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Summing Dirichlet series

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- Fourier series and Dirichlet series

This example illustrates the impact of harmonic analysis on number theory.

We first exploit the Plancherel theorem for Fourier series, namely, that

$$\int_0^1 |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |\langle f, \psi_n \rangle|^2$$

where $\langle f, g \rangle = \int_0^1 f \bar{g}$ and $\psi_n(x) = e^{2\pi i n x}$. Applied to polynomials (restricted to $[0, 1]$) gives a systematic approach to summing series^[1] such as

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots = \frac{\pi^2}{6}$$

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \dots = \frac{\pi^4}{90}$$

These are **special values** of the Riemann-Euler **zeta function**

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

which is an example of a **Dirichlet series**

$$\sum_{n \geq 1} \frac{a_n}{n^s} \quad (a_n \text{ in } \mathbb{C})$$

Certain more complicated Dirichlet series with *periodic coefficients* a_n , meaning that

$$a_m = a_n \quad \text{for } m = n \bmod N \text{ (for some fixed } N)$$

can be summed by exploiting the fact (proven earlier) that Fourier series of (finitely) piecewise C^1 functions do converge pointwise at C^1 points.

Note that more elementary methods *can* produce a few results in this direction. For example, from the geometric series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots$$

we integrate to obtain

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

^[1] Euler was the first to successfully sum these series, *circa* 1750, although he had more a *heuristic* than *proof*. But even a heuristic was something that had eluded the Bernouillis, and Euler's success was one of the things that made a big impression on the Bernouillis, who were very influential in mathematics in Europe and Russia in those times.

from which we have, by letting $x = 1$, the old result

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

But these completely elementary methods were powerless to evaluate $\zeta(2)$.

1. Plancherel and values of zeta

The *sawtooth function*

$$f(x) = x - \llbracket x \rrbracket - \frac{1}{2}$$

(where $\llbracket x \rrbracket$ is the greatest integer $\leq x$) has Fourier coefficients readily computed with one integration by parts, namely (for $n \neq 0$)

$$\widehat{f}(n) = \int_0^1 \left(x - \frac{1}{2}\right) e^{-2\pi i n x} dx = \left[\frac{\left(x - \frac{1}{2}\right)^2}{2} \frac{e^{-2\pi i n x}}{-2\pi i n} \right]_0^1 - \int_0^1 \frac{e^{-2\pi i n x}}{-2\pi i n} dx = \frac{1}{-2\pi i n}$$

The subtraction of the $1/2$ is designed to make the 0^{th} Fourier coefficient be 0. Then Plancherel immediately gives

$$\frac{1}{4\pi^2} \sum_{0 \neq n \in \mathbb{Z}} \frac{1}{n^2} = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \left[\frac{\left(x - \frac{1}{2}\right)^3}{3} \right]_0^1 = \frac{1}{12}$$

Keeping in mind that the sum over n is over not only the positive integers, but *all* non-zero integers,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$$

The next example would need a *quadratic* polynomial $B_2(x)$ such that

$$\int_0^1 B_2(x) dx = 0 \quad \text{to make} \quad \langle B_2, \psi_0 \rangle = 0$$

and whose derivative is $x - \frac{1}{2}$, which we have seen integrates against $\overline{\psi}_n$ nicely. Thus, take

$$B_2(x) = \int_0^x \left(t - \frac{1}{2}\right) dt - \int_0^1 \int_0^u \left(t - \frac{1}{2}\right) dt du = \frac{\left(x - \frac{1}{2}\right)^2}{2} - \left[\frac{\left(u - \frac{1}{2}\right)^3}{6} \right]_0^1 = \frac{\left(x - \frac{1}{2}\right)^2}{2} - \frac{1}{24} \quad (0 \leq x < 1)$$

Then compute the n^{th} Fourier coefficient integrating by parts

$$\int_0^1 B_2(x) \overline{\psi}_n(x) dx = \left[B_2(x) \frac{\overline{\psi}_n(x)}{-2\pi i n} \right]_0^1 - \int_0^1 \left(x - \frac{1}{2}\right) \frac{\overline{\psi}_n(x)}{-2\pi i n} dx = \frac{B_2(1) - B_2(0)}{-2\pi i n} - \frac{1}{(-2\pi i n)^2} = \frac{1}{(2\pi n)^2}$$

since we already computed that $\langle x - \frac{1}{2}, \psi_n \rangle = -1/2\pi i n$, and since

