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Harmonic analysis on spheres

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- Properties of calculus on spheres
- Existence of the spherical Laplacian
- Polynomial eigenvectors for the spherical Laplacian
- Complete determination of eigenvectors
- Existence of invariant integrals on spheres
- L^2 spectral decompositions on spheres
- Pointwise convergence of Fourier series on S^{n-1}
- Sobolev inequalitites on S^{n-1}

In review:

Harmonic analysis on the *circle* is the theory of Fourier series, which studies the expressibility of functions and *generalized* functions as sums of the fundamental exponential functions. The exponential functions are *simpler* functions, and are both *eigenfunctions* of the translation-invariant differential operator $\frac{d}{dx}$, and *group homomorphisms* to $\mathbb{C}^\times$.

Harmonic analysis on the *line* is the theory of Fourier *transforms*, more complicated than the circle case due to the line's non-compactness. On $\mathbb{R}$ the exponential functions, while still eigenfunctions for $\frac{d}{dx}$ and still giving group homomorphisms, are no longer in $L^2(\mathbb{R})$. Entangled with this point is the fact that Fourier inversions expresses functions as *integrals* of exponential functions, not *sums*.

Both the circle and the real line are *groups*, and are *abelian*. These two features lend simplicity to their harmonic analysis. By contrast, unsurprisingly, *non-abelian* groups and associated spaces have a more complicated harmonic analysis.

One immediate complication is that for non-abelian G not all subgroups are normal, so not all quotients G/H (with subgroup H) are *groups*. This already has consequences for quotients of *finite* groups. By contrast, the quotient $\mathbb{R}/\mathbb{Z}$ of the real line by the integers presents the circle as a *group*, not merely as a *quotient space* of a group.

Beyond *finite* groups and their quotients, a more typical situation is illuminated by spheres $S^{n-1} \subset \mathbb{R}^n$ which are quotients [1]

$$S^{n-1} \approx SO(n-1) \setminus SO(n)$$

of rotation groups $SO(n)$. Spheres are rarely groups, but are acted-upon transitively by rotation groups. An essential of simplicity *is* preserved, the *compactness*.

Fourier series of functions on spheres are sometimes called *Laplace series*.

Later examples of harmonic analysis related to *non-compact* non-abelian groups are vastly more complicated than the compact (non-abelian) compact case here.

Functions on spheres have surprising and non-obvious connections to the harmonic analysis of certain *non-compact* groups, such as $SL_2(\mathbb{R})$. [2]

[1] Rotation groups as *orthogonal groups* and *special orthogonal groups* are reviewed below.

[2] Relevant keywords for this unobvious connection are *oscillator representation* and *Weil representation*.

1. *Properties of calculus on spheres*

To emphasize and exploit the rotational symmetry of spheres, we want *eigenfunctions* for rotation-invariant *differential operators* on spheres, and expect that these eigenfunctions will be the proper analogues of exponential functions on the circle or line. Thus, we must identify rotation-invariant differential operators on spheres. We will need a rotation-invariant *measure* or *integral* on spheres. Rather than writing formulas or giving constructions, we describe these objects by their desired properties.

For a positive integer n let $S = S^{n-1}$ be the usual unit $(n-1)$-sphere

$$S = S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_1^2 + \ldots + x_n^2 = 1\}$$

Write Δ^S for the desired rotation-invariant second-order [3] differential operator on functions on S, and let $\int_S f$ denote the desired rotation-invariant (positive) integral. We call Δ^S the **Laplacian** on the sphere. All functions here are indefinitely differentiable. [4]

Two desired properties are

$$\begin{array}{lcll} \int_S (\Delta^S f) \cdot \varphi & = & \int_S f \cdot (\Delta^S \varphi) & (\textbf{self-adjointness}) \\ \int_S (\Delta^S f) \cdot \overline{f} & \leq & 0 & (\textbf{definiteness}) \end{array}$$

with equality only for f *constant*. We also assume that Δ^S has *real coefficients*, in the abstracted sense that

$$\overline{\Delta^S f} = \Delta^S \, \overline{f}$$

We have the natural complex hermitian inner product

$$\langle f, g \rangle = \int_S f \cdot \overline{g}$$

at least for on differentiable functions on S.

We have a typical linear algebra conclusion, via a typical argument:

[1.0.1] **Corollary:** Granting existence of invariant Δ^S and invariant measure on S^{n-1}, with the self-adjointness and definiteness properties, eigenvectors f, g for Δ^S with *distinct* eigenvalues are orthogonal with respect to the inner product $\langle , \rangle$. Any eigenvalues are *negative real* numbers.

Proof: Let $\Delta^S f = \lambda \cdot f$ and $\Delta^S g = \mu \cdot g$. Assume $\lambda \neq 0$ (or else interchange the roles of λ and μ). Then

$$\langle f, f \rangle = \frac{1}{\lambda} \int_S (\Delta^S f) \cdot \overline{f} = \frac{1}{\lambda} \int_S f \, \overline{\Delta^S f} = \frac{\overline{\lambda}\lambda}{\lambda} \int_S f \, \overline{f}$$

[3] Unlike the circle and line, there are no rotation-invariant *first-order* differential operators on higher-dimensional spheres. By contrast, suitably symmetric *second-order* operators are ubiquitous.

[4] The notion of *differentiability* for functions on a sphere can be given in several ways, all equivalent. At one extreme, the most pedestrian is to declare a function f on S^{n-1} differentiable if the function $F(x) = f(x/|x|)$ on $\mathbb{R}^n - 0$ is differentiable on $\mathbb{R}^n - 0$. At the other extreme one gives S^{n-1} its usual structure of *smooth manifold*, which incorporates a notion of differentiable function. But, happily, the choice of definition doesn't matter much for us, since we won't be attempting to directly *compute* any derivatives, but only use *properties* of the process of differentiation.

