak{g})$ this tensor maps to

$$\Omega = \sum_i x_i \, x_i^* \in U(\mathfrak{g})$$

[5] The *Adjoint* action.

[6] Is well defined on ...

Thus, since the element of $T(\mathfrak{g})$ was independent of basis and was $GL(n,\mathbb{C})$-invariant, its image is also. That is, the Casimir element can be so expressed via any basis, and is invariant under the natural $GL(n,\mathbb{C})$ action on $U(\mathfrak{g})$

$$\Omega = \sum_i x_i\, x_i^* = \sum_i g x_i g^{-1}\, g x_i^* g^{-1} \in U(\mathfrak{g})$$

In effect we take the derivative of the $GL(n,\mathbb{C})$-invariance relation to show that Ω is in the center of the enveloping algebra, as follows. With $g = e^{\varepsilon z}$ with ε small and $z \in \mathfrak{g}$, as complex n-by-n matrices

$$gxg^{-1} = x + \varepsilon(zx - xz) + O(\varepsilon^2)$$

Thus, using the $GL(n,\mathbb{C})$-invariance of Ω we have, modulo $O(\varepsilon^2)$,

$$\sum_i x_i\, x_i^* = \Omega = \sum_i g x_i g^{-1}\, g x_i^* g^{-1} = \sum_i x_i\, x_i^* + \varepsilon \sum_i (z x_i - x_i z)\, x_i^* + \varepsilon \sum_i x_i\, (z x_i^* - x_i^* z)$$

Thus,

$$0 = \sum_i \left((z x_i - x_i z)\, x_i^* + x_i\, (z x_i^* - x_i^* z)\right)$$

Inside $U(\mathfrak{g})$, for any $x, y, z \in \mathfrak{g}$, we have

$$(zx - xz)\, y + x\, (zy - yz) = zxy - xzy + xzy - xyz = zxy - xyz = z(xy) - (xy)z$$

Therefore, applying this termwise to the previous equality,

$$0 = z\,\Omega - \Omega\, z$$

for all $z \in \mathfrak{g}$, which shows that Ω is in the center of the enveloping algebra $U(\mathfrak{g})$. ///

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3. *Complete reducibility*

The following theorem is a very special case, and most of the features of the theorem are immediately applicable more generally. The missing item would be sufficient detail about the Casimir operator on finite dimensional representations of a more general Lie algebra $\mathfrak{g}$, so for the moment we content ourselves with appearing to treat only $\mathfrak{sl}(2)$.

[3.0.1] Theorem: Any finite-dimensional complex representation of $\mathfrak{sl}(2)$ is a direct sum of irreducibles.

[3.0.2] Remark: The assertion of the theorem is that any finite-dimensional complex representation of $\mathfrak{sl}(2)$ is **completely reducible**.

Proof: General argument (at the end of this proof) leads one to consider the slightly odd-seeming situation treated in the first part of the proof.

First, it is useful to observe that on a one-dimensional representation V of $\mathfrak{g}$ the action of $\mathfrak{g}$ is 0. Indeed, the trace of any commutator $ST - TS$ of endomorphisms on a finite-dimensional space is necessarily 0. Since $[H, X] = 2X$, $[H, Y] = -2Y$, and $[X, Y] = H$, every element of $\mathfrak{g} = \mathfrak{sl}(2)$ is a (linear combination of) commutators, so the trace of any element of $\mathfrak{g}$ on a finite-dimensional representation is 0. In particular, the image of $\mathfrak{g}$ in the endomorphisms of a *one*-dimensional space must be 0.

Now consider the family of situations wherein V is a representation with a codimension-one subrepresentation. That is, by the previous paragraph, we have an exact sequence

$$0 \to W \to V \to \mathbb{C} \to 0$$

where the $\mathbb{C}$ is the one-dimensional representation of $\mathfrak{g}$. We claim that this *splits*, in the sense that there is $s : \mathbb{C} \to V$ such that

$$V = s(\mathbb{C}) \oplus W$$

We do induction on the dimension of W. If W is *reducible*, let U be a proper submodule. Then

$$0 \to W/U \to V/U \to \mathbb{C} \to 0$$

is still exact, and $\dim W/U < \dim W$, so there is a section $s : \mathbb{C} \to V/U$. Let $s(\mathbb{C}) = W'/U$ for a subspace W' of V. Then we have another exact sequence of the same sort

$$0 \to U \to W' \to \mathbb{C} \to 0$$

Since $\dim W' < \dim W$, the exact sequence splits, giving $t : \mathbb{C} \to W'$ such that $W' = U \oplus t(\mathbb{C})$. Because $s(\mathbb{C}) = W'/U$ complemented W/U in V/U, and because $t(\mathbb{C}) \cap U = 0$ inside W', we find that $t(\mathbb{C}) \cap W = 0$. That is,

$$V = W \oplus t(\mathbb{C})$$

giving the induction step for the splitting for W reducible.

Still in the family of cases

$$0 \to W \to V \to \mathbb{C} \to 0$$

we are left with the case of the induction step (on the dimension of W) for W *irreducible*. Again, then $\mathfrak{g}$ acts by 0 on V/W, so $\mathfrak{g} \cdot V \subset W$, and, thus, the Casimir Ω acts by 0 on V/W. On the other hand, we saw earlier that the Casimir operator Ω acts by the scalar $n(n-1)/2$ on an irreducible W of dimension n. If $\dim W > 1$ then this scalar is not 0. Thus, the $U(\mathfrak{g})$-endomorphism

$$\Omega : V \to V$$

is non-zero on the codimension-one subspace W, and maps the whole space V to W (since Ω is 0 on V/W). That is, Ω has a non-trivial kernel on V, necessarily one-dimensional since Ω is a non-zero scalar on the codimension-one subspace W. Since Ω commutes with $U(\mathfrak{g})$ this kernel is a subrepresentation of V, providing the complement to W.

We ought not overlook the extreme case that the codimension-one irreducible W is one-dimensional. Again, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, so $\mathfrak{g} \cdot V = 0$. Thus, for this two-dimensional V with a one-dimensional subrepresentation W, *any* one-dimensional vector subspace U of V complementary to W gives $V = U \oplus W$, for trivial reasons.

This completes the induction (on $\dim W$) step for *codimension one* subrepresentations W.

Now consider an exact sequence of finite-dimensional $\mathfrak{g}$-modules

$$0 \to W \to V \to V/W \to 0$$

We will exhibit a $\mathfrak{g}$-complement to W in V as the kernel of a $\mathfrak{g}$-homomorphism $V \to W$. To this end, put a $\mathfrak{g}$-module structure [7] on $\mathrm{Hom}_{\mathbb{C}}(V, W)$ by

$$(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v)$$

[7] The computational verification that this is a Lie algebra representation is peculiar, but straightforward. First, for $x, y \in \mathfrak{g}$ and f a $\mathbb{C}$-linear map $V \to W$,

$$(xy \cdot f))(v) = x((yf)(v)) - (yf)(xv) = x(y\,f(v) - f(yv)) - (y\,f)(xv) - f(yx\,v)) = xy\,f(v) - xf(yv) - y\,f(xv) + f(yx\,v))$$

Note that the middle two terms are symmetrical in x, y. Thus,

$$([x,y]\,f))(v) = xyf(v) + f(yxv) - yxf(v) - f(xyv) = [x,y]f(v) - f([x,y]v)$$

for $x \in \mathfrak{g}$ and $f \in \mathrm{Hom}_{\mathbb{C}}(V, W)$. Among the merely $\mathbb{C}$-linear homomorphisms the $U(\mathfrak{g})$-morphisms $V \to W$ are exactly the ones which are *annihilated* by this action of $\mathfrak{g}$. [8] Consider, though, the (somewhat larger) collection

$$B = \{f \in \mathrm{Hom}_{\mathbb{C}}(V, W) : f \text{ is a scalar on } W\}$$

and submodule

$$A = \{f \in \mathrm{Hom}_{\mathbb{C}}(V, W) : f \text{ is } 0 \text{ on } W\}$$

And B is readily verified [9] to be a $\mathfrak{g}$-submodule of $\mathrm{Hom}_{\mathbb{C}}(V, W)$, as is A. The scalar by which $f \in B$ acts on W determines f modulo A, so the quotient of B by the $U(\mathfrak{g})$-homomorphisms is at most one-dimensional. On the other hand, among merely $\mathbb{C}$-linear maps of V to W there certainly is one that is the identity on W, hence non-zero. Thus, the quotient B/A is exactly one-dimensional, and we have an exact sequence

$$0 \to A \to B \to \mathbb{C} \to 0$$

where the one-dimensionality of the quotient $\mathbb{C}$ assures (as above) that it is a trivial $\mathfrak{g}$-module. Thus, from above, there is a $U(\mathfrak{g})$-homomorphism $s : \mathbb{C} \to B$ such that

$$B = A \oplus s(\mathbb{C})$$

and take $f \in s(\mathbb{C})$. Since f is not in A it is scalar on W but not 0 on W. Thus, adjust f by a constant to make it be the identity on W. The fact that f is in the trivial $\mathfrak{g}$-module $s(\mathbb{C})$ is that $\mathfrak{g} \cdot f = 0$, which (with the action of $\mathfrak{g}$ on endomorphisms) asserts exactly that f is a $\mathfrak{g}$-homomorphism. Thus, $\ker f$ is a $\mathfrak{g}$-submodule of V. Because the modules are finite dimensional and f is the identity on W, the kernel of f is a complement to W. ///

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4. *Verma modules for* $\mathfrak{sl}(2)$

Given an eigenvalue $\lambda \in \mathbb{C}$ for H, we make the *universal* [10] $U(\mathfrak{g})$-module $M_\lambda^{\mathrm{naive}}$ (first in a naive normalization, then, later, corrected) possessing a distinguished vector $v \neq 0$ such that [11]

$$Xv = 0 \qquad Hv = \lambda v$$

and no other relations. Before giving a careful construction, we observe that, by Poincaré-Birkhoff-Witt, $M_\lambda^{\mathrm{naive}}$ will apparently have a $\mathbb{C}$-vectorspace basis consisting of the *infinite* list $v, Yv, Y^2v, \ldots$, since no power

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[8] The annihilation condition $x \cdot f(v) - f(x \cdot v) = 0$ for $x \in \mathfrak{g}$ and f an endomorphism immediately yields the $\mathfrak{g}$-homomorphism condition $x \cdot f(v) = f(x \cdot v)$.

[9] For $f \in B$ let c be the scalar by which it acts on W, take $x \in \mathfrak{g}$ and $w \in W$ and compute

$$(x \cdot f)(w) = x \cdot f(w) - f(x \cdot w) = x \cdot cw - cx \cdot w = cx \cdot w - cx \cdot w = 0$$

Note that this does *not* assert that $\mathfrak{g}$ annihilates f, only that $\mathfrak{g} \cdot f$ is 0 when restricted to W. And, since W is a $\mathfrak{g}$-subspace of V, B is stable under the action of $\mathfrak{g}$.

[10] By standard elementary category-theoretic arguments, there exists *at most one* $U(\mathfrak{g})$-module M generated by a vector $m \neq 0$ with $Xm = 0$ and $Hm = \lambda \cdot m$ with given $\lambda \in \mathbb{C}$, such that for any $U(\mathfrak{g})$-module V generated by a vector $v \neq 0$ with with $Xv = 0$ and $Hv = \lambda \cdot v$, there exists a unique $U(\mathfrak{g})$-homomorphism $M \to V$ taking m to v. The pair (M, m) is unique up to unique isomorphism, as usual in these stories.

