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Exercises 01

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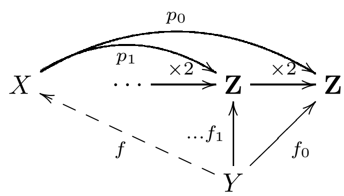
[1.1] Let X (with projections p_i) be the (projective) limit of abelian groups

$$\ldots \xrightarrow{\times 2} \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z}$$

That is, given an abelian group Y and group homomorphisms $f_n : Y \longrightarrow \mathbf{Z}$ compatible with the transition maps (meaning that $2 \cdot f_n(y) = f_{n-1}(y)$ for $y \in Y$), there is a unique $f : Y \longrightarrow X$ making a commutative diagram



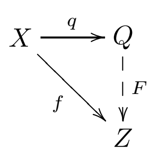
Prove that $X \approx \{0\}$.

[1.2] Let X (with projections p_i) be the (projective) limit of abelian groups

$$\ldots \xrightarrow{\times 2} \mathbf{Q} \xrightarrow{\times 2} \mathbf{Q}$$

Determine X and the projections.

[1.3] Let X be a topological space, and let $\sim$ be an equivalence relation[1] on X. The **quotient** $Q = X/\sim$ of X by $\sim$ as a *set* is the set of equivalence classes with respect to $\sim$, and there is the natural **quotient map** $q : X \longrightarrow Q$. A mapping-property definition of the **quotient topology** on Q is that, for every continuous map $f : X \longrightarrow Z$ of X to another topological space Z, if f is *constant* on equivalence classes[2] of $\sim$ in X, then f *factors through* q (uniquely), in the sense that there is a (unique) continuous $F : Q \longrightarrow Z$ such that $f = F \circ q$. That is, in a diagram,



Thus, by the usual diagrammatic argument(s), there is at most one such topology on Q, up to unique isomorphism. Prove that the usual *construction*[3] of quotient topology on Q does meet the condition.

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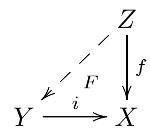
[1] The notion of *equivalence relation* is very general, important, and basic, which gives us all the more reason to review it: a *relation* R on a set X is just a subset of $X \times X$, where $(x, y) \in R$ means that *the relation holds* between x and y. A more suggestive notation is xRy when the relation holds. A relation is an *equivalent relation*, and written $x \sim y$ instead of xRy, if it resembles the relation of *equality*, in the following senses: it is *reflexive*, meaning that $x \sim x$ for all x; it is *symmetric*, meaning that $x \sim y$ implies $y \sim x$; it is *transitive* in the sense that $x \sim y$ and $y \sim z$ implies $x \sim z$.

[2] A function f being constant on equivalence classes means what it sounds like: if $x \sim y$, then $f(x) = f(y)$.

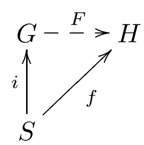
[3] Recall that the usual construction of the quotient topology on an image $q : X \longrightarrow Q$ of X is that a set U is open in Q if and only if $q^{-1}(U)$ is open in X.

[1.4] Let Y be a subset of a topological space X, with inclusion map $i : Y \longrightarrow X$. The mapping-property definition of the **subset topology** is that, given a continuous map $f : Z \longrightarrow X$ of a topological space Z to X, with image $f(Z)$ contained in $i(Y)$, there is a unique continuous map $F : Z \longrightarrow Y$ such that $f = i \circ F$. In a diagram,



Prove that this diagrammatic definition is fulfilled by the usual *construction* of the subset topology.

[1.5] By definition, **free group on a set** S is a group G and a set map $i : S \longrightarrow G$ such that, for every group H and set map $f : S \longrightarrow H$ there is a unique *group homomorphism* $F : G \longrightarrow H$ such that $f = F \circ i$, that is, such that we have a commutative diagram



Prove that, as usual, there is at most one free group on a given set S. Prove that the map $i : S \longrightarrow G$ is *injective* as a set map.

[1.6] Prove that the following construction of a free group $i : S \longrightarrow G$ on a *finite* set succeeds.[4] First, show that for any set map $f : S \longrightarrow H$, the subgroup $\langle f(S) \rangle$ of H generated by[5] the image $f(S)$ is *countable* (either finite or countable infinite). Show that there are finitely-many groups of a given finite cardinality n, and countably-many countably-infinite groups. Then show that the collection of *isomorphism classes of* set maps $j : S \longrightarrow G$ with G a group generated by $j(S)$ is a *set* I. Then let $G = \prod_{H \in I} H$ be the group product over (representatives[6] for) isomorphism classes $f_H : S \longrightarrow H$ in I, with $j : S \longrightarrow G$ defined by

$$\mathrm{pr}_H \circ j = f_H$$

where $\mathrm{pr}_H : G \longrightarrow H$ is the projection for $H \in I$. Then show that this $j : S \longrightarrow G$ is a free group on S.

[1.7] Prove that the not-necessarily-abelian group coproduct of two groups G and H exists, as a quotient of a suitable free group.

[1.8] Let A_i (i in an index set I) be abelian groups, and describe the natural map from the set coproduct $\coprod_i^{\text{sets}} A_i$ to the abelian group coproduct $\coprod_i^{\text{ab gps}} A_i$, and observe that it is very rarely a bijection (unless the index set has just one element, or maybe playing on coincidences like $2 + 2 = 2 \cdot 2$).

[1.9] Why is it *not* paradoxical that the underlying set of a (abelian-group) coproduct of abelian groups is *not* the disjoint union (set coproduct) of the underlying sets?

[1.10] Prove that the 2-solenoid $\lim_n \mathbf{R}/2^n\mathbf{Z}$ is *not* homeomorphic to a product $\mathbf{R}/\mathbf{Z} \times \mathbf{Z}_2$ (where $\mathbf{Z}_2 = \lim_n \mathbf{Z}/2^n$).

[4] A version of this appears in Lang's *Algebra*, and is attributed to Jacques Tits. The same approach succeeds for arbitrary sets S with a few additional complications.

[5] A usual, the subgroup *generated by* a subset is the intersection of all subgroups containing that subset.

[6] Yes, a choice of representatives uses the Axiom of Choice in some form. The fact that this lends an air of indeterminacy to the construction is merely an artifact, though, since the uniqueness of a free group, if it exists at all, is already clear.