Since $\lambda \neq 0$, f is not identically 0, so the integral of $f \cdot \overline{f}$ is not 0, and $\lambda = \overline{\lambda}$, so λ is *real.* Then the negative definiteness of Δ^S (and positive-ness of the invariant measure on S) gives

$$\lambda \cdot \langle f, f \rangle = \int_S (\Delta^S f) \cdot \overline{f} < 0$$

so $\lambda < 0$. Next,

$$\langle f, g \rangle = \frac{1}{\lambda} \int_S (\Delta^S f) \cdot \overline{g} = \frac{1}{\lambda} \int_S f \cdot \overline{\Delta^S g} = \frac{\overline{\mu}}{\lambda} \int_S f \cdot \overline{g}$$

We know that the eigenvalues λ, μ are real, so for $\mu/\lambda \neq 1$ necessarily the integral is 0. ///

The standard **special orthogonal group** [5] is

$$SO(n) = \{g \in GL_n(\mathbb{R}) \,:\, g^\top g = 1_n \quad \text{and} \quad \det g = 1\}$$

and acts on S by *right* [6] matrix multiplication,

$$k \times x \to xk \qquad (\text{for } x \in S^{n-1} \text{ and } k \in O(n))$$

considering elements of $\mathbb{R}^n$ as *row* vectors. We will refer to the action of elements of $SO(n)$ on $S = S^{n-1}$ as **rotations**. [7] It is useful to note that for $g \in SO(n)$, inverting both sides of $g^\top g = 1$ gives $g^{-1}(g^\top)^{-1} = 1$, and then $1 = gg^\top$.

Since the usual inner product $\langle , \rangle$ on $\mathbb{R}^n$ is given by

$$\langle x, y \rangle = xy^\top \qquad (\text{for } \textit{row} \text{ vectors } x, y)$$

we see that $SO(n)$ preserves lengths (and angles):

$$\langle xk, yk \rangle = (xk)(yk)^\top = x\,(kk^\top)\,y^\top = x\,y^\top = \langle x, y \rangle$$

Also, observe that for any k such that $k^\top k = 1$

$$(\det k)^2 = \det k^\top \cdot \det k = \det(k^\top k) = \det 1_n = 1$$

Thus, $\det k = \pm 1$.

[1.0.2] Claim: The action of $SO(n)$ on S^{n-1} is *transitive.* [8]

[5] The *full* orthogonal group is defined by the condition $g^\top g = 1_n$ *without* the determinant condition. We want the determinant condition so that *orientation* is preserved, however one cares to define it. The modifier *special* refers to the determinant condition.

[6] The choice of *right* matrix multiplication on *row* vectors is not essential, but is made with some forethought. Specifically, since we will discuss *functions on* the sphere, this has minor advantages that will become clear. Additionally, the contemporary conventions about groups acting on *functions on* a set fairly strongly chooses the action on the *set* to be on the right. As late as the mid-1960's the convention was ambiguous, but it is not ambiguous currently.

[7] To *prove* that $SO(n)$ is exactly all the rotations, one would need a precise definition of *rotation*, which we avoid.

[8] Recall that the definition of *transitivity* is that for all $x, y \in S$ there is $g \in O(n)$ such that $gx = y$. If we believe that $SO(n)$ really is all rotations of the sphere, then our physical intuition seems to make the transitivity clear. But this supposed triviality requires *two* assumptions, first that all rotations are given by $SO(n)$, and that our intuition about rotations is good even in higher dimensions. We can do better than this.

Proof: We show that, given $x \in S$ there is $k \in SO(n)$ such that $e_1 k = x$, where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$. That is, we construct $k \in SO(n)$ such that the top row of k is x. Indeed, complete x to an $\mathbb{R}$-basis $x, x_2, x_3, \ldots, x_n$ for $\mathbb{R}^n$. Then apply the *Gram-Schmidt* process [9] to find an orthonormal (with respect to the standard inner product) basis $x, v_2, \ldots, v_n$ for $\mathbb{R}^n$. The condition $kk^\top = 1$, when expanded, is exactly the assertion that the rows of k form an orthonormal basis, so taking $x, v_2, \ldots, v_n$ as the rows of k, we have k such that $e_1 k = x$. As noted just above, the determinant of this k is ± 1. To ensure that it is 1, replace v_n by $-v_n$ if necessary. This still gives $e_1 k = x$, giving the transitivity. ///

The *isotropy group* $SO(n)_{e_n}$ of the last standard basis vector $e_n = (0, \ldots, 0, 1)$ is

$$\text{(isotropy group)} \;=SO(n)_{e_n} = \{ \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \;:\; A \in SO(n-1)\} \;\approx\; SO(n-1)$$

Thus, by transitivity, as $SO(n)$-spaces

$$S^{n-1} \approx SO(n-1)\backslash SO(n)$$

Now define the action of $k \in SO(n)$ on *functions* f on the sphere $S = S^{n-1}$ (or on the ambient $\mathbb{R}^n$) by [10]

$$(k \cdot f)(x) = f(xk)$$

Then our rotation invariance conditions are

$$\begin{aligned} \int_S k \cdot f &= \int_S f && \text{(for } k \in SO(n)) \\ \Delta^S (k \cdot f) &= k \cdot (\Delta^S f) && \text{(for } k \in SO(n)) \end{aligned}$$

[1.0.3] Remark: To prove *existence* of an invariant integral and Laplacian, there are several approaches, of varying technical expense, and of varying quality. For our immediate purposes, we will choose a most pedestrian *ad hoc* approach, despite the ugliness of this approach. We will give a more sensible and prettier argument later, but it will require more technique.

[1.0.4] Remark: One difficulty here is that we do not have an adequate language or technical set-up to prove any sort of *uniqueness* of either the invariant integral or the invariant second-order operator. This leaves us in the awkward position of worrying that varying constructions may give varying results.

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[9] Recall that, given a basis $v_1, \ldots, v_n$ for a (real or complex) vector space with an inner product (real-symmetric or complex hermitian), the Gram-Schmidt process produces an *orthogonal* or *orthonormal* basis, as follows. Replace v_1 by $v_1/|v_1$ to give it length 1. Then replace v_2 first by $v_2 - \langle v_2, v_1\rangle v_1$ to make it orthogonal to v_1 and then by $v_2/|v_2|$ to give it length 1. Then replace v_3 first by $v_3 - \langle v_3, v_1\rangle v_1$ to make it orthogonal to v_1, then by $v_3 - \langle v_3, v_2\rangle v_2$ to make it orthogonal to v_2, and then by $v_3/|v_3|$ to give it length 1. And so on.