[11] With hindsight, this will be renormalized. Since the correct normalization can only be seen *after* an initial computation is done, it seems reasonable to approach the computation honestly rather than inject an unguessable normalization at the outset.

of Y annihilates v. Then,[12] Verma modules yield all the *finite-dimensional* irreducibles as quotients, since (by elementary linear algebra, above) every finite-dimensional representation has a highest weight vector. We will determine the condition on λ such that M_λ^{naive} admits a finite-dimensional quotient. [13]

More carefully, still in the innocent normalization, let I_λ^{naive} be the left $U(\mathfrak{g})$ ideal generated by $H-\lambda$ and by X, namely

$$I_\lambda^{\text{naive}} = U(\mathfrak{g})\cdot(H-\lambda) + U(\mathfrak{g})\cdot X$$

and put

$$M_\lambda^{\text{naive}} = U(\mathfrak{g})/I_\lambda^{\text{naive}}$$

The distinguished *(highest weight)* vector in this model is

$$v = 1 + I_\lambda^{\text{naive}}$$

By construction, indeed, $Xv = 0$ and $Hv = \lambda v$.

An equivalent construction having a different appearance is as follows. Let $\mathbf{b}$ be the (Borel) Lie subalgebra $\mathbb{C}\cdot H + \mathbb{C}\cdot X$ of $\mathfrak{g}$. Make $\mathbb{C}$ into a $U(\mathbf{b})$-module $\mathbb{C}_\lambda$ by

$$X\cdot\alpha = 0 \qquad H\cdot\alpha = \lambda\cdot\alpha$$

By Poincaré-Birkhoff-Witt, $U(\mathbf{b})$ injects to $U(\mathfrak{g})$, and we may consider $U(\mathfrak{g})$ a right $U(\mathbf{b})$-module. Then define

$$M_\lambda^{\text{naive}} = U(\mathfrak{g})\otimes_{U(\mathbf{b})}\mathbb{C}_\lambda$$

Let $\mathfrak{n}^- = \mathbb{C}\cdot Y$. Then $U(\mathfrak{n}^-) = \mathbb{C}[Y]$. In either construction of M_λ^{naive} Poincaré-Birkhoff-Witt shows that the natural map

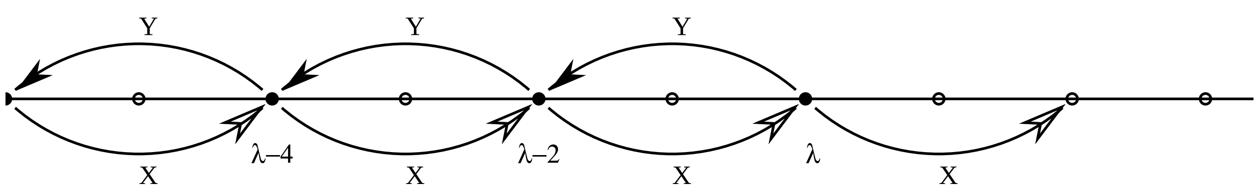
$$\mathbb{C}[Y] = U(\mathfrak{n}^-) \to M_\lambda^{\text{naive}}$$

defined by

$$Y^n \to Y^n\cdot v \qquad (0\le n\in\mathbb{Z})$$

is a $U(\mathfrak{n}^-)$-isomorphism. In particular, M_λ^{naive} has a $\mathbb{C}$-vectorspace basis consisting of

$$Y^n\cdot v \qquad (0\le n\in\mathbb{Z})$$



[4.0.1] Theorem: *(Naive form)* In the innocent normalization as above,

$$\dim_{\mathbb{C}}\text{Hom}_{U(\mathfrak{g})}(M_\mu^{\text{naive}}, M_\lambda^{\text{naive}}) = \begin{cases} 1 & (\mu=\lambda)\\ 1 & (\mu = -\lambda-2 \text{ and } 0\le\lambda\in\mathbb{Z})\\ 0 & (\text{otherwise})\end{cases}$$

[12] mildly ironically

[13] The later renormalization allows the question of morphisms among these modules and existence of finite-dimensional quotients to be given in a more symmetrical and attractive form.

Renormalize: The second point in the theorem suggests the proper normalization for a symmetrical result. Instead of the naive forms above, let

$$I_\lambda = U(\mathfrak{g}) \cdot (H - (\lambda - 1)) + U(\mathfrak{g}) \cdot X$$

$$M_\lambda = U(\mathfrak{g})/I_\lambda$$

or, in terms of tensor products,

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathbf{b})} \mathbb{C}_{\lambda-1}$$

Then we have

[4.0.2] Theorem:

$$\dim_{\mathbb{C}} \mathrm{Hom}_{U(\mathfrak{g})}(M_\mu, M_\lambda) = \begin{cases} 1 & (\mu = \lambda) \\ 1 & (\mu = -\lambda \text{ and } 1 \le \lambda \in \mathbb{Z}) \\ 0 & (\text{otherwise}) \end{cases}$$

And any non-zero homomorphism is *injective*.

Proof: In the proof, we use the naive normalization. With any $U(\mathfrak{g})$-homomorphism $f : M_\mu^{\text{naive}} \to M_\lambda^{\text{naive}}$, the image of the highest-weight vector from M_μ must be an H-eigenvector annihilated by X. Since $[H, Y] = -2Y$, in M_λ^{naive}, by an easy induction,[14]

$$H \cdot Y^n v = (\lambda - 2n) \cdot Y^n v$$

Thus, the various vectors $Y^n v$ have distinct eigenvalues $\lambda - 2n$. Thus, any image of an H-eigenvector must be a multiple of one of these. Another induction[15] shows that

$$X \cdot Y^n v = n(\lambda - (n-1)) \cdot Y^n v$$

For this to be zero $\lambda = n - 1$. That is, M_λ^{naive} contains a vector annihilated by X other than multiples of the highest weight vector if and only if λ is a non-negative integer.

Here (because we are in the *universal* $U(\mathfrak{g})$-module with highest weight λ) the vector $Y^n v$ is not 0, has weight $\lambda - 2n$, and X annihilates it if and only if $\lambda = n - 1$. That is, $n = \lambda + 1$. Thus, $Y^n v$ has weight

$$\text{weight } Y^n v = \lambda - 2n = \lambda - 2(\lambda + 1) = -\lambda - 2$$

That is, there is a non-zero $U(\mathfrak{g})$-homomorphism of $M_{-\lambda-2}^{\text{naive}}$ to M_λ^{naive} if and only if $0 \le \lambda \in \mathbb{Z}$.

For injectivity, use the fact that

$$M_\lambda^{\text{naive}} = U(\mathbb{C} \cdot Y) \cdot v = U(\mathfrak{n}^-) \cdot v = \mathbb{C}[Y] \cdot v$$

where v is the highest-weight vector. Let $Y^n v$ be the vector annihilated by X, as just above. Suppose that the map of $M_{-\lambda-2}^{\text{naive}}$ to M_λ^{naive} sending the highest-weight vector of $M_{\lambda-2}^{\text{naive}}$ to Y^n were not injective. In particular, suppose some $f(Y) \cdot Y^n v = 0$ for $f(Y) \in \mathbb{C}[Y] = U(\mathfrak{n}^-)$. Then

$$(f(Y) Y^n) \cdot v = 0$$

Since $g(Y) \to g(Y) \cdot v$ is a linear isomorphism of $U(\mathfrak{n}^-)$ to M_λ^{naive}, necessarily $f(Y) Y^n = 0$ in $U(\mathfrak{n}^-) = \mathbb{C}[Y]$, so $f(Y) = 0$. This proves the injectivity. ///

[14] This is the same computation as earlier, using the fact that $XY \cdot Y^n v = [X, Y] \cdot Y^n v + YX \cdot Y^n v$.

[15] Again the same computation as above in the discussion of highest weights, moving X across the Ys.

[4.0.3] Remark: Since the symmetry in the naive normalization is

$$\lambda \to -\lambda - 2$$

renormalize by replacing λ by $\lambda + c$ with suitable constant c. The symmetry

$$\lambda + c \to -(\lambda + c) - 2$$

is

$$\lambda \to -\lambda - 2c - 2$$

The prettiest outcome would be to have no constant appearing, take $c = -1$. This hindsight gives the revised version of the theorem.

Now we can produce an irreducible quotient of M_λ for $1 \le \lambda \in \mathbb{Z}$. In fact, the above discussion makes clear that $M_\lambda / M_{-\lambda}$ is finite dimensional, but there is a lighter and more broadly applicable argument:

[4.0.4] Lemma: Let M be a $U(\mathfrak{g})$ module in which every vector is a sum of H-eigenvectors. Let v be an element of a $U(\mathfrak{g})$-submodule of M. Let $v = \sum_\mu v_\mu$ where v_μ is a μ-eigenvector for H (and the complex numbers μ appearing in the sum are distinct). Then each v_μ is also in N.

Proof: Let

$$v'_\mu = \left(\prod_{\nu \neq \mu} (H - \nu) \right) \cdot v \in \mathbb{C}[H] \cdot v \subset N$$

Then

$$v'_\mu = \left(\prod_{\nu \neq \mu} (\mu - \nu) \right) \cdot v_\mu$$

and the leading constant is not 1. ///

[4.0.5] Theorem: For arbitrary $\lambda \in \mathbb{C}$, the Verma module M_λ has a unique maximal proper submodule I_λ, and I_λ does *not* contain the highest weight vector v_λ of M_λ. Thus, M_λ has a unique irreducible quotient, $M_\lambda / I\lambda$, the image of v_λ in this quotient is non-zero, and (therefore) this quotient has highest weight $\lambda - 1$. [16]

Proof: First, no proper submodule can contain the highest weight vector v, because v generates the whole module. Further, no proper submodule can contain a linear combination

$$cv + \sum_i c_i \, v_i$$

with v_i of other weights $\lambda_i \neq \lambda$, or else v lies in that submodule (by the lemma), and this vector would generate the whole module. Therefore, no sum of proper submodules contains v. Thus, the sum I_λ of *all* proper submodules does not contain v, so is proper. That is, I_λ is the unique *maximal* proper submodule. The kernel of any mapping to an irreducible must be maximal proper, so M_λ / I_λ is the unique irreducible quotient. Since the image of the highest weight vector from M_λ is not 0, it generates the quotient, and certainly still has the highest weight $\lambda - 1$. ///

[4.0.6] Remark: This proof of uniqueness of irreducible quotient applies to *all* Verma modules, not just with $1 \le \lambda \in \mathbb{Z}$.

[4.0.7] Remark: The theme of *unique irreducible quotient* of naturally constructed and parametrized object recurs in many genres of representation theory. The present instance is surely one of the simplest.

[16] This is the normalized version.

If $\lambda \geq 1$ say λ is **dominant**. If $\lambda \in \mathbb{Z}$ say λ is **integral**.

[4.0.8] Corollary: For dominant integral λ, the unique irreducible quotient of the Verma module M_λ is finite dimensional. ///

Proof: For $\mathfrak{sl}(2)$, the finite-dimensionality is visible from the explicit information we have about the dimensions of the H-eigenspaces in M_λ. That is, for λ dominant integral, the weights of M_λ are

$$\text{weights } M_\lambda = \lambda - 1,\ \lambda - 3,\ \ldots, 3 - \lambda,\ 1 - \lambda,\ -1 - \lambda,\ -3 - \lambda,\ \ldots$$

and each such weight-space has dimension exactly 1. The weights of $M_{-\lambda}$ are

$$\text{weights } M_{-\lambda} = -\lambda - 1,\ -\lambda - 3,\ \ldots$$

with dimensions 1. Thus, in $M_\lambda / M_{-\lambda}$ the weight spaces with weights

$$-\lambda - 1,\ -\lambda - 3,\ \ldots$$

cancel out, leaving exactly

$$\text{weights } M_\lambda / M_{-\lambda} = \lambda - 1,\ \lambda - 3,\ \ldots, 3 - \lambda,\ 1 - \lambda$$

which is visibly finite-dimensional. ///

[4.0.9] Remark: Part of the point is that the weight λ uniquely determines the isomorphism class of an irreducible with highest weight $\lambda - 1$.