$$B_2(1) - B_2(0) = \int_0^1 \left(x - \frac{1}{2}\right) dx = 0$$

Then Plancherel gives

$$\frac{1}{(2\pi)^4} \sum_{0 \neq n \in \mathbb{Z}} \frac{1}{n^4} = \int_0^1 B_2(x)^2 dx = \int_0^1 \left(\frac{\left(x - \frac{1}{2}\right)^2}{2} - \frac{1}{24} \right)^2 dx = \int_0^1 \left(\frac{\left(x - \frac{1}{2}\right)^4}{4} - \frac{\left(x - \frac{1}{2}\right)^2}{24} + \frac{1}{576} \right) dx$$

so

$$\frac{1}{8\pi^4} \sum_{0 < n \in \mathbb{Z}} \frac{1}{n^4} = \left[\frac{(x - \frac{1}{2})^5}{20} - \frac{(x - \frac{1}{2})^3}{72} + \frac{(x - \frac{1}{2})}{576} \right]_0^1 = 2 \cdot \left(\frac{1}{2^5 \cdot 20} - \frac{1}{2^3 \cdot 72} + \frac{1}{2 \cdot 576} \right)$$

Moving the $8\pi^4$ from the other side, we have

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \dots = \pi^4 \cdot \left(\frac{1}{40} - \frac{1}{36} + \frac{1}{72} \right) = \frac{\pi^4}{90}$$

One can readily arrange an induction to find a sequence of polynomials^[2] $B_k(x)$ with *rational* coefficients, with properties

$$B_1(x) = x - \frac{1}{2} \quad \frac{d}{dx} B_k(x) = B_{k-1}(x) \quad \int_0^1 B_k(x) dx = 0$$

Claim: The Fourier series of the periodicized version of $B_k(x)$ is

$$B_k(x - \llbracket x \rrbracket) \sim \frac{-1}{(2\pi i)^k} \sum_{0 \neq n \in \mathbb{Z}} \frac{1}{n^k} \psi_n(x)$$

Proof: Integrating by parts,

$$\begin{aligned} \int_0^1 B_k(x) \bar{\psi}_n(x) dx &= \left[B_k(x) \frac{\bar{\psi}_n(x)}{-2\pi i n} \right]_0^1 - \int_0^1 B_{k-1}(x) \frac{\bar{\psi}_n(x)}{-2\pi i n} dx \\ &= \frac{B_k(1) - B_k(0)}{-2\pi i n} - \frac{-1}{(2\pi i n)^{k-1}} \cdot \frac{1}{-2\pi i n} = \frac{-1}{(2\pi i n)^k} \end{aligned}$$

since

$$B_k(1) - B_k(0) = \int_0^1 B_{k-1}(t) dt = 0$$

by induction. ///

To determine the polynomials more systematically, we use a **generating function** trick:

Claim: We have the identity

$$1 + t B_1(x) + t^2 B_2(x) + t^3 B_3(x) + \dots = \frac{t \cdot e^{tx}}{e^t - 1}$$

Remark: This gives yet another proof that the polynomials $B_k(x)$ have *rational* coefficients.

Proof: Let

$$f(x) = f_t(x) = 1 + t B_1(x) + t^2 B_2(x) + t^3 B_3(x) + \dots$$

Applying a standard idea, we differentiate in x and see what happens. We find that

$$\frac{\partial}{\partial x} f(x) = t \cdot 1 + t^2 B_1(x) + t^3 B_2(x) + \dots = t \cdot f(x)$$

^[2] These polynomials are roughly the so-called *Bernoulli polynomials*. We are not worrying about conventional normalization or indexing.

Therefore,

$$f(x) = C_t \cdot e^{tx} \quad (\text{for some } C_t)$$

The conditions $\int_0^1 B_k(x) dx = 0$ give, on one hand,

$$\int_0^1 C_t \cdot e^{tx} dx = \int_0^1 (1 + t B_1(t) + \dots) dx = 1$$

On the other hand

$$\int_0^1 C_t \cdot e^{tx} dx = C_t \cdot \frac{e^t - 1}{t}$$

Thus,

$$C_t = \frac{t}{e^t - 1}$$

Therefore,

$$1 + t B_1(x) + t^2 B_2(x) + \dots = \frac{t \cdot e^{tx}}{e^t - 1}$$

as claimed. ///

2. Fourier series pointwise and values of zeta

Now we use the pointwise convergence of Fourier series to evaluate *certain* Dirichlet series

$$\sum_{n \geq 1} \frac{a_n}{n^k}$$

with *periodic* coefficients a_n . It turns out that there is a profound restriction on which of these we can evaluate, in terms of the *parity* of k . The present section evaluates $\zeta(2k)$ again, by the method that the *next* section extends to more complicated periodic coefficients.