[10] Here is one benefit of letting the group act on the *right* on the *set*. If, instead, we try to let the group act by *left* multiplication, and then try to define $(k \cdot f)(x) = f(kx)$, we have a problem, as follows. For $g, h \in SO(n)$, *associativity* fails, since

$$((gh) \cdot f)(x) = f((gh)\,x) = f(g(hx)) = (g \cdot f)(hx) = (h \cdot (g \cdot f))(x)$$

instead of $g \cdot (h \cdot f)$. If we insist on a left action on the set, then we *can* succeed at some cost, by defining $(k \cdot f)(x) = f(k^{-1}x)$. One does see this in the literature, and, indeed, there's nothing wrong with this. In fact, often a set has groups acting both on left and right, and the inverse is inevitable. However, with just a one-sided action as we have here, avoiding the inverse is a pleasant (if minor) economy.

2. *Existence of the spherical Laplacian*

For one direct proof of *existence* of the invariant Laplacian onm $S = S^{n-1}$ we *cheat* by using the way that the sphere $S = S^{n-1}$ is imbedded in $\mathbb{R}^n$. [11] We grant that the usual Euclidean Laplacian

$$\Delta = \left(\frac{\partial}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial}{\partial x_n}\right)^2$$

is $SO(n)$-invariant [12] on differentiable functions on $\mathbb{R}^n$. For a function f on S, create a function F on $\mathbb{R}^n - 0$ by $F(x) = f(x/|x|)$, and define

$$\Delta^S f = (\text{restriction back to } S \text{ of }) \, \Delta F$$

The map $f \to F$ that creates from f on S the degree-zero [13] positive-homogeneous [14] function F on $\mathbb{R}^n - 0$ *commutes with* the action of $SO(n)$. [15] From the definition, it is clear that

$$\Delta^S \, \overline{f} = \overline{\Delta^S \, f}$$

The $SO(n)$-invariance of the spherical Laplacian follows from the $SO(n)$-invariance of the usual Laplacian: for $k \in SO(n)$

$$\Delta^S(k \cdot f) = (\Delta(k \cdot F)) = k \cdot (\Delta F) = k \cdot (\Delta F)|_S$$

since restriction to the sphere commutes with $SO(n)$, as does $f \to F$. Thus, Δ^S is $SO(n)$-invariant.

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[11] The imbedding of S^{n-1} in $\mathbb{R}^n$ does have subtler ramifications, too, so taking advantage of relatively *obvious* aspects is not so unconscionable.

[12] To verify that the usual Euclidean Laplacian is $SO(n)$-invariant is at worst a matter of direct computation. In fact, the Laplacian is invariant under the *full* orthogonal group $O(N)$. Let $k \in O(n)$ with ij^{th} entry k_{ij}. For F on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$,

$$\Delta(k \cdot F)(x) = \sum_\ell (\frac{\partial}{\partial x_\ell})^2 F(\ldots, \sum_i x_i k_{ij}, \ldots) = \sum_\ell \frac{\partial}{\partial x_\ell} \sum_s k_{\ell s} F_s(\ldots, \sum_j x_i k_{ij}, \ldots)$$

where F_s is the partial derivative of F with respect to its s^{th} argument. Taking the next derivative gives $\sum_\ell \sum_{s,t} k_{\ell s} k_{\ell t} F_{st}(\ldots, \sum_i x_i k_{ij}, \ldots)$. Interchange the order of the sums and observe that $\sum_\ell k_{\ell s} k_{\ell t}$ is the $(s,t)^{th}$ entry of $k^\top \cdot k$, which is the $(s,t)^{th}$ entry of 1_n. Thus, the whole is

$$\sum_s F_{ss}(\ldots, \sum_i x_i k_{ij}, \ldots) = (\Delta F)(xk) = (k \cdot (\Delta F))(x)$$

[13] We make $F(x) = f(x/|x|)$ be homogeneous of degree 0, rather than $|x|^s \, f(x/|x|)$ of degree s, in order that the *constant* functions on the sphere become constant functions on $\mathbb{R}^n - 0$, and thus be *annihilated* by any differential operator.

[14] A function F on $\mathbb{R}^m$ is *positive-homogeneous* of degree $s \in \mathbb{C}$ if, for all $t > 0$ and for all $x \in \mathbb{R}^m$, $F(tx) = t^s \cdot F(x)$.

[15] The map $f \to F$ commutes with the action $(k \cdot F)(x) = F(xk)$ of $SO(n)$ on functions because $F(xk) = f(xd/|xk|) = f((x/|x|) \cdot k) = (kf)(x/|x|)$.

3. *Polynomial eigenvectors for the spherical Laplacian*

Once one has the idea of looking at positive-homogeneous functions of various degrees as a device to extend functions from the sphere to the ambient Euclidean space, one might also run things in the opposite direction, as an *experiment* to see how the spherical Laplacian behaves. As usual, a function φ on $\mathbb{R}^n$ (or an open subset) is **harmonic** if it is annihilated by the Euclidean Laplacian.

Indeed, this experiment works so well that, if we use some basic results about tempered distributions, we can completely classify the spherical Laplacian's eigenfunctions, which we do in the next section.

[3.0.1] **Claim:** For f positive-homogeneous of degree s on $\mathbb{R}^n - 0$

$$\Delta(|x|^{-s} f) = -s(s+n-2)|x|^{-(s+2)} f + |x|^{-s} \Delta f$$

[3.0.2] **Corollary:** For f positive-homogeneous of degree s and *harmonic*, the restriction $f|_S$ of f to the sphere S^{n-1} is an *eigenfunction* for Δ^S,

$$\Delta^S(f|_S) = -s(s+n-2)\cdot(f|_S)$$

with eigenvalue $-s(s+n-2)$. ///

Proof: *(of claim)* Computing directly, letting $r = |x|$, and letting f_i be the partial derivative with respect to the i^{th} argument,

$$\Delta^S(f|_S) = \Delta f(x/|x|) = \Delta\left(|x|^{-s}\cdot f(x)\right) = \sum_i \frac{\partial}{\partial x_i}^2 \left((r^2)^{-\frac{s}{2}}\cdot f(x)\right)$$