[4.0.10] Remark: The question of the **multiplicities** of the weight spaces, that is, the dimension of various eigenspaces of H in an irreducible finite-dimensional representation with highest weight $\lambda - 1$, is trivial here. Specifically, the only non-trivial weight spaces are those with H-eigenvalue between $\lambda - 1$ and $1 - \lambda$ inclusive, of the same parity as $\lambda - 1$, and these are 1-dimensional.

[4.0.11] Remark: An important question is the decomposition of tensor products of finite-dimensional irreducibles, or, more extravagantly, of $M_\lambda \otimes_{\mathbb{C}} M_\mu$.

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5. *Harish-Chandra homomorphism for* $\mathfrak{sl}(2)$

The goal here is to understand the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{sl}(2)$.

By Poincaré-Birkhoff-Witt the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{sl}(2)$ has a basis of monomials

$$Y^a\, H^b\, X^c$$

Let ρ be a representation of $\mathfrak{g}$ with highest weight λ, with (non-zero) highest weight vector v. In this small example, we take this to mean that $Hv = \lambda \cdot v$, with $\lambda \in \mathbb{C}$. On one hand, a monomial in $U(\mathfrak{g})$ as above acts on the highest weight vector v by *annihilating* it if any X actually occurs. And elements of of $\mathbb{C}[H] \subset U(\mathfrak{g})$ act on v by the (multiplicative extension of) the highest weight.

We claim that if a sum of monomials $Y^a H^b X^c$ is in the *center* $Z(\mathfrak{g})$ of $U(\mathfrak{g})$, then in each monomial occuring in the sum Y cannot occur unless X also occurs.

Using this claim (cast as the lemma just below), and using the fact (a form[17] of Schur's lemma) that the center $Z(\mathfrak{g})$ acts by scalars on an irreducible of $\mathfrak{g}$, we can compute the *eigenvalues* of elements $z \in Z(\mathfrak{g})$

[17] A version of Schur's lemma that applies to not necessarily finite-dimensional irreducibles, due to [Dixmier 1977], is as follows. By Poincaré-Birkhoff-Witt, the $\mathbb{C}$-dimension of $U(\mathfrak{g})$ is *countable*, so the dimension of an irreducible is countable. Then the dimension of the endomorphisms of an irreducible is countable, because the image of each of countably many basis elements is determined by countably many coefficients (in terms of that basis). As usual, the endomorphism algebra of an irreducible is a division ring. Since $\mathbb{C}$ is algebraically closed, if the endomorphism ring of an irreducible is strictly larger than $\mathbb{C}$, then it contains t *transcendental* over $\mathbb{C}$. But then the uncountably-many endomorphisms $1/(t - \alpha)$ for $\alpha \in \mathbb{C}$ are linearly independent over $\mathbb{C}$, contradiction. Thus, the endomorphism algebra of an irreducible cannot be larger than $\mathbb{C}$ itself.

on an irreducible with highest weight by evaluating z on the highest weight vector, and (by the claim) the values depend only upon the effect of H on v.

[5.0.1] Lemma: If a linear combination z of monomials (as above) lies in $Z(\mathfrak{g})$ then in every monomial where Y occurs X occurs also.

Proof: Each such monomial is an eigenvector for $\text{ad}H$ (actually, its extension[18] to $U(\mathfrak{g})$), with eigenvalue

$$-2a + 2c$$

Because $(\text{ad}H)z = 0$, for each monomial occurring in the expression for z

$$-2a + 2c = 0$$

since these monomials are linearly independent (by Poincaré-Birkhoff-Witt). That is, if Y appears X appears. ///

[5.0.2] Corollary: The eigenvalues of elements of the center $Z(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ on an irreducible representation V of $\mathfrak{g}$ with a highest weight λ are completely determined by λ.

Proof: By Schur's lemma, an element z of the center $Z(\mathfrak{g})$ acts on an irreducible by a scalar $c(z)$. To determine the scalar it suffices to compute zv for a highest weight vector. By the lemma, for $z \in Z(\mathfrak{g})$, expressed as a sum of monomials as above, every monomial not in $\mathbb{C}[H]$ has X occuring in it, and so annihilates v. Any monomial

$$M = H^b$$

acts on the highest weight vector v by the scalar

$$\lambda^b$$

Thus, with

$$z = \sum_i c_i H^{b_i} + (\text{terms with } X \text{ appearing})$$

we have

$$zv = \Big(\sum_i c_i \cdot \lambda^{b_i}\Big) \cdot v$$

That is, the constant $c(z)$ such that $zv = c(z) \cdot v$ is completely determined by λ. ///

More can be said in the direction of the previous lemma and corollary. Let

$$I = U(\mathfrak{g}) \cdot X$$

be the left ideal in $U(\mathfrak{g})$ generated by X. By Poincaré-Birkhoff-Witt, we know that this consists exactly of all linear combinations of monomials, as above, in which X appears.

[5.0.3] Lemma: $I \cap \mathbb{C}[H] = 0$ (where by Poincaré-Birkhoff-Witt $\mathbb{C}[H]$ injects into $U(\mathfrak{g})$).

Proof: For $x \in I$ we have $xv = 0$ for all highest-weight vectors v in all finite-dimensional irreducibles of $\mathfrak{g}$. On the other hand, for $h \in \mathbb{C}[H]$, we have $hv = (\lambda h)v$ (where we extend $H \to \lambda$ to an algebra homomorphism $\lambda : \mathbb{C}[H] \to \mathbb{C}$). Thus, since there *exists* a finite-dimensional irreducible representation of $\mathfrak{g}$ with highest

[18] As usual, on $\mathfrak{g}$ the operator $\text{ad}H$ is just bracketing with H. For example, $(\text{ad}H)(X) = HX - XH = 2X$, $\text{ad}H(H) = [H,H] = 0$, and $\text{ad}H(Y) = -2Y$. The extension of $\text{ad}H$ is as a *derivation*, meaning $(\text{ad}H)(xy) = (\text{ad}H(x))y + x(\text{ad}H(y))$ for $x, y \in U(\mathfrak{g})$. Thus, for example, $\text{ad}H(X^c) = 2cX^c$.

weight λ for every *dominant integral*[19] λ, we have $\lambda h = 0$ for all dominant integral λ. Since h is really a polynomial $P(H)$ in H, the vanishing $\lambda h = 0$ is the assertion that $P(\lambda) = 0$ for $0 \le \lambda \in \mathbb{Z}$. A polynomial in one variable with infinitely many zeros must be identically zero. ///

[5.0.4] Lemma: $Z(\mathfrak{g}) \subset \mathbb{C}[H] + I$

Proof: This is a restatement of the first lemma above: writing $z \in Z(\mathfrak{g})$ as a sum of monomials in our current style, in each such monomial, if Y occurs then X occurs. That is, if the monomial is not already in $\mathbb{C}[H]$ then it is in I. ///

Thus, the sum $\mathbb{C}[H] + I$ is *direct*. Let

$$\gamma_o = \text{ projection of } Z(\mathfrak{g}) \text{ to the } \mathbb{C}[H] \text{ summand}$$

Study of intertwining operators among Verma modules (as above) [20] led Harish-Chandra to renormalize this. [21] Define a *linear* map $\sigma : \mathbb{C} \cdot H \to \mathbb{C}[H]$ by

$$\sigma(H) = H - 1$$

where 1 is the 1 in $U(\mathfrak{g})$. [22] Extend σ by multiplicativity to an *associative algebra* homomorphism

$$\sigma : \mathbb{C}[H] \to \mathbb{C}[H]$$

That is, for a polynomial $P(H)$ in H,

$$\sigma P(H) = P(H-1)$$

Then define the **Harish-Chandra homomorphism**[23]

$$\gamma = \sigma \circ \gamma_o : Z(\mathfrak{g}) \to \mathbb{C}[H] = U(\mathbb{C} \cdot H)$$

Especially as a prototype for later, more complicated, examples, we consider the map $H \to -H$ in a more structured manner. That is, define the **Weyl group**[24]

$$W = \{ \text{ permutation matrices in } GL(2,\mathbb{C}) = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & cr0 \end{pmatrix}, \}$$

Conjugation[25] by $w \in W$ stabilizes $\mathfrak{g} = \mathfrak{sl}(2)$, and behaves well with respect to the Lie bracket in $\mathfrak{g}$ for obvious reasons, namely, for $x, y \in \mathfrak{g}$

$$[wxw^{-1}, wyw^{-1}] = wxw^{-1} \cdot wyw^{-1} - wyw^{-1} \cdot wxw^{-1} = w(xy - yx)w^{-1} = w[x,y]w^{-1}$$

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[19] Again, in this simple situation λ *dominant integral* means that $0 \le \lambda(H) \in \mathbb{Z}$. We were originally using λ to denote the image $\lambda(H) \in \mathbb{C}$, which does completely determine the algebra homomorphism $\lambda : \mathbb{C}[H] \to \mathbb{C}$ denoted by the same symbol.

[20] Yes, Harish-Chandra's 1951 study of intertwining operators among Verma modules occurred 15 years before Verma's 1966 Yale thesis.

[21] Recall from above that in the naive normalization it was $M^{\text{naive}}_{-\lambda-2}$ that had a non-trivial homomorphism to $M^{\text{naive}}_{\lambda}$ for λ dominant integral, motivating Harish-Chandra's renormalization replacing (in this little example) λ by $\lambda - 1$.

[22] It bears repeating that this 1 appears because it is half of the 2 that appears in $[H, X] = 2X$, and also appears in the naive computation of homomorphisms of Verma modules among each other, above. Thus, this renormalization will be somewhat more involved in larger examples.

[23] Yes, in this presentation it is not clear how much this depends on all the choices made.

[24] In this example we will not attempt to delineate what data are needed to specify the Weyl group.

[25] This conjugation action is *Ad* adjoint action, but we do not need to contemplate this.

In particular, the non-trivial element of W interchanges X and Y, and maps $H \to -H$.

[5.0.5] Theorem: The Harish-Chandra map γ above is an isomorphism of $Z(\mathfrak{g})$ to the subalgebra $\mathbb{C}[H]^W$ of the universal enveloping algebra $\mathbb{C}[H]$ of the Cartan subalgebra[26] $\mathbb{C} \cdot H$ invariant under the Weyl group

Proof: First, prove that γ is multiplicative. Since σ is *defined* to be multiplicative, it suffices to prove that the original projection map γ_o is multiplicative. Let I be the left ideal generated by X, as above. Let γ_o also denote the projection from the whole of $I + \mathbb{C}[H]$ to $\mathbb{C}[H]$. Thus, for any $u \in \mathbb{C}[H] + I$, the image $\gamma_o(u)$ is the unique element of $U(\mathfrak{h})$ such that

$$u - \gamma_o u \in I$$

For $z, z' \in Z(\mathfrak{g})$

$$zz' - \gamma_o(z)\gamma_o(z') = z(z' - \gamma_o z') + \gamma_o(z')(z - \gamma_o z)$$

where we use the fact that $\mathbb{C}[H]$ is commutative to interchange $\gamma_o(z)$ and $\gamma_o(z')$. The right-hand side is in the ideal I, so

$$\gamma_o(zz') = \gamma_o(z) \cdot \gamma_o(z')$$

as claimed.