First, we use the *pointwise* convergence result for piecewise C^∞ functions at points where left and right derivatives exist. For $k > 1$ this applies to the periodic versions $B_k(x - \llbracket x \rrbracket)$ of the polynomials $B_k(x)$ constructed just above. Thus, using the computation of Fourier coefficients from the previous section,

$$B_k(x - \llbracket x \rrbracket) = \frac{-1}{(2\pi i)^k} \sum_{0 \neq n \in \mathbb{Z}} \frac{1}{n^k} \cdot \psi_n(x)$$

Evaluating this at $x = 0$, for $k > 1$,

$$B_k(0) = \frac{-1}{(2\pi i)^k} \sum_{0 \neq n \in \mathbb{Z}} \frac{1}{n^k}$$

For k odd, we have a boring outcome: the right-hand side is 0, due to cancellation of the n and $-n$ terms. But for k even we get something:

Claim:

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = B_{2k}(0) \cdot \frac{-(2\pi i)^{2k}}{2}$$

for even integers $2k$ with $2k \geq 2$. And

$$t^2 \cdot \frac{2\zeta(2)}{(2\pi i)^2} + t^4 \cdot \frac{2\zeta(4)}{(2\pi i)^4} + t^6 \cdot \frac{2\zeta(6)}{(2\pi i)^6} + t^8 \cdot \frac{2\zeta(8)}{(2\pi i)^8} + \dots = 1 - \frac{t}{2} - \frac{t}{e^t - 1}$$

Proof: The first assertion is just a rearrangement of the Fourier series for $B_k(x)$ evaluated at $x = 0$. Then subtract $1 + tB_1(x) = 1 + t(x - \frac{1}{2})$ from both sides of the identity

$$1 + tB_1(x) + t^2B_2(x) + t^3B_3(x) + \dots = \frac{t \cdot e^{tx}}{e^t - 1}$$

evaluate at $x = 0$, and multiply through by -1 , noting that $B_k(0) = 0$ for odd $k > 1$. ///

Remark: Note that this approach simply gives no information about the values of the zeta function at positive odd integers.

Next

3. Fourier series and Dirichlet series

Next, consider Dirichlet series $\sum a_n/n^s$ with *periodic* coefficients a_n . Specifically, suppose that there is a positive integer N such that $a_m = a_n$ for $m = n \pmod N$. It is convenient to think of the function $n \rightarrow a_n$ as being defined for all $n \in \mathbb{Z}$, factoring through the quotient $\mathbb{Z} \rightarrow \mathbb{Z}/N$.

The finite group \mathbb{Z}/N has an obvious symmetry $x \rightarrow -x \pmod N$. It turns out that the behavior of the coefficient function $n \rightarrow a_n$ under $n \rightarrow -n \pmod N$ is decisive for summing $\sum a_n/n^k$. Say

$$\begin{aligned} n \rightarrow a_n \text{ is } \mathbf{odd} & \quad \text{when } a_m = -a_n \text{ for } m = -n \pmod N \\ n \rightarrow a_n \text{ is } \mathbf{even} & \quad \text{when } a_m = +a_n \text{ for } m = -n \pmod N \end{aligned}$$

Then, roughly,

$$\text{we can only evaluate } \sum_n \frac{a_n}{n^k} \quad \text{for } k \text{ odd} \quad \text{when } n \rightarrow a_n \text{ is odd}$$

$$\text{we can only evaluate } \sum_n \frac{a_n}{n^k} \quad \text{for } k \text{ even} \quad \text{when } n \rightarrow a_n \text{ is even}$$

That is, the literal parity of the integer k must ^[3] match the *oddness* or *evenness* of the function $n \rightarrow a_n$ under the automorphism $n \rightarrow -n$ of \mathbb{Z}/N .

It is easiest to treat the case that N is a prime p .

[... iou ...] more later

[3] This apparent parity restriction certainly arises in any of the known devices to evaluate these special values (at integers) of Dirichlet series with periodic coefficients. Around 1977 Deligne created systematic conjectures which include these and other examples of special value results. All special value results over the intervening years appear to be compatible with Deligne's conjectures. The authoritative source (though not easy to read) is P. Deligne, *Valeurs des fonctions L et périodes d'intégrales*, in Automorphic Forms, Representations, and L-functions, Proc. Symp. Pure Math. vol. 33 part II, pp. 313-346.