$$= \sum_i \frac{\partial}{\partial x_i}\Big(-\frac{s}{2}\,(2x_i)\,(r^2)^{-(\frac{s}{2}+1)}\,f(x) + (r^2)^{-s/2}\,f_i(x)\Big) = \sum_i \frac{\partial}{\partial x_i}\Big(-sx_i\,(r^2)^{-(\frac{s}{2}+1)}\,f(x) + (r^2)^{-s/2}\,f_i(x)\Big)$$

$$= \sum_i \Big(-s\,(r^2)^{-(\frac{s}{2}+1)}\,f(x) + sx_i\,(\frac{s}{2}+1)(2x_i)\,(r^2)^{-(\frac{s}{2}+2)}\,f(x) - 2sx_i(r^2)^{-(\frac{s}{2}+1)}\,f_i(x) + (r^2)^{-s/2}\,f_{ii}(x)\Big)$$

$$= -ns\,(r^2)^{-(\frac{s}{2}+1)}\,f(x) + sr^2\,(\frac{s}{2}+1)\,2\,(r^2)^{-(\frac{s}{2}+2)}\,f(x) - 2s(r^2)^{-(\frac{s}{2}+1)}\,sf(x) + (r^2)^{-s/2}\,\Delta f(x)$$

by using Euler's identity[16] that for positive-homogeneous f of degree s,

$$\sum_i x_i\, f_i(x) = s\cdot f$$

as well as the obvious $\sum_i x_i^2 = r^2$. Simplifying, we have

$$\Delta(|x|^{-s}\cdot f(x)) = -ns\,r^{-(s+2)}\,f(x) + s(s+2)\,r^{-(s+2)}\,f(x) - 2s\,r^{-(s+2)}\,sf(x) + r^{-s}\,\Delta f(x)$$

$$= -s(n-(s+2)+2s)\,r^{-(s+2)}\,f(x) + r^{-s}\,\Delta f(x) = -s(n+s-2)\,r^{-(s+2)}\,f(x) + r^{-s}\,\Delta f(x)$$

as asserted. ///

[16] Euler's identity is readily proven by considering the function $g(t) = G(tx)$ for $t > 0$, differentiating with respect to t, and evaluating at $t = 1$.

[3.0.3] Remark: The most tractable homogeneous functions are homogeneous *polynomials*, so we look for *harmonic* homogeneous polynomials before doing anything subtler. [17]

Let

$$\mathcal{H}^d = \{\text{homogeneous (total) degree } d \text{ harmonic polynomials in } \mathbb{C}[x_1, \ldots, x_n]\}$$

Let $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$ be the *homogeneous* polynomials of degree d. Introduce a temporary complex-hermitian form [18]

$$(,) : \mathbb{C}[x_1, \ldots, x_n] \times \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}$$

by

$$(P, Q) = \overline{Q}(\partial)\,(P(x))\,|_{x=0}$$

where $Q(\partial)$ means to replace x_i by $\partial/\partial x_i$ in a polynomial, and $R|_{x=0}$ means to evaluate R at $x = 0$. The main point of this contruction is the immediate identity

$$(\Delta f, g) = (f, r^2 g)$$

where $r^2 = x_1^2 + \ldots + x_n^2$. That is, we have made multiplication by r^2 *adjoint* to application of Δ. This will be useful only after we prove that the form $(,)$ is suitably non-degenerate.

[3.0.4] Claim: The form $(,)$ is positive-definite hermitian.

Proof: By looking at *monomials*, a basis for the $\mathbb{C}$-vectorspace $\mathbb{C}[x_1, \ldots, x_n]$, we see that $(P, Q) = 0$ for *homogeneous* polynomials P, Q *unless* P, Q are of the same degree. And, when restricted to the *homogeneous* polynomials $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$ of degree d, the form $(,)$ has an *orthogonal basis* of *distinct monomials*, since

$$\left(\frac{\partial}{\partial x_1}^{m_1} \ldots \frac{\partial}{\partial x_n}^{m_n}\right)(x_1^{e_1} \ldots x_n^{e_n})\Big|_{x=0} = \begin{cases} 0 & \text{(if } \textit{any } m_i \neq e_i) \\ m_1! \ldots m_n! & \text{(if } \textit{every } m_i = e_i) \end{cases}$$

Further, by looking at the orthogonal basis of monomials, we see that $(,)$ is *hermitian* and *positive definite* on $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$. ///

[3.0.5] Claim: The map

$$\Delta : \mathbb{C}[x_1, \ldots, x_n]^{(d)} \to \mathbb{C}[x_1, \ldots, x_n]^{(d-2)}$$

is *surjective*. Harmonic polynomials f in $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$ are *orthogonal* to polynomials $r^2 h$ (with $h \in \mathbb{C}[x_1, \ldots, x_n]^{(d-2)}$) with respect to $(,)$.

Proof: For $h \in \mathbb{C}[x_1, \ldots, x_n]^{(d-2)}$ if $(\Delta f, h) = 0$ for all $f \in \mathbb{C}[x_1, \ldots, x_n]^{(d)}$ then

$$0 = (\Delta f, h) = (f, r^2 h)$$

for all f, so $r^2 h = 0$, so $h = 0$, by the positive-definiteness of $(,)$. This also proves the second assertion. ///

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[17] And, gratifyingly, a slightly more sophisticated argument in the next section proves that there are no *other* eigenfunctions of the spherical Laplacian.

[18] This hermitian form is not as *ad hoc* as it may look. In effect, it pairs polynomials against *Fourier transforms* of polynomials, which are derivatives of Dirac delta at 0, which are *compactly-supported* distributions, so can be evaluated on polynomials, which are smooth. One need not think in these terms to use the properties here, which are much more elementary to prove. A more sophisticated interpretation *is* resurrected in the next section to give a definitive classification of eigenvectors.

Iterating the claim just proven, we have

$$\mathbb{C}[x_1,\ldots,x_n]^{(d)} = \mathcal{H}^d \oplus r^2\mathcal{H}^{d-2} \oplus r^4\mathcal{H}^{d-4} + \ldots$$

[3.0.6] Claim: Polynomials restricted to the n-sphere are equal to linear combinations of *harmonic* polynomials.