Weyl group invariance. Next, prove that the image of γ is inside $\mathbb{C}[H]^W$, that is, is invariant under $H \to -H$. It suffices to evaluate these polynomials on λ dominant integral, that is, on $0 \leq \lambda \in \mathbb{Z}$. [27] Let M_λ be the Verma module for λ, that is, the *universal* $\mathfrak{g}$-module with highest weight $\lambda - 1$, with highest-weight vector v. Since the ideal I annihilates v, z acts on v by its projection to $\mathbb{C}[H]$, namely $\gamma_o(z)$, which by the definition of M_λ acts on v by

$$\gamma_o(z)(\lambda - 1) = (\text{polynomial } \gamma_o(z) \text{ evaluated at } \lambda - 1)$$

Since v generates M_λ and z is in the center of $U(\mathfrak{g})$, z acts on all of M_λ by the same scalar.

From the study of intertwining operators among Verma modules above, for λ *dominant integral*[28] there exists a non-zero intertwining operator

$$M_{-\lambda} \to M_\lambda$$

So the scalar by which z acts on $M_{-\lambda}$ is the same as that by which z acts on M_λ. The scalar by which z acts on $M_{-\lambda}$ is the left-hand side of the desired equality, and the scalar by which it acts on M_λ is the right-hand side. This proves the Weyl group invariance of elements in the image of the Harish-Chandra map γ.

Surjectivity. For $\mathfrak{sl}(2)$, with $\mathfrak{h} = \mathbb{C} \cdot H$, the Weyl group acts on $U(\mathfrak{h}) = \mathbb{C}[H]$ by sending $H \to -H$. Thus, the invariants are the polynomials in H^2. Recall from above that the un-normalized Harish-Chandra map γ_o is the composite

$$Z(\mathfrak{g}) \subset U(\mathfrak{h}) \oplus U(\mathfrak{g}) \cdot X \to U(\mathfrak{h})$$

Thus, rewriting the Casimir element

$$\Omega = \frac{1}{2}H^2 + XY + YX = \frac{1}{2}H^2 + H + 2YX$$

so that it is a sum of an element of $U(\mathfrak{h})$ and an element in the ideal $U(\mathfrak{g})X$ generated by X, we see that

$$\gamma_o(\Omega) = \frac{1}{2}H^2 + H$$

[26] For the moment, one can take $\mathbb{C} \cdot H$ as the definition of *Cartan subalgebra*.

[27] Polynomials with coefficients in $\mathbb{C}$ in one variable are uniquely determined by their values at positive integers.

[28] Again, this condition is $0 < \lambda \in \mathbb{Z}$.

Then $\sigma(H) = H - 1$ sends this to

$$\gamma(\Omega) = (\sigma \circ \gamma_o)(\Omega) = \sigma(\frac{1}{2}H^2 + H) = \frac{1}{2}(H-1)^2 + (H-1) = \frac{1}{2}H^2 - H + \frac{1}{2} + H - 1 = \frac{1}{2}H^2 - \frac{1}{2}$$

Thus, $2\gamma(\Omega) + 1 = H^2$, so $\gamma(Z(\mathfrak{g}))$ includes all of

$$\mathcal{U}(\mathfrak{h})^W = \mathbb{C}[H]^W = \mathbb{C}[H^2]$$

That is, the Harish-Chandra map is *surjective*.

Injectivity. [29] Since $\mathfrak{g} \oplus \mathbb{C}$ injects to $U(\mathfrak{g})$, the renormalizing algebra homomorphism σ is an algebra injection. Thus, it suffices to show that γ_o is injective. Keep in mind that γ_o is composition

$$Z(\mathfrak{g}) \subset U(\mathfrak{h}) \oplus U(\mathfrak{g})X \to U(\mathfrak{h})$$

where the last map is the projection along the direct sum. Thus, on a highest weight vector v in a any representation, and for any $z \in Z(\mathfrak{g})$,

$$z \cdot v = \gamma_o(z) \cdot v$$

since the component of z in the left ideal $U(\mathfrak{g})X$ annihilates v. From above, all finite dimensional irreducibles for $\mathfrak{sl}(2)$ have a highest weight vector, so $\gamma_o(z) = 0$ for $z \in Z(\mathfrak{g})$ implies that z acts by 0 on any finite dimensional irreducible. By *complete reducibility* of arbitrary finite-dimensional representations of $\mathfrak{g}$, this implies that z acts by 0 on every finite dimensional representation of $\mathfrak{g}$.

We will extract enough finite-dimensional representations from the adjoint action of $\mathfrak{g}$ on $U(\mathfrak{g})$ itself to be able to prove that an element $u \in U(\mathfrak{g})$ [30] acting trivially on all these representations is 0.

Let

$$T^i(\mathfrak{g}) = \underbrace{\mathfrak{g} \otimes \ldots \otimes \mathfrak{g}}_{i}$$

be the degree i part of the tensor algebra of $\mathfrak{g}$, and

$$U^n(\mathfrak{g}) = \text{ image in } U(\mathfrak{g}) \text{ of } \oplus_{0 \le i \le n} T^i(\mathfrak{g})$$

the degree $\le n$ part [31] of the universal enveloping algebra $U(\mathfrak{g})$. Let $x \in \mathfrak{g}$ act on $v \in U(\mathfrak{g})$ by [32]

$$x \cdot v = xv - vx$$

Note that for v a monomial

$$v = X_1 X_2 \ldots X_n$$

we have [33]

$$x \cdot v = xv - vx = [x, X_1]X_2 \ldots X_n + X_1[x, X_2]X_3 \ldots X_n + \ldots + X_1 \ldots [x, X_{n-1}]X_n + X_1 \ldots Xn - 1[x, X_n]$$

This action stabilizes each $U^n(\mathfrak{g})$, so gives a well-defined action of $\mathfrak{g}$ on the quotient

$$F^n = U^n(\mathfrak{g})/U^{n-1}(\mathfrak{g})$$

[29] This part of the argument follows [Knapp 1996].

[30] It turns out that there is no great advantage to assuming that u is in the *center* of $U(\mathfrak{g})$.

[31] Since the elements $xy - yx - [x, y]$ generating the kernel of the quotient map from the tensor algebra to the enveloping algebra are not *homogeneous*, the image in $U(\mathfrak{g})$ of $T^n(\mathfrak{g})$ is not closed under multiplication.

[32] This is the extension to $U(\mathfrak{g})$ of the adjoint action of $\mathfrak{g}$ on itself.

[33] By a typical *telescoping* effect.

In the latter quotient, we can rearrange factors in a given monomial. Thus, with $a+b+c=n$,

$$x \cdot Y^a H^b X^c = a[x,Y]Y^{a-1}H^b X^c + Y^a[x,H]H^{b-1}X^c + Y^a H^b [x,X]X^{c-1} \bmod U^{n-1}(\mathfrak{g})$$

EDIT: ... complete this argument...

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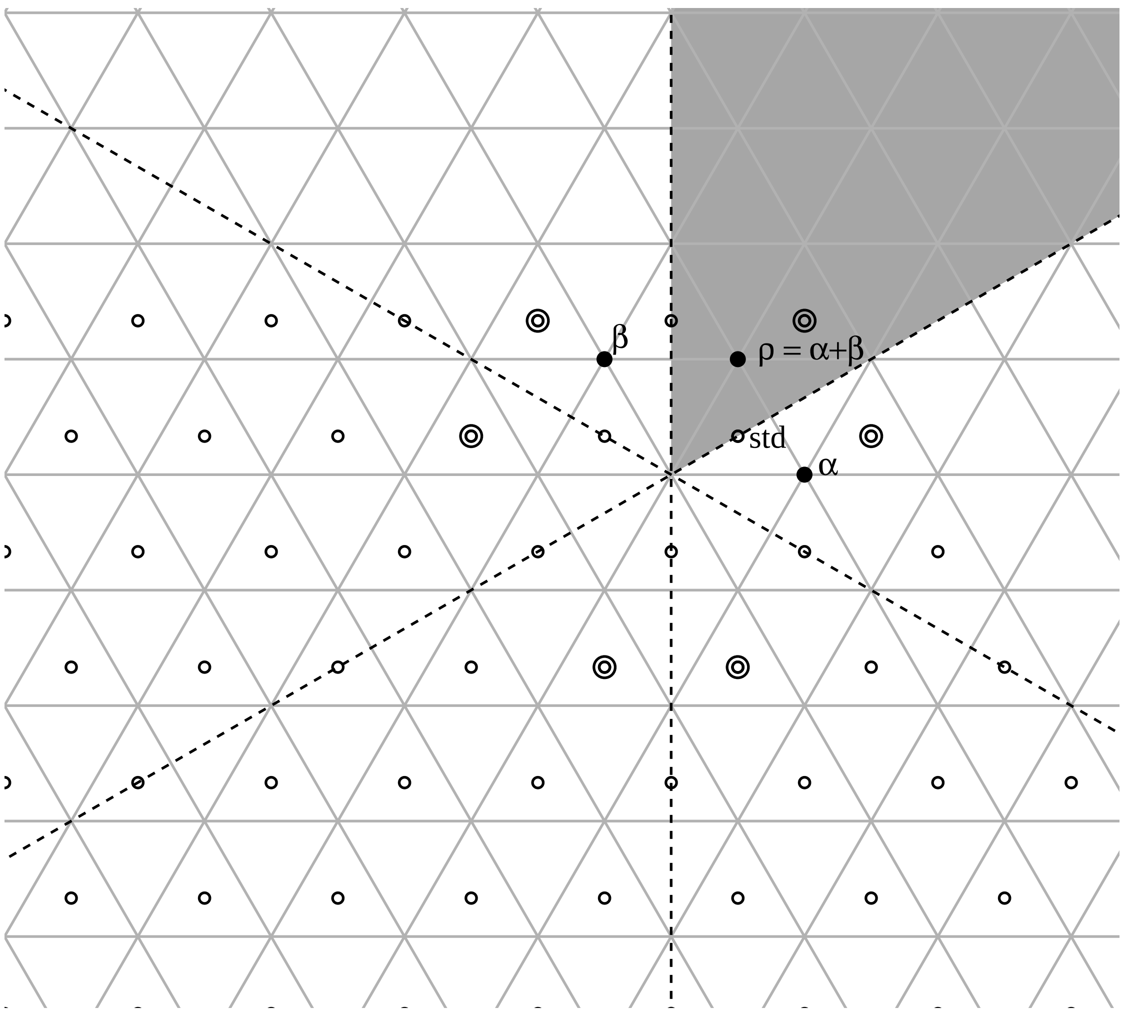
6. *Highest weights for* $\mathfrak{sl}(3)$



Now consider $\mathfrak{g} = \mathfrak{sl}(3)$, the trace-zero 3-by-3 matrices. [34] Let

$$H_\alpha = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad H_\beta = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \qquad H_{\alpha+\beta} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

[34] Which, for our purposes, may as well be *complex*, since we will only consider representations as complex-linear endomorphisms of complex vector spaces.

$$X_\alpha = \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix} \qquad X_\beta = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{pmatrix} \qquad X_{\alpha+\beta} = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ & & 0 \end{pmatrix}$$

$$Y_\alpha = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \qquad Y_\beta = \begin{pmatrix} 0 & & \\ & 0 & \\ & 1 & 0 \end{pmatrix} \qquad Y_{\alpha+\beta} = \begin{pmatrix} 0 & & \\ & 0 & \\ 1 & & 0 \end{pmatrix}$$