Proof: Use the observation just made to write

$$f = f_0 + r^2 f_2 + r^4 f_4 + \ldots$$

with each f_i harmonic. When restricted to the sphere we have

$$f|_S = (f_0 + r^2 f_2 + r^4 f_4 + \ldots)|_S = (f_0 + f_2 + f_4 + \ldots)|_S$$

since $r^2 = 1$ on the sphere. ///

[3.0.7] Remark: Again, from computations above,

$$\Delta^S f = -d(d+n-2)\cdot f \quad (\text{for } f \in \mathcal{H}^d)$$

Since the degree d is non-negative,

$$\lambda_d = -d(d+n-2) = -(d+\frac{n-2}{2})^2 + (\frac{n-2}{2})^2$$

the eigenvalues $\lambda_d = -d(d+n-2)$ are non-positive, and 0 only for degree $d = 0$. Also, as $d \to +\infty$, the eigenvalues go to $-\infty$. Indeed, $\lambda_{d'} < \lambda_d \le 0$ for $d' > d$, so the spaces $\mathcal{H}^d$ are *distinguished* by their eigenvalues for the spherical Laplacian.

[3.0.8] Remark: In the case of the circle S^1, the 0-eigenspace is 1-dimensional and for $d > 0$ the $(-d^2)$-eigenspace is just 2-dimensional, with basis $(x \pm iy)^d$. By contrast, for $n > 1$ the dimensions of eigenspaces are *unbounded* as the degree d goes to $+\infty$. Specifically,

[3.0.9] Claim: The dimension of $\mathcal{H}^d$ is

$$\dim_{\mathbb{C}} \mathcal{H}^d = \dim \mathbb{C}[x_1,\ldots,x_n]^{(d)} - \dim \mathbb{C}[x_1,\ldots,x_n]^{(d-2)} = \binom{n+d-1}{n} - \binom{n+d-3}{n}$$

Proof: From above,

$$\Delta : \mathbb{C}[x_1,\ldots,x_n]^{(d)} \to \mathbb{C}[x_1,\ldots,x_n]^{(d-2)}$$

is surjective, so the dimension of $\mathcal{H}^d$ is indeed the indicated difference of dimensions.

The dimension of the space of degree d polynomials in n variables can be counted as follows. The number of monomials $x_1^{e_1}\ldots x_n^{e_n}$ with $\sum_i e_i = d$ can be counted as follows. Imagine each exponent as consisting of the corresponding number of marks, lined up, with $n-1$ additional marks to separate the marks corresponding to the n distinct variables x_i, for a total of $n+d-1$. The choice of location of the separating marks is the binomial coefficient. ///

[3.0.10] Corollary: The dimension of $\mathcal{H}^d$ grows like d^{n-2} as $d \to +\infty$. ///

[3.0.11] Corollary: The dimensions of the polynomial λ-eigenspaces for the spherical Laplacian Δ grow like $\lambda^{\frac{n}{2}-1}$. ///

4. *Complete determination of eigenvectors*

The palpable reversibility of the computation of the spherical Laplacian on restrictions of positive-homogeneous functions suggests a converse:

[4.0.1] **Claim:** Let f be an eigenvector for Δ^S on $S = S^{n-1}$, with eigenvalue λ. For any complex s such that $-s(s+n-2) = \lambda$, the function

$$F(x) = |x|^s \cdot f(x/|x|)$$

on $\mathbb{R}^n - 0$ is *harmonic.*

Proof: The hypothesis is that

$$|x|^2 \cdot \Delta f(x/|x|) = -s(s+n-2) \cdot f(x/|x|)$$

with the factor of $|x|^2$ since Δ maps positive-homogeneous degree t functions to positive-homogeneous functions of degree $t-2$. By the computation of the previous section, since F is positive homogeneous of degree s,

$$|x|^2 \cdot \Delta(|x|^{-s}\, F(x)) = -s(s+n-2)\, |x|^{-s} F(x) + |x|^{-s+2} \Delta F(x)$$

Putting these together, since f and F agree on the sphere,

$$-s(s+n-2)\, |x|^{-s} F(x) = -s(s+n-2)\, F(x/|x|) = -s(s+n-2)\, f(x/|x|) = |x|^2 \cdot \Delta f(x/|x|)$$

$$= |x|^2 \cdot \Delta F(x/|x|) = |x|^2 \cdot \Delta |x|^{-s}\, F(x) = -s(s+n-2)\, |x|^{-s} F(x) + |x|^{-s+2} \Delta F(x)$$

Cancelling,

$$0 = |x|^{-s+2} \Delta F(x)$$

so F is harmonic. ///

Recall our earlier linear-algebra observation (from self-adjointness and definiteness) that all eigenvalues λ of Δ^S are *negative* real numbers (apart from $\lambda = 0$ for constants). Thus, given $\lambda \le 0$, solving for $s \in \mathbb{C}$ such that $-s(s+n-2) = \lambda$ gives

$$-\lambda = |\lambda| = s(s+n-2) = s^2 + (n-2)s = (s + \frac{n-2}{2})^2 - (\frac{n-2}{2})^2$$

Thus,

$$(s + \frac{n-2}{2})^2 = (\frac{n-2}{2})^2 - \lambda$$

and

$$s = -\frac{n-2}{2} \pm \sqrt{(\frac{n-2}{2})^2 + |\lambda|}$$

Thus, there are choices $s > 0$ and $s < 0$. The $s > 0$ choice makes $F(x) = |x|^s\, f(x/|x|)$ at least *continuous* at $x = 0$.

[4.0.2] **Lemma:** A function $F(x) = |x|^s\, f(x/|x|)$ with $s > 0$ and f continuous on S gives a *tempered distribution* (by *integrating against* it).

Proof: Since f is continuous on the compact S, it is *bounded*, say by a constant C. Then $|F(x)| \le C \cdot |x|^s$ is of polynomial growth on $\mathbb{R}^n$. Further, the same inequality shows that as $|x| \to 0$ the function $F(x)$ goes to 0. Thus, F is *continuous* on $\mathbb{R}^n$, so certainly *locally integrable.* [19] Local integrability is sufficient for

[19] As usual, *local integrability* of a function F is the condition that the integral of $|F|$ on an arbitrary compact K is finite.