These are nicely related to each other via Lie brackets. For example,

$$[H_\alpha, X_\alpha] = 2X_\alpha \qquad [H_\beta, X_\beta] = 2X_\beta \qquad [H_\alpha, X_\beta] = -X_\beta \qquad [H_\beta, X_\alpha] = -X_\alpha$$

$$[H_\alpha, Y_\alpha] = -2Y_\alpha \qquad [H_\beta, Y_\beta] = -2Y_\beta \qquad [H_\alpha, Y_\beta] = Y_\beta \qquad [H_\beta, Y_\alpha] = Y_\alpha$$

$$[X_\alpha, X_\beta] = X_{\alpha+\beta} \qquad [Y_\alpha, Y_\beta] = -Y_{\alpha+\beta} \qquad H_\alpha + H_\beta = H_{\alpha+\beta}$$

$$[X_\alpha, Y_\alpha] = H_\alpha \qquad [X_\beta, Y_\beta] = H_\beta \qquad [X_{\alpha+\beta}, Y_{\alpha+\beta}] = H_{\alpha+\beta} = H_\alpha + H_\beta$$

(The sign in $[Y_\alpha, Y_\beta] = -Y_{\alpha+\beta}$ is harmless.) To organize these relations, think in terms of the eigenspaces on $\mathfrak{g}$ for H_α and H_β. Let $\mathfrak{h} = \mathbb{C}\cdot H_\alpha + \mathbb{C}\cdot H_\beta \subset \mathfrak{g}$. [35] From a linear map

$$\lambda : \mathfrak{h} \to \mathbb{C}$$

we obtain an algebra homomorphism of the (commutative) subalgebra [36]

$$\lambda : \mathbb{C}[H_\alpha, H_\beta] \approx U(\mathfrak{h}) \to \mathbb{C}$$

Of course, the pair $(\lambda H_\alpha, \lambda H_\beta)$ of complex numbers determines λ completely. Call such λ a **weight** (or **root**, if one wants, when the ambient vector space is $\mathfrak{g}$). Let $\mathfrak{g}_\lambda$ be the λ-eigenspace [37] in $\mathfrak{g}$ itself. We distinguish two important weights α and β such that for all $H \in \mathfrak{h}$

$$HX_\alpha = \alpha(H)\cdot X_\alpha$$

$$HX_\beta = \beta(H)\cdot X_\alpha$$

namely,

$$\alpha(H_\alpha) = 2 \qquad \alpha(H_\beta) = -1$$

$$\beta(H_\alpha) = -1 \qquad \beta(H_\beta) = 2$$

Then

$$\begin{array}{ccccc} \alpha \text{ eigenspace} & = & \mathbb{C}\cdot X_\alpha & = & \mathfrak{g}_\alpha \\ \beta \text{ eigenspace} & = & \mathbb{C}\cdot X_\beta & = & \mathfrak{g}_\beta \\ (\alpha+\beta) \text{ eigenspace} & = & \mathbb{C}\cdot X_{\alpha+\beta} & = & \mathfrak{g}_{\alpha+\beta} \\ -\alpha \text{ eigenspace} & = & \mathbb{C}\cdot Y_\alpha & = & \mathfrak{g}_{-\alpha} \\ -\beta \text{ eigenspace} & = & \mathbb{C}\cdot Y_\beta & = & \mathfrak{g}_{-\beta} \\ -(\alpha+\beta) \text{ eigenspace} & = & \mathbb{C}\cdot Y_{\alpha+\beta} & = & \mathfrak{g}_{-\alpha-\beta} \\ 0 \text{ eigenspace} & = & \mathbb{C}\cdot H_\alpha + \mathbb{C}\cdot H_\beta & = & \mathfrak{g}_0 \end{array}$$

[35] Since $H_{\alpha+\beta} = H_\alpha + H_\beta$ any eigenvector for both H_α and H_β will be an eigenvector for $H_{\alpha+\beta}$.

[36] Poincaré-Birkhoff-Witt implies that the universal enveloping algebra of a Lie subalgebra *injects* to the universal enveloping algebra of the larger Lie algebra.

[37] Also called **λ-rootspace** because it is in $\mathfrak{g}$. It is an instance of a **weight space**.

The list of eigenvalues is the list of **roots** of $\mathfrak{g}$ (with respect to $\mathfrak{h} = \mathbb{C}\cdot H_\alpha + \mathbb{C}\cdot H_\beta$). The **positive simple roots** are [38] α and β. [39] The other positive (non-simple) root is $\alpha+\beta$. The *negative* roots are the negatives of the positive ones. A useful elementary structural fact is that

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$$

which allows us to anticipate the vanishing of many Lie brackets without re-doing matrix computations. The proof in this context is direct: $hx - xh = \lambda x$ and $hy - yh = \mu y$ imply that

$$[h,[x,y]] = h[x,y] - [x,y]h = h(xy-yx) - (xy-yx)h = hxy - hyx - xyh + yxh$$

$$= [h,x]y + xhy - [h,y]x - yhx - xyh + yxh = [h,x]y + x[h,y] - [h,y]x - y[h,x]$$

$$= [[h,x],y] + [x,[h,y]] = [\lambda x, y] + [x, \mu y] = (\lambda+\mu)\cdot[x,y]$$

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Consider *finite-dimensional* $U(\mathfrak{g})$ modules V. Since $[H_\alpha, H_\beta] = 0$, and by the finite-dimensionality, there is at least one eigenvector $v \neq 0$ for both H_α and H_β. Let $\lambda : \mathfrak{h} \to \mathbb{C}$ be the eigenvalue (**weight**) of v. Since $[H_\alpha, X_\alpha] = 2X_\alpha$ and $[H_\beta, X_\alpha] = -X_\alpha$

$$H_\alpha\cdot(X_\alpha\cdot v) = ([H_\alpha, X_\alpha] + X_\alpha H_\alpha)\cdot v = (2X_\alpha + X_\alpha H_\alpha)\cdot v = (\alpha(H_\alpha)\cdot X_\alpha + X_\alpha H_\alpha)\cdot v$$

$$= \alpha(H_\alpha)\cdot X_\alpha v + X_\alpha(\lambda(H_\alpha)\cdot v) = \alpha(H_\alpha)\cdot X_\alpha v + \lambda(H_\alpha)\cdot X_\alpha v = (\alpha+\lambda)(H_\alpha)\cdot X_\alpha v$$

$$H_\beta\cdot(X_\alpha\cdot v) = ([H_\beta, X_\alpha] + X_\alpha H_\beta)\cdot v = (-H_\beta\cdot X_\alpha + X_\alpha H_\beta)\cdot v = (\alpha(H_\beta)\cdot X_\alpha + X_\alpha H_\beta)\cdot v$$

$$= \alpha(H_\beta)\cdot X_\alpha v + X_\alpha(\lambda(H_\beta)\cdot v) = \alpha(H_\beta)\cdot X_\alpha v + \lambda(H_\beta)\cdot X_\alpha v = (\alpha+\lambda)(H_\beta)\cdot X_\alpha v$$

Similarly, [40]

$$H_\alpha\cdot(X_\beta\cdot v) = (\beta+\lambda)(H_\alpha)\cdot X_\beta v$$

$$H_\beta\cdot(X_\beta\cdot v) = (\beta+\lambda)(H_\beta)\cdot X_\beta v$$

and

$$H_\alpha\cdot(X_{\alpha+\beta}\cdot v) = (\alpha+\beta+\lambda)(H_\alpha)\cdot X_\beta v$$

$$H_\beta\cdot(X_{\alpha+\beta}\cdot v) = (\alpha+\beta+\lambda)(H_\beta)\cdot X_\beta v$$

Indeed, with just a little hindsight, we see that we have, for *any* $H \in \mathfrak{h} = \mathbb{C}\cdot H_\alpha + \mathbb{C}\cdot H_\beta$,

$$\begin{aligned} H\cdot(X_\alpha\cdot v) &= (\alpha+\lambda)(H)\cdot X_\alpha v \\ H\cdot(X_\beta\cdot v) &= (\beta+\lambda)(H)\cdot X_\beta v \\ H\cdot(X_{\alpha+\beta}\cdot v) &= (\alpha+\beta+\lambda)(H)\cdot X_{\alpha+\beta} v \end{aligned}$$

By finite-dimensionality, there is an eigenvector $v \neq 0$ for H_α and H_β which is *annihilated* by X_α, X_β, and $X_{\alpha+\beta}$. This v is a **highest-weight vector** in V.

Poincaré-Birkhoff-Witt says that the monomials

$$Y_\alpha^a\, Y_\beta^b\, Y_{\alpha+\beta}^c\, H_\alpha^d\, H_\beta^e\, X_\alpha^f\, X_\beta^g\, X_{\alpha+\beta}^h$$

[38] In addition to the choice of abelian subalgebra $\mathfrak{h}$ with respect to which to decompose, there are additional choices needed to determine *positive* roots. For the moment, we will not worry about this.

[39] Our indexing of X_α, X_β, $X_{\alpha+\beta}$, Y_α, Y_β, and $Y_{\alpha+\beta}$ in terms of α and β makes perfect sense insofar as the subscript tells in what weight-space in $\mathfrak{g}$ they lie. The indexing for H_α and H_β is best explained by the subsequent discussion, though at least we can say that $[X_\alpha, Y_\alpha] = H_\alpha$.

[40] One might note that the way we've written these identities the redundancy of separate consideration of H_α and H_β becomes visible. Still, it is convenient to have named these elements of $\mathfrak{h} \subset \mathfrak{g}$.

(with non-negative integers a, b, c, d, e, f, g, h) are a $\mathbb{C}$-basis for $U(\mathfrak{g})$, so, given a highest-weight vector v in V, the $U(\mathfrak{g})$ submodule generated by v is *spanned* by vectors

$$\{Y_\alpha^a\, Y_\beta^b\, Y_{\alpha+\beta}^c\, v:\ a, b, c \geq 0\}$$

The weight (eigenvalue) of $Y_\alpha^a Y_\beta^b Y_{\alpha+\beta}^c v$ is

$$\text{weight}\ \left(Y_\alpha^a\, Y_\beta^b\, Y_{\alpha+\beta}^c\, v\right) = (\text{weight}\ v) - a\cdot\alpha - b\cdot\beta - c(\alpha+\beta) = (\text{weight}\ v) - (a+c)\cdot\alpha - (b+c)\cdot\beta$$

Thus, by finite-dimensionality, both $a+c$ and $b+c$ are bounded, and therefore only finitely-many of these vectors can be non-zero. [41]

At this point we exploit the fact that the triples $H_\alpha, X_\alpha, Y_\alpha$ and $H_\beta, X_\beta, Y_\beta$ are copies of the corresponding basis elements for $\mathfrak{sl}(2)$, for which we have decisive computational results. [42] In particular, let a be the least non-negative integer such that $Y_\alpha^a v \neq 0$ but $Y_\alpha^{a+1} v = 0$. Let $\lambda = (\lambda H_\alpha, \lambda H_\beta)$ be the weight of v. Then [43]

$$\begin{aligned} 0 = X_\alpha \cdot 0 = X_\alpha \cdot Y_\alpha^{a+1} v &= ([X_\alpha, Y_\alpha] + Y_\alpha X_\alpha = 0pt2pt) \cdot Y_\alpha^a v = (H_\alpha + Y_\alpha X_\alpha)\cdot Y_\alpha^a v \\ &= (\lambda H_\alpha - 2a)\cdot Y_\alpha^a v + Y_\alpha (X_\alpha \cdot Y_\alpha\, Y_\alpha^{a-1} v) \end{aligned}$$

Do an induction to re-express $X_\alpha \cdot Y_\alpha^a v$ by moving the X_α across the Y_αs

$$X_\alpha Y_\alpha v = [X_\alpha, Y_\alpha] v + Y_\alpha X_\alpha v = H_\alpha v + 0 = \lambda H_\alpha \cdot v$$

$$X_\alpha Y_\alpha^2 = (X_\alpha Y_\alpha) Y_\alpha v = (H_\alpha + Y_\alpha X_\alpha) Y_\alpha v = (\lambda H_\alpha - 2) Y_\alpha v + Y_\alpha (\lambda H_\alpha \cdot v) = (2\lambda H_\alpha - 2) Y_\alpha v$$

$$X_\alpha Y_\alpha^3 = (X_\alpha Y_\alpha) Y_\alpha^2 v = H_\alpha Y_\alpha^2 v + Y_\alpha (X_\alpha Y_\alpha^2 v) = (\lambda H_\alpha - 2\cdot 2) Y_\alpha^2 v + Y_\alpha ((2\lambda H_\alpha - 2)\cdot v) = (3\lambda H_\alpha - 6) Y_\alpha^2 v$$

$$X_\alpha Y_\alpha^4 = (X_\alpha Y_\alpha) Y_\alpha^3 v = H_\alpha Y_\alpha^3 v + Y_\alpha (X_\alpha Y_\alpha^3 v) = (\lambda H_\alpha - 6) Y_\alpha^3 v + Y_\alpha ((3\lambda H_\alpha - 6)\cdot v) = (4\lambda H_\alpha - 12) Y_\alpha^3 v$$

Generally, by induction,

$$X_\alpha Y_\alpha^\ell v = \ell \left(\lambda H_\alpha - (\ell - 1)\right) Y_\alpha^{\ell-1} v$$

Thus, for $Y_\alpha^{a+1} v = 0$ and $Y_\alpha^a v \neq 0$ it must be that

$$\lambda H_\alpha - a = 0$$

This is a duplicate of the $\mathfrak{sl}(2)$ computation.