(integration against) F to be a *distribution*. The moderate growth then assures that (integration against) F is a *tempered* distribution. The moderate growth. ///

Use notation $(t \cdot \varphi)(x) = \varphi(xt)$ for $t \in \mathbb{R}^\times$ and φ continuous on $\mathbb{R}^n$. Say that a distribution u on $\mathbb{R}^n$ is *positive-homogeneous* of degree s when

$$u(t^{-1} \cdot \varphi) = t^{n+s}\, u(\varphi) \qquad (\text{for } \varphi \in \mathscr{S}(\mathbb{R}^n) \text{ and } t > 0)$$

This is compatible with homogeneity of *functions*, since for $u = u_f$ being *integration against* a positive-homogeneous function f of degree s, we compute

$$u_f(t^{-1} \cdot \varphi) = \int_{\mathbb{R}^n} f(x)\, \varphi(t^{-1}x)\, dx = t^n \int_{\mathbb{R}^n} f(tx)\, \varphi(x)\, dx = t^{n+s} \int_{\mathbb{R}^n} f(x)\, \varphi(x)\, dx = t^{n+s}\, u_f(\varphi)$$

The extra exponent n arises from the change of measure.

One may recall that something like the following is true, but a reprise of the proof helps us be sure that we correctly recall how the exponent behaves.

[4.0.3] Claim: Let u be a tempered distribution u positive-homogeneous of degree s. Then $\widehat{F}$ is positive homogeneous of degree $-(s+n)$.

Proof: Using the definition,

$$\widehat{F}(t^{-1} \cdot \varphi) = u(\widehat{t^{-1}\varphi}) = t^{-n}\, u(t\, \widehat{\varphi}) = t^{-n}\, t^{-n}\, u(\widehat{\varphi}) = t^{-n}\, t^{-n}\, \widehat{F}(\varphi) = t^{-(s+n)+n}\, \widehat{F}(\varphi)$$

which, with the normalization above to match the integration-against homogeneous functions, is as desired. ///

Then a degree s positive-homogeneous F with $\Delta F = 0$ has a Fourier transform $\widehat{F}$ which is positive homogeneous of degree $-(s+n)$ and satisfies

$$(-2\pi i)^2 \cdot |x|^2 \cdot \widehat{F} = 0$$

since the Fourier transform converts the differential operator $\frac{\partial}{\partial x_j}$ into multiplication by $-2\pi i x_j$.

This immediately shows that $\widehat{F}$ *has support* $\{0\}$. One may have classified distributions supported at 0, essentially by the definitions and existence of *Taylor-Maclaurin expansions*: these are exactly derivatives of the Dirac delta (at 0).

[4.0.4] Claim: The collection of distributions supported at 0 is the direct sum

$$\bigoplus_{d=0,-1,-2,\ldots} \left(\oplus_{-|\alpha|=d}\, \mathbb{C} \cdot \delta^{(\alpha)} \right)$$

over $d = 0, -1, -2, -3, \ldots$ of positive-homogeneous distributions of degree d

$$\text{positive-homogeneous degree } d \;=\; \oplus_{|\alpha|=d}\, \mathbb{C} \cdot \delta^{(\alpha)}$$

where α is summed over multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| = \sum_i |\alpha_i|$.

Proof: We grant ourselves that all distributions at 0 are finite linear combinations of derivatives of δ. The homogeneity is easy to gauge:

$$\delta^{(\alpha)}(t^{-1} \cdot \varphi) = (-1)^{|\alpha|}\, (t^{-1} \cdot \varphi)^{(\alpha)}(0) = (-1)^{|\alpha|}\, t^{-|\alpha|}\, \varphi^{(\alpha)}(0) = t^{-|\alpha|}\, \delta^{(\alpha)}(\varphi)$$

Thus, the positive-homogeneous degree d distributions are the sum over α with $-|\alpha| = d$, as asserted.
///

[4.0.5] **Corollary:** The only possible degrees of positive-homogeneity of (tempered or not) distributions supported at 0 are $0, -1, -2, -3, \ldots$. ///

[4.0.6] **Corollary:** The only possible degrees $s > 0$ for positive-homogeneous harmonic F are $0 < s \in \mathbb{Z}$, and, further, F is a harmonic *polynomial*.

Proof: Fourier transforms of derivatives of Dirac delta are polynomials. ///

[4.0.7] **Corollary:** The eigenfunctions for Δ^S are exactly the restrictions to S^{n-1} of homogeneous harmonic polynomials of degree d, with eigenvalue $-d(d+n-2)$. ///

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5. *Existence of invariant integrals on spheres*

So far, we used the assumed *existence* of an $SO(n)$-invariant integral on S^{n-1} to be sure that eigenvalues for the spherical Laplacian Δ^S are non-positive, in determining all eigenvectors.

To *prove* existence of an invariant integral we can write a *formula* [20] as follows, using the fact that the measure on $\mathbb{R}^n$ is $SO(n)$-invariant, since the absolute value of the determinant of an element of $SO(n)$ is 1. For a continuous function f on S, define

$$\int_S f = \int_{R^n} \gamma(|x|^2)\; f(x/|x|)\; dx$$

where γ is a fixed smooth non-negative function on $[0, \infty)$ with

$$\int_{\mathbb{R}^n} \gamma(|x|^2)\, dx = 1$$

For convenience, we may at some moments suppose that γ has compact support and vanishes identically on a neighborhood of 0. [21] Then for $k \in SO(n)$ we do have the $SO(n)$-invariance of the integral:

$$\int_S k \cdot f = \int_{R^n} \gamma(|x|^2)\, f(\frac{xk}{|xk|})\, dx = \int_{R^n} \gamma(|xk^{-1}|^2)\, f(\frac{x}{|x|})\, dx = \int_{R^n} \gamma(|x|^2)\, f(\frac{x}{|x|})\, dx = \int_S f$$

by making the change of variables to replace x by xk^{-1}, and using the fact that $|xk^{-1}| = |x|$. Less trivial is proof of the desired integration-by-parts-twice result from this clunky viewpoint:

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[20] If we were already happy with *Haar measure* on $SO(n)$ and invariant measures on quotient spaces such as S^{n-1}, then we would not need an explicit construction. We rarely need more than the *existence* (and essential uniqueness) of such an integral.

[21] Compact support and vanishing condition would imply that there would be no boundary terms in when we integrate by parts on $\mathbb{R}^n$.

[5.0.1] **Proposition:** For differentiable functions f, φ on S^n,

$$\int_S (\Delta^S f) \cdot \varphi = \int_S f \cdot \Delta^S \varphi$$

Further, Δ^S is *negative-definite* in the sense that

$$\int_S (\Delta^S f) \cdot \overline{f} \leq 0$$

with equality only for f *constant*.

Proof: Let $F(x) = f(x/r)$ and $\Phi(x) = \varphi(x/r)$, where $r = |x|$. By definition,

$$\int_S (\Delta^S f) \cdot \varphi = \int_{\mathbb{R}^n} \gamma(r^2)\, r^2 \cdot (\Delta F)(x)\, \Phi(x)\, dx$$

where the r^2 is inserted so that $r^2 \Delta F$ is positive-homogeneous of degree 0 as required by the integration formula. [22] Integrating by parts, this is

$$-\int_{\mathbb{R}^n} \sum_i \frac{\partial F}{\partial x_i} \frac{\partial}{\partial x_i} \left(r^2 \cdot \gamma(r^2)\, \Phi(x) \right) dx$$

Let $\delta(r^2) = r^2 \gamma(r^2)$. Then

$$\frac{\partial}{\partial x_i} \left[r^2 \cdot \gamma(r^2)\, \Phi(x) \right] = \frac{\partial}{\partial x_i} \left[\delta(r^2)\, \Phi(x) \right] = 2x_i \delta'(r^2)\Phi(x) + \delta(r^2) \frac{\partial \Phi}{\partial x_i}$$

Thus, the whole is

$$-\int_{\mathbb{R}^n} \sum_i \frac{\partial F}{\partial x_i} \left[2x_i \delta'(r^2)\Phi(x) + \delta(r^2) \frac{\partial \Phi}{\partial x_i} \right] dx = -\int_{\mathbb{R}^n} \sum_i \frac{\partial F}{\partial x_i}\, \delta(r^2) \frac{\partial \Phi}{\partial x_i}\, dx$$

since Euler's identity asserts that

$$\sum_i x_i \frac{\partial F}{\partial x_i} = 0$$

because F is positive-homogeneous of degree 0. That last expression for the integral is symmetric in F and Φ, giving the desired integration-by-parts result. Finally, with $\Phi = \overline{F}$, the last expression for the integral is visibly non-positive, and is 0 only if $\partial F / \partial x_i = 0$ for all i, only if F is constant, only if f is constant. ///

[5.0.2] **Remark:** This argument *should* be unsatisfying, since it does not readily extend to more general situations. We will give a more universal existence argument a bit later.

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6. L^2 *spectral decompositions on spheres*

The idea of **spectral decomposition** on the sphere is that functions on the sphere should be *sums of eigenfunctions* for the spherical Laplacian. For L^2 functions the convergence should be in L^2. For *smooth*

[22] Since $F(x) = f(x/r)$ is positive-homogeneous of degree 0, ΔF is positive-homogeneous of degree -2, so we need to adjust it by r^2.

functions, the sum should converge well in a *pointwise* sense. As usual, L^2 convergence is not likely to imply pointwise convergence.

[6.0.1] **Theorem:** The collection of finite linear combinations of homogeneous harmonic polynomials restricted to S^{n-1} is *dense* in $L^2(S^{n-1})$.

Proof: To prove *completeness*, we will prove that restrictions to the sphere of harmonic polyomials are dense in $C^o(S^{n-1})$, and then invoke Urysohn's lemma to know that $C^o(S^{n-1})$ is dense in $L^2(S^{n-1})$.

To prove the density of (restrictions of) polynomials in $C^o(S^{n-1})$, construct an **approximate identity** (on the sphere) from polynomials. That is, we find a sequence φ_n of (harmonic) polynomials whose values on the sphere are *non-negative*, whose *integrals* on the sphere are 1, and such that, for every $\varepsilon > 0$ and $\delta > 0$, there is N such that for $n \geq N$

$$\int_{x\in S^{n-1}:\ |x-e_1|>\delta} \varphi_n(x)\,dx < \varepsilon$$

That is, the masses of the φ_n bunch up at $e_1 \in S^{n-1}$. Then by acting by $SO(n)$ we could make a sequence of functions with masses bunching up at any desired point. [23]

[... iou ...]

Thus, every L^2 function f on S^n has at least an L^2 Fourier expansion

$$f = \sum_{d=0}^{\infty} f_d \qquad \text{(equality in an } L^2 \text{ sense)}$$

where f_d is the orthogonal (in L^2) projection of f to the space $\mathcal{H}^d$ of homogeneous degree d harmonic polynomials (restricted to the sphere).

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7. *Pointwise convergence of Fourier series on* S^n

To assess the *pointwise* convergence of Fourier series on the sphere, we must first be aware that, unlike the functions $e^{in\theta}$ on circle S^1, for $f \in \mathcal{H}^d$ on S^n with $n > 1$ there is *no* instantaneous comparison [24] of the two norms

$$|f|_{C^o} = \sup_{x\in S^{n-1}} |f(x)| \qquad |f|_{L^2} = \left(\int_S |f(x)|^2\,dx\right)^{1/2}$$

Nevertheless, a little work *does* give a useful comparison: [25]

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[23] Construction of such an approximate identity in effect gives a conceptual argument for Stone-Weierstraß-type results in this highly structured situation.

[24] Of course, since the spaces $\mathcal{H}^d$ are finite-dimensional, each one has a unique topology compatible with the vector space structure. (The latter fact is not trivial to prove, but is not too hard.) Thus, the fact that the sup-norm and L^2-norm are comparable on these finite-dimensioal spaces is clear *a priori.* But this qualitative fact is irrelevant to the issue at hand. What is *not* clear is how the comparison constants grow in the parameter d, and this greater precision is needed for Sobolev-type estimates.