Similarly, let b be the smallest positive integer such that $Y_\beta^{b+1} v = 0$ but $Y_\beta^b v \neq 0$. Then (using X_β in the role of X_α in the previous computation)

$$\lambda H_\beta - b = 0$$

That is, the highest weight λ has

$$(\lambda H_\alpha, \lambda H_\beta) = (a, b)$$

and thus is completely determined by the structural constants $0 \leq a \in \mathbb{Z}$ and $0 \leq b \in \mathbb{Z}$. [44]

[41] At this point we cannot presume any linear independence that does not follow from eigenvalue (weight) considerations. When we look at the *free* module with specified highest weight, we *can* presume linear independence from the corresponding linear independence in $U(\mathfrak{g})$, from Poincaré-Birkhoff-Witt.

[42] The marvel is that a systematic exploitation of this is adequate to understand so much about representation theory of larger Lie algebras. Perhaps one demystification is that it is exactly the semi-simple Lie algebras, whose exemplar $\mathfrak{sl}(3)$ we consider here, that yield to this sort of analysis.

[43] This is a reprise of the argument for $\mathfrak{sl}(2)$, with notation trivially adapted to the present context.

[44] The argument so far shows that these integrality and positivity conditions are *necessary* for existence of a finite-dimensional representation with highest weight (a, b).

[6.0.1] Remark: It seems difficult to do much further in this primitive context. The next section recasts these ideas into a more effective and useful form.

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7. *Verma modules for* $\mathfrak{sl}(3)$

As done earlier for $\mathfrak{sl}(2)$, we consider the *universal* $U(\mathfrak{g})$-module with a highest-weight vector v with eigenvalue $\lambda : \mathfrak{h} \to \mathbb{C}$. As above, λ is completely determined by its values λH_α, λH_β. [45] We carry out the discussion first in a *naive* normalization, and then renormalize when we see how better to describe the symmetries which become visible.

There are two constructions. First, given complex numbers λH_α and λH_β, let I_λ^{naive} be the left ideal in $U(\mathfrak{g})$ generated by X_α, X_β, $X_{\alpha+\beta}$, and $H_\alpha - \lambda H_\alpha$ and $H_\beta - \lambda H_\beta$. Let

$$M_\lambda^{\text{naive}} = U(\mathfrak{g})/I_\lambda^{\text{naive}}$$

Or let $\mathbf{b}$ be the Borel subalgebra of $\mathfrak{g}$ generated by H_α, H_β, X_α, X_β, and $X_{\alpha+\beta}$, with one-dimensional representation $\mathbb{C}_\lambda$ of $\mathbf{b}$ on $\mathbb{C}$ with X_α, X_β, and $X_{\alpha+\beta}$ acting by 0, and H_α acting by λH_α and H_β acting by λH_β. Then define

$$M_\lambda^{\text{naive}} = U(\mathfrak{g}) \otimes_{U(\mathbf{b})} \mathbb{C}_\lambda$$

The image of 1 in the first construction, or, equivalently, the image of $1 \otimes 1$ in the second construction, is the highest weight vector $v \neq 0$ (annihilated by X_α, X_β, and $X_{\alpha+\beta}$). In either construction, it is clear that, by Poincaré-Birkhoff-Witt, the vectors

$$Y_\alpha^a \, Y_\beta^b \, Y_{\alpha+\beta}^c \, v$$

are a *basis* for M_λ^{naive}, with weights

$$\lambda - a\alpha - b\beta - c(\alpha+\beta)$$

[7.0.1] Proto-Theorem: Let λ be a weight. For $0 \leq a \in \mathbb{Z}$

$$\text{Hom}_{U(\mathfrak{g})}(M_{\lambda-(a+1)\alpha}^{\text{naive}}, M_\lambda^{\text{naive}}) \neq 0 \quad \text{if and only if} \quad \lambda H_\alpha = a$$

Similarly, for $0 \leq b \in \mathbb{Z}$,

$$\text{Hom}_{U(\mathfrak{g})}(M_{\lambda-(b+1)\beta}^{\text{naive}}, M_\lambda^{\text{naive}}) \neq 0 \quad \text{if and only if} \quad \lambda H_\beta = b$$

Proof: Let v be a highest-weight vector in M_λ^{naive}. From our observations of the consequences of Poincaré-Birkhoff-Witt, the only way a weight $\lambda - (a+1)\alpha$ occurs is as $Y_\alpha^{a+1} v$. The computation of the previous section shows that $X_\alpha Y_\alpha^{a+1} v = 0$ if and only if $\lambda H_\alpha = a$. The assertion for β follows similarly. What remains is to check that X_β and $X_{\alpha+\beta}$ also annihilate $Y_\alpha^{a+1} v$ if X_α does (and, similarly, that X_α and $X_{\alpha+\beta}$ annihilate $Y_\beta^{b+1} v$ if X_β does).

Now $[X_\alpha, Y_\beta] = 0$ since there are no $(\alpha-\beta)$-weight elements in $\mathfrak{g}$, that is, since

$$\mathfrak{g}_{\alpha-\beta} = 0$$

This is good, since, in other words, X_β and Y_α *commute*, and

$$X_\beta \cdot Y_\alpha^{a+1} v = Y_\alpha^{a+1} \cdot X_\beta v = Y_\alpha^a \cdot 0 = 0$$

[45] As for $\mathfrak{sl}(2)$, we observe that the usual category-theoretic argument prove that there is at most one such thing, up to unique isomorphism. Thus, the remaining issue is a construction. The categorical uniqueness argument promises us that our particular choice of construction does not affect the outcome.

And, further,

$$X_{\alpha+\beta} \cdot Y_\alpha^{a+1} v = [X_\alpha, X_\beta] \cdot Y_\alpha^{a+1} v = (X_\alpha X_\beta - X_\beta X_\alpha) \cdot Y_\alpha^{a+1} v$$

$$= X_\alpha Y_\alpha^{a+1} X_\beta v - X_\beta (a+1)(\lambda H_\alpha - a) Y_\alpha^a v = 0 - (a+1)(\lambda H_\alpha - a) Y_\alpha^a X_\beta v = 0$$

since we already know that $X_\alpha v = 0$ (and $X_\beta v = 0$). Thus, $Y_\alpha^{a+1} v$ is a highest-weight vector and has weight $\lambda - (a+1)\alpha$. By the universality of $M^{\text{naive}}_{\lambda-(a+1)\alpha}$, there is a unique (up to constant) non-zero map of it to M^{naive}_λ sending the highest-weight vector in $M^{\text{naive}}_{\lambda-(a+1)\alpha}$ to $Y_\alpha^{a+1} v$. A similar argument applies to β. ///

[7.0.2] Remark: As for $\mathfrak{sl}(2)$, we see that there is room for improvement of the normalization of the parametrization of the universal modules. That is, the more attractively normalized **Verma module** M_λ is the universal $U(\mathfrak{g})$-module with highest weight described by

$$H_\alpha \cdot v = \Big(\lambda(H_\alpha) - 1\Big) \cdot v$$

$$H_\beta \cdot v = \Big(\lambda(H_\beta) - 1\Big) \cdot v$$

Solving for $\rho = a\alpha + b\beta$ such that

$$(\lambda - \rho)H_\alpha = \lambda(H_\alpha) - 1 \qquad (\lambda - \rho)H_\beta = \lambda(H_\beta) - 1$$

gives the system

$$2a - b = 1 \qquad -a + 2b = 1$$

which gives $a = b = 1$, so

$$\rho = \alpha + \beta$$

Thus, the highest weight of the (renormalized) Verma module M_λ should be $\lambda - (\alpha + \beta)$. Thus, restating, [46]

[7.0.3] Proto-Theorem: Let M_λ be the universal highest-weight module with highest weight $\lambda - (\alpha + \beta)$. For $1 \le a \in \mathbb{Z}$

$$\text{Hom}_{U(\mathfrak{g})}(M_{\lambda - a\alpha}, M_\lambda) \neq 0 \quad \text{if and only if} \quad \lambda H_\alpha = a$$

Similarly, for $1 \le b \in \mathbb{Z}$,

$$\text{Hom}_{U(\mathfrak{g})}(M_{\lambda - b\beta}, M_\lambda) \neq 0 \quad \text{if and only if} \quad \lambda H_\beta = b$$

Proof: The thing to check is the effect of the renormalization. Let λ_{old} be the old λ, which is exactly the highest weight, and

$$\lambda_{\text{new}} = \lambda_{\text{old}} - \rho = \lambda_{\text{old}} - (\alpha + \beta)$$

Then the condition

$$\lambda_{\text{old}} H_\alpha = a$$

becomes

$$(\lambda_{\text{new}} - \alpha - \beta) H_\alpha = a$$

Since $\alpha H_\alpha = 2$ and $\beta H_\alpha = -1$, this is

$$\lambda_{\text{new}} H_\alpha = a + 1$$

Thus, replace a by $a - 1$ and b by $b - 1$ in the statement of the previous proto-theorem. ///

And, also, unlike the $\mathfrak{sl}(2)$ case, there is a perhaps prior issue of coordinates to use on functionals on $\mathfrak{h} = \mathbb{C} \cdot H_\alpha + \mathbb{C} \cdot H_\beta$. Since the actions of Y_α, Y_β, and $Y_{\alpha+\beta}$ on weights are naturally described in terms of

[46] But this is still unfinished.

the simple roots [47] α and β, it might be reasonable to think of λ as a linear combination of α and β, rather than (as above) telling its effect on H_α and H_β. [48] Even better, with a reasonable non-degenerate bilinear form on $\mathfrak{h}$, we could use it to identify linear functionals with elements of the space itself. Take

$$\langle x, y\rangle = \mathrm{tr}(xy)$$

gives a non-degenerate bilinear form on $\mathfrak{g}$, with natural properties we can exploit. Restricting $\langle,\rangle$ to $\mathfrak{h}$ is the bilinear form we want. [49] Notice that [50]

$$\langle H_\alpha, H_\alpha\rangle = 2 \qquad \langle H_\beta, H_\beta\rangle = 2 \qquad \langle H_\alpha, H_\beta\rangle = -1$$

The small miracle, whether one views it as pre-arranged or a natural accident, is that α and H_α are naturally identified via $\langle,\rangle$, as are β and H_β, because

$$\alpha H_\alpha = 2 = \langle H_\alpha, H_\alpha\rangle \qquad \alpha H_\beta = -1 = \langle H_\beta, H_\alpha\rangle$$

(and symmetrically in α and β). Thus, via the duality given by $\langle,\rangle$, we may often identify $\alpha \sim H_\alpha$ and $\beta \sim H_\beta$. [51]

The geometry of this non-degenerate form is respected by the **Weyl group** W here consisting of all 3-by-3 permutation matrices, acting on $\mathfrak{g}$ by conjugation [52]

$$w(x) = wxw^{-1}$$

The $\mathfrak{sl}(2)$ versions of this associated to the simple roots α and β are

$$s_\alpha = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \qquad s_\beta = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

which lie in W. Indeed, it is an elementary fact that W is generated by these two elements. Note that, first, as expected,

$$s_\alpha H_\alpha = -H_\alpha \qquad s_\beta H_\beta = -H_\beta$$

but that the other interactions are not trivial:

$$s_\alpha H_\beta = H_{\alpha+\beta} = H_\alpha + H_\beta \qquad s_\beta H_\alpha = H_{\alpha+\beta} = H_\alpha + H_\beta$$

If we make the identifications $\alpha = H_\alpha$ and $\beta = H_\beta$, via $\langle,\rangle$, this is [53]

$$s_\alpha\alpha = -\alpha \qquad s_\beta\beta = -\beta \quad \text{and} \quad s_\alpha\beta = \alpha+\beta \qquad s_\beta\alpha = \alpha+\beta$$

[47] Though we have not given a definition of what *simple roots* are in general, these α and β would be seen to meet that definition, and in any case we can call them by this name.

[48] The latter approach in effect expresses λ in terms of a dual basis H_α^*, H_β^* to H_α, H_β.

[49] As for $\mathfrak{sl}(2)$, this form is a constant multiple of the more intrinsically defined *Killing form*, but we do not need to worry about this.

[50] The fact that these good choices of H_α and H_β have the same lengths $\langle H_\alpha, H_\alpha\rangle$, $\langle H_\beta, H_\beta\rangle$ is sometimes coyly expressed by saying that $\mathfrak{g}$ is *simply-laced*. I do not know any explanation of this terminology.

[51] In any case, using such an identification is traditional.

[52] More intrinsically, again, this is the *Ad* adjoint action.

[53] There is the minor economy of symbols achieved by replacing H_α by α, and so on, which probably doesn't hurt anything as long as we remember the duality.

Yet further,[54] these order-two linear maps s_α and s_β are expressible as **reflections**[55]

$$s_\alpha(x) = x - \frac{2\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle} \cdot \alpha$$

With this apparatus available, and having renormalized the parametrization as already suggested in the $\mathfrak{sl}(2)$ case, we rewrite the proto-theorem again, to

[7.0.4] Theorem:

$$\mathrm{Hom}_{U(\mathfrak{g})}(M_{s_\alpha \lambda}, M_\lambda) \neq 0 \quad \text{if and only if} \quad 0 < \langle \lambda, \alpha\rangle \in \mathbb{Z}$$

Similarly, and symmetrically,

$$\mathrm{Hom}_{U(\mathfrak{g})}(M_{s_\beta \lambda}, M_\lambda) \neq 0 \quad \text{if and only if} \quad 0 < \langle \lambda, \beta\rangle \in \mathbb{Z}$$

A weight λ meeting the conditions

$$0 < \langle \lambda, \alpha\rangle \in \mathbb{Z} \quad \text{and} \quad 0 < \langle \lambda, \beta\rangle \in \mathbb{Z}$$

is **dominant integral**.[56]

Proof: The condition $\lambda H_\alpha = a$ of the last proto-theorem becomes $\langle \lambda, \alpha\rangle = a$ if we use the pairing *and* the identification of H_α and α.

$$\lambda - a\alpha = \lambda - \frac{2\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \cdot \alpha = s_\alpha \lambda$$

Similarly for β. ///

[7.0.5] Remark: The thing to reflect upon is the possibility of iterating the reflections. The somewhat enhanced notational set-up (and renormalization) make this issue far more palatable than in the more raw initial version.

For example, suppose that λ is a dominant integral weight, that is, such that *both*

$$0 < \langle \lambda, \alpha\rangle \in \mathbb{Z} \quad \text{and} \quad 0 < \langle \lambda, \beta\rangle \in \mathbb{Z}$$

What can be said in this vein about $s_\alpha \lambda$? Using the convenient fact that s_α is an isometry, we have

$$\langle s_\alpha \lambda, \alpha\rangle = \langle \lambda, s_\alpha \alpha\rangle = -\langle \lambda, \alpha\rangle < 0$$

But

$$\langle s_\alpha \lambda, \beta\rangle = \langle \lambda, s_\alpha \beta\rangle = \langle \lambda, \alpha + \beta\rangle = \langle \lambda, \alpha\rangle + \langle \lambda, \beta\rangle > 0$$

and integrality is not lost. Thus, with *both* conditions met, still

$$\mathrm{Hom}_{U(\mathfrak{g})}(M_{s_\beta s_\alpha \lambda},\, M_{s_\alpha \lambda}) \neq 0$$

Symmetrically,

$$\mathrm{Hom}_{U(\mathfrak{g})}(M_{s_\alpha s_\beta \lambda},\, M_{s_\beta \lambda}) \neq 0$$

[54] Whether by luck or skill.

[55] We do not need any fancier definition of *reflection* than that the reflection s through the plane orthogonal to a vector v in $\mathbb{C}\alpha + \mathbb{C}\beta$ is given by the indicated formula. Whether by luck or by skill, the restriction of $\langle , \rangle$ to $\mathbb{R}H_\alpha + \mathbb{R}H_\beta$ is positive definite, so the usual style of discussion of the elementary geometry is not misleading.

[56] *Integral* for the fact that the inner product values are in $\mathbb{Z}$, and *dominant* for the fact that the values are positive.

And, then, by composing homomorphisms $M_{s_\alpha s_\beta \lambda} \to M_{s_\beta \lambda}$ with $M_{s_\beta \lambda} \to M_\lambda$, for λ dominant integral we have

$$\mathrm{Hom}_{U(\mathfrak{g})}(M_{s_\beta s_\alpha \lambda},\, M_\lambda) \neq 0$$

Continuing,

$$\langle s_\beta s_\alpha \lambda, \alpha\rangle = \langle s_\alpha \lambda, s_\beta \alpha\rangle = \langle s_\alpha \lambda, \alpha+\beta\rangle = \langle \lambda, s_\alpha \alpha + s_\alpha \beta\rangle = \langle \lambda, -\alpha+\alpha+\beta\rangle = \langle \lambda, \beta\rangle > 0$$

(and integrality still holds), so dominant integral λ gives

$$\mathrm{Hom}_{U(\mathfrak{g})}(M_{s_\alpha s_\beta s_\alpha \lambda},\, M_{s_\beta s_\alpha \lambda}) \neq 0$$

Similarly, [57] for dominant integral λ

$$\mathrm{Hom}_{U(\mathfrak{g})}(M_{s_\beta s_\alpha s_\beta \lambda},\, M_{s_\alpha s_\beta \lambda}) \neq 0$$

But

$$\langle s_\alpha s_\beta s_\alpha \lambda, \beta\rangle = \langle \lambda, s_\alpha s_\beta s_\alpha \beta\rangle = \langle \lambda, s_\alpha s_\beta(\alpha+\beta)\rangle = \langle \lambda, s_\alpha(\alpha+\beta-\beta)\rangle = \langle \lambda, -\alpha\rangle < 0$$

and similarly

$$\langle s_\beta s_\alpha s_\beta \lambda, \alpha\rangle < 0$$

so we obtain no further implied existence of non-trivial homomorphisms $M_{w\lambda} \to M_\lambda$ for $w \in W$.

But we have many non-trivial homomorphisms already. Indeed, it is elementary that

$$W = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha\}$$

Thus, we have

[7.0.6] Corollary: For λ dominant integral, for each $w \in W$

$$\mathrm{Hom}_{U(\mathfrak{g})}(M_{w\lambda}, M_\lambda) \neq 0$$

We should look at the W-orbits on weights λ. It is already clear that if λ is *integral* then any image $w\lambda$ is integral.

Less trivial is the issue of dominance. Restrict attention to weights λ such that $\lambda H_\alpha \in \mathbb{R}$ and $\lambda H_\beta \in \mathbb{R}$. The **positive Weyl chamber** is

$$C = \text{ positive chamber } = \{\lambda : \langle \lambda, \alpha\rangle > 0 \text{ and } \langle \lambda, \alpha\rangle > 0\}$$

[7.0.7] Proposition: Given a weight λ such that $\lambda H_\alpha \in \mathbb{R}$ and $\lambda H_\beta \in \mathbb{R}$, there is $w \in W$ such that $w\lambda$ is in the closure $\overline{C}$ (in $\mathbb{R}^2$) of the positive chamber. If $w\lambda$ is in the interior of C, then w is uniquely determined by this condition.

[7.0.8] Remark: In the particular example of $\mathfrak{sl}(3)$, one can prove this directly in a pedestrian if *ad hoc* manner. A pictorial argument seems to suffice. By contrast, an argument sufficiently general to apply to $\mathfrak{sl}(n)$ would require more preparation is desirable at the moment.

[7.0.9] Remark: We have shown that for dominant integral λ *at least* the Verma modules $M_{w\lambda}$ for $w \in W$ have non-trivial homomorphisms to M_λ. We have *not* proven that no *other* highest weights occur. For $\mathfrak{sl}(2)$

[57] Though the perspicacious reader will already have noted that $s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$, so the gain in the following observation is not so much that another submodule of M_λ is uncovered, but that a different chain of inclusions, via $M_{s_\alpha s_\beta \lambda}$, is found.

the picture was so simple that this was clear, but already for $\mathfrak{sl}(3)$ it is mildly implausible that we will try to compute things explicitly. Instead, as in the next section, we use ideas of Harish-Chandra and approach this issue indirectly.

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8. *Harish-Chandra homomorphism for* $\mathfrak{sl}(3)$

By Poincaré-Birkhoff-Witt the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{sl}(3)$ is spanned by monomials

$$Y_\alpha^a Y_\beta^b Y_{\alpha+\beta}^c H_\alpha^d H_\beta^e H_{\alpha+\beta}^f X_\alpha^g X_\beta^h X_{\alpha+\beta}^i \qquad (a, b, c, d, e, f, g, h, i \text{ non-negative integers})$$

Let ρ be a representation of $\mathfrak{g}$ with highest weight, with (non-zero) highest weight vector v. On one hand, a monomial in $U(\mathfrak{g})$ as above acts on the highest weight vector v by *annihilating* it if any X_α, X_β, or $X_{\alpha+\beta}$ actually occurs. The elements of of $U(\mathfrak{h}) = \mathbb{C}[H_\alpha, H_\beta, H_{\alpha+\beta}]$ act on v by the (multiplicative extension of) the highest weight.

We claim that if a sum of monomials is in the *center* $Z(\mathfrak{g})$ of $U(\mathfrak{g})$, then in each summand none of Y_α, Y_β, yc occurs unless some $X_\alpha, X_\beta, X_{\alpha+\beta}$ also occurs.

Using this claim (cast as the lemma just below), and using the fact (Schur's lemma) that the center $Z(\mathfrak{g})$ acts by scalars on an irreducible of $\mathfrak{g}$, we can compute the *eigenvalues* of elements $z \in Z(\mathfrak{g})$ on an irreducible with highest weight by evaluating z on the highest weight vector, and (by the claim) the values depend only upon the effect of elements $H_\alpha, H_\beta, H_{\alpha+\beta}$ of $\mathfrak{h}$ on v. (Poincaré-Birkhoff-Witt implies that $U(\mathfrak{h})$ imbeds into $U(\mathfrak{g})$.)