[25] In fact, the proof of the following proposition uses few of the specifics of this situation, and, indeed, the same argument proves that for a compact group K acting transitively on a set X, with a finite-dimensional K-stable space V of functions on X, for $f \in V$ we have the same comparison of sup-norm and L^2-norm, namely $|f|_{C^o} \leq \sqrt{(\dim V)/\mathrm{vol}\,(X)} \cdot |f|_{L^2}$. This inequality is interesting even for *finite* groups.

[7.0.1] Proposition: Let $f \in \mathcal{H}^d$. Then

$$|f|_{C^o} \le \sqrt{\frac{\dim \mathcal{H}^d}{\mathrm{vol}\,(S^{n-1})}} \cdot |f|_{L^2}$$

And the estimate is *sharp*, in the sense that there is a function in $\mathcal{H}^d$ for which equality holds.

[7.0.2] Remark: The occurrence of the square root of the total measure of S^{n-1} compensates for the fact that the square root of the total measure enters in the L^2-norm, and does *not* enter in the sup norm.

[7.0.3] Remark: Using the dimension computation from above, as $d \to +\infty$

$$\sqrt{\dim \mathcal{H}^d} \sim \sqrt{\binom{n+d-1}{n} - \binom{n+d-3}{n}} \sim d^{\frac{n}{2}-1}$$

Proof: For $x \in S$, from the Riesz-Fischer theorem, the functional $f \to f(x)$ is necessarily given by

$$f(x) = \langle f, F_x \rangle$$

for some $F_x \in \mathcal{H}^d$. Then the Cauchy-Schwarz-Bunyakowsky inequality gives

$$|f(x)| = |\langle f, F_x \rangle| \le |f|_{L^2} \cdot |F_x|_{L^2}$$

which bounds the value $f(x)$ in terms of its L^2-norm with constant being the L^2-norm of F_x. In particular,

$$|F_x(y)| = |\langle F_x, F_y \rangle| \le |F_x|_{L^2} \cdot |F_y|_{L^2}$$

and for $x = y$

$$F_x(x) = \langle F_x, F_x \rangle = |F_x|^2_{L^2}$$

This shows that

$$|F_x|_{C^o} = F_x(x) = |f\mathbb{F}_x|^2_{L^2}$$

We will show that $F_x(x)$ is independent of $x \in S$, which will allow us to determine it precisely.

Let $k \in SO(n)$, with action $(k \cdot f)(x) = f(xk)$. [26] By design, the action of $k \in SO(n)$ on functions is *unitary* in the sense that

$$\langle k \cdot f, k \cdot F \rangle = \int_X f(xk)\, \overline{F(xk)}\, dx = \int_X f(x)\, \overline{F(x)}\, dx = \langle f, F \rangle$$

by replacing x by xk^{-1} in the integral. Then the functions F_x have natural relations among themselves, namely,

$$\langle f, F_{xk} \rangle = f(xk) = (k \cdot f)(x) = \langle k \cdot f, F_x \rangle = \langle f, k^{-1} \cdot F_x \rangle$$

Thus, the relation is

$$F_{xk} = k^{-1} \cdot F_x$$

In particular, the L^2-norm of the function F_x is the same for every $x \in S$.

[26] As usual, the inverse in the definition of the action on *functions* on the space is to have the associativity $(hk) \cdot f = h \cdot (k \cdot f)$ for $h, k \in SO(n)$.

Now comes the trick. Express F_x in terms of an orthonormal basis $\{f_i\}$ for $\mathcal{H}^d$, as usual, by

$$F_x = \sum_i \langle F_x, f_i \rangle \cdot f_i$$

Evaluating both sides at x gives

$$F_x(x) = \sum_i \langle F_x, f_i \rangle \cdot f_i(x) = \sum_i \overline{f_i(x)} \cdot f_i(x) = \sum_i |f_i(x)|^2$$

In fact, the value $F_x(x)$ is independent of x, since for $k \in SO(n)$ and $x \in S$

$$F_{xk}(xk) = \langle F_{xk}, F_{xk} \rangle = \langle k \cdot F_x, k \cdot F_x \rangle = \langle F_x, F_x \rangle = F_x(x)$$

by the unitariness of the action. Integrating $F_x(x) = \sum_i |f_i(x)|^2$ over S, using the fact that $F_x(x)$ is independent of $x \in S$,

$$\text{vol}\,(S) \cdot F_x(x) = \dim_{\mathbb{C}} \mathcal{H}^d$$

Then

$$F_x(x) = |F_x|_{C^o} = |F_x|^2_{L^2}$$

gives

$$|F_x|_{L^2} = \sqrt{F_x(x)} = \sqrt{\frac{\dim \mathcal{H}^d}{\text{vol}\,(S)}}$$

Combining this with $|f(x)|\ e|F_x|_{L^2} \cdot |f|_{L^2}$ from above, we have

$$|f|_{C^o} \leq |F_x|_{L^2} \cdot |f|_{L^2} \sqrt{\frac{\dim \mathcal{H}^d}{\text{vol}\,(S)}} \cdot |f|_{L^2}$$

as claimed by the proposition.

Further,

$$|F_x|_{C^o} = |F_x|_{L^2} \cdot |F_x|_{L^2} = \sqrt{\frac{\dim \mathcal{H}^d}{\text{vol}\,(S)}} \cdot |F_x|_{L^2}$$

so the estimate is sharp. ///

[7.0.4] **Corollary:** In an L^2-expansion $f = \sum_d f_d$ with $f_d \in \mathcal{H}^d$, the convergence is (uniformly) *pointwise* (so the partial sums converge to a continuous function) when

$$\sum_d d^{(n-1)/2} \cdot |f_d|_{L^2} < \infty$$

Proof: From the proposition, and from the formula for the dimension of spaces of harmonic polynomials, there is a uniform constant C such that for all degrees d

$$|f_d|_{C^o} \leq C \cdot d^{(n-1)/2} \cdot |f_d|_{L^2}$$

Then the convergence assumption assures that the sequence $\sum_{d \leq N} f_d$ of continuous functions on S converges in sup norm, so converges uniformly pointwise to a continuous function. ///

[7.0.5] **Remark:** Note that in the sum in the corollary the