[8.0.1] Lemma: If a linear combination z of monomials (as above) lies in $Z(\mathfrak{g})$ then in every monomial where some ya, yb, or $Y_{\alpha+\beta}$ occurs some one of X_α, X_β, or $X_{\alpha+\beta}$ occurs.

Proof: For $H \in \mathfrak{h}$, each such monomial is an eigenvector for ad$H$, with eigenvalue

$$-a\alpha(H) - b\beta(H) - c(\alpha+\beta)(H) + g\alpha(H) + h\beta(H) + i(\alpha+\beta)(H)$$

Because $(\mathrm{ad}H)z = 0$, for each monomial

$$-a\alpha(H) - b\beta(H) - c(\alpha+\beta)(H) + g\alpha(H) + h\beta(H) + i(\alpha+\beta)(H)$$

for all $H \in \mathfrak{h}$ since these monomials are linearly independent (by Poincaré-Birkhoff-Witt). Since H_α and H_β are a basis for the vectorspace $\mathfrak{h}$, the corresponding equalities for $H = H_\alpha$ and $H = H_\beta$ imply

$$-a\alpha - b\beta - c(\alpha+\beta) + g\alpha + h\beta + i(\alpha+\beta) = 0$$

Then it is a weak conclusion that if any of $Y_\alpha, Y_\beta, Y_{\alpha+\beta}$ appears then one of $X_\alpha, X_\beta, X_{\alpha+\beta}$ appears. ///

[8.0.2] Corollary: The eigenvalues of elements of the center $Z(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ on an irreducible representation V of $\mathfrak{g}$ with a highest weight λ are completely determined by λ.

Proof: By Schur's lemma, an element z of the center $Z(\mathfrak{g})$ acts on an irreducible by a scalar $c(z)$. To determine the scalar it suffices to compute $zv = c(v) \cdot v$ for a highest weight vector. By the lemma, for $z \in Z(\mathfrak{g})$, expressed as a sum of monomials as above, every monomial not in $U(\mathfrak{h})$ has one of$X_\alpha, X_\beta, X_{\alpha+\beta}$ occuring in it, and so annihilates v. Any monomial

$$M = H_\alpha^d \, H_\beta^e \, H_{\alpha+\beta}^f$$

in $U(\mathfrak{h})$ acts on the highest weight vector v by a scalar

$$\lambda(H_\alpha)^d \, \lambda(H_\beta)^e \, \lambda(H_{\alpha+\beta})^f = \lambda(H_\alpha)^d \, \lambda(H_\beta)^e \, \left(\lambda(H_\alpha) + \lambda(H_\beta)\right)^f$$

Not at all claiming that this value is the same for every such monomial occurring in z, nevertheless the constant $c(z)$ such that $zv = c(z) \cdot v$ is completely determined by λ. ///

As above, let

$$\mathfrak{h} = \mathbb{C} \cdot H_\alpha + \mathbb{C} \cdot H_\beta + \mathbb{C} \cdot H_{\alpha+\beta}$$

and also

$$\mathfrak{n}_- = \mathbb{C} \cdot Y_\alpha + \mathbb{C} \cdot Y_\beta + \mathbb{C} \cdot Y_{\alpha+\beta}$$

$$\mathfrak{n}_+ = \mathbb{C} \cdot X_\alpha + \mathbb{C} \cdot X_\beta + \mathbb{C} \cdot X_{\alpha+\beta}$$

Let

$$I = U(\mathfrak{g}) \cdot \mathfrak{n}_+ = \{\sum_i u_j\, X_j : u_j \in U(\mathfrak{g}), X_j \in \mathfrak{n}_+\}$$

be the left ideal in $U(\mathfrak{g})$ generated by X_α, X_β, $X_{\alpha+\beta}$, that is, by $\mathfrak{n}_+$. Again by Poincaré-Birkhoff-Witt, we know that this consists exactly of all linear combinations of monomials (ordered as above) in which some one of $X_\alpha, X_\beta, X_{\alpha+\beta}$ does occur.

[8.0.3] **Remark:** To make the proof of the following lemma work, we already need the *existence* of a finite-dimensional representation with a given dominant integral highest weight. We do have existence, since we know that these occur as the unique irreducible quotients of the corresponding Verma modules.

[8.0.4] **Lemma:** $I \cap U(\mathfrak{h}) = 0$

Proof: For $X \in I$ we would have $Xv = 0$ for any highest-weight vector v in any representation of $\mathfrak{g}$.

On the other hand, for $X \in U(\mathfrak{h})$, we have $Xv = (\lambda X)v$ (where we extend λ to an algebra homomorphism $U(\mathfrak{H}) \to \mathbb{C}$). Since each Verma module $M_{\lambda+\rho}$ (with $\rho = \alpha + \beta$, as above) has a unique irreducible quotient with highest weight λ, we have $\lambda X = 0$ for all $\lambda \in \mathfrak{h}^*$. The algebra $U(\mathfrak{h})$ can be identified with polynomial functions on $\mathfrak{h}^*$, so the fact that $X = 0$ as a function on $\mathfrak{h}^*$ implies that $X = 0$ in $U(\mathfrak{h}^*)$. ///

[8.0.5] **Lemma:** $Z(\mathfrak{g}) \subset U(\mathfrak{h}) + I$

Proof: This is a restatement of the first lemma above: writing $z \in Z(\mathfrak{g})$ as a sum of monomials in our current style, in each such monomial, if any $Y_\alpha, Y_\beta, Y_{\alpha+\beta}$ occurs then some $X_\alpha, X_\beta, X_{\alpha+\beta}$ occurs. That is, if the monomial is not already in $U(\mathfrak{h})$ then it is in I. ///

Thus, the sum $U(\mathfrak{h}) + I$ is *direct*. Let

$$\gamma_o = \text{ projection of } Z(\mathfrak{g}) \text{ to the } U(\mathfrak{h}) \text{ summand}$$

A subsequent study of intertwining operators led Harish-Chandra to renormalize this. Define a *linear* map $\sigma : \mathfrak{h} \to U(\mathfrak{h})$ by

$$\sigma(H) = H - \delta(H) \cdot 1$$

where 1 is the 1 in $U(\mathfrak{g})$ and δ is half the sum of the positive roots

$$\delta = \frac{1}{2} \cdot \sum_{\alpha>0} \alpha$$

Extend σ by multiplicativity to an *associative algebra* homomorphism

$$\sigma : U(\mathfrak{h}) \to U(\mathfrak{h})$$

Then define the **Harish-Chandra homomorphism**

$$\gamma = \sigma \circ \gamma_o$$

Thus, for $\lambda \in \mathfrak{h}^*$, $z \in Z(\mathfrak{g})$, implicitly taking the multiplicative extension of any linear map $\lambda : \mathfrak{h} \to U(\mathfrak{h})$,

$$\gamma(Z)(\lambda) = \gamma_o(Z)(\lambda - \delta) = \lambda(\gamma_o(Z)) - \delta(\gamma_o(Z))$$

This all appears to depend upon a choice of the positive roots, but we will see that the normalized γ does *not* depend upon any such choices.

[8.0.6] Theorem: The Harish-Chandra map γ above is an isomorphism of $Z(\mathfrak{g})$ to the subalgebra $U(\mathfrak{h})^W$ of the universal enveloping algebra $U(\mathfrak{h})$ of the Cartan subalgebra $\mathfrak{h}$ invariant under the Weyl group W.

Proof: First, prove that γ is multiplicative. Since σ is *defined* to be multiplicative, it suffices to prove that the original projection map γ_o is multiplicative. Let I be the left ideal generated by $\mathfrak{n}_+$, as above. Let γ_o also denote the projection from the whole of $I + U(\mathfrak{h})$ to $U(\mathfrak{h})$. Thus, for any $u \in U(\mathfrak{g})$, the image $\gamma_o(u)$ is the unique element of $U(\mathfrak{h})$ such that

$$u - \gamma_o u \in I$$

For $z, z' \in Z(\mathfrak{g})$

$$zz' - \gamma_o(z)\gamma_o(z') = z(z' - \gamma_o z') + \gamma_o(z')(z - \gamma_o z)$$

where we use the fact that $U(\mathfrak{h})$ is commutative to interchange $\gamma_o(z)$ and $\gamma_o(z')$. The right-hand side is in the ideal I, so

$$\gamma_o(zz') = \gamma_o(z) \cdot \gamma_o(z')$$

as claimed.

Next, prove that the image of γ is inside $U(\mathfrak{h})^W$. It suffices to prove s-invariance for simple reflections $s \in W$. Viewing elements of $U(\mathfrak{h})$ as polynomials on $\mathfrak{h}^*$, to prove equality it suffices to consider λ dominant integral in $\mathfrak{h}^*$. That is, we assume that $\lambda - \delta$ is dominant integral and show that

$$\gamma_o(Z)(\lambda - \delta) = \gamma_o(Z)(s\lambda - \delta)$$

Let $V(\lambda)$ be the Verma module for λ, that is, the *universal* $\mathfrak{g}$-module with highest weight $\lambda - \delta$ and highest-weight vector v. Since the ideal I annihilates v, z acts on v by its projection to $U(\mathfrak{h})$, namely $\gamma_o(z) \in U(\mathfrak{h})$, which by definition of $V(\lambda)$ acts on v by

$$\gamma_o(z)(\lambda - \delta)$$

Since v generates $V(\lambda)$ and z is in the center of $U(\mathfrak{g})$, z acts on all of $V(\lambda)$ by the same scalar.

From the study of intertwining operators among Verma modules, we know that (for dominant integral λ) there exists an intertwining operator

$$V(s\lambda) \to V(\lambda)$$

That is, the scalar by which z acts on $V(s\lambda)$ is the left-hand side of the desired equality, and the scalar by which it acts on $V(\lambda)$ is the right-hand side, which are necessarily equal since this intertwining operator is non-zero. This proves the Weyl group stability of the image of the Harish-Chandra map γ.

It seems that the argument for **surjectivity** is not any simpler for $\mathfrak{sl}(3)$ than for $\mathfrak{sl}(n)$, and we will give that argument later.

Next, **injectivity**. Since $\mathfrak{g} \oplus \mathbb{C}$ injects to $U(\mathfrak{g})$, the algebra homomorphism σ is an algebra *injection*. Thus, it suffices to show that γ_o is injective. If $\gamma_o(z) = 0$ for $z \in Z(\mathfrak{g})$ then on any finite-dimensional irreducible V with highest weight λ the element z acts by $z(\lambda) = 0$. Since finite-dimensional representations of $\mathfrak{g}$ are completely reducible, this implies that z acts by 0 on every finite-dimensional representation of $\mathfrak{g}$.

At this point, one has choices about how to proceed...

EDIT: more later ...

[8.0.7] Corollary: If $\mu \in \mathfrak{h}^*$ is not in the Weyl group orbit $W \cdot \lambda$ of $\lambda \in \mathfrak{h}^*$, then the Verma module M_μ admits no non-zero $\mathfrak{g}$-homomorphism to M_λ.

Proof:

///

9. *Multiplicities, existence of finite-dimensional irreducibles*

So far, we have seen *necessary* conditions (integrality, dominance) for λ to be a highest weight of a finite-dimensional irreducible. We have not proven *existence*. We will prove existence by a more careful analysis of the dimensions of the weight spaces in Verma modules, by now knowing when there do or do not exist homomorphisms among them, and then conclude that for integral dominant λ the unique irreducible quotient of M_λ is *finite-dimensional*. [58]

EDIT: ... write this ...

[58] Yes, this is fairly round-about, and it might seem ironic that we prove existence of finite-dimensional irreducibles by a careful discussion of these infinite-dimensional Verma modules and the maps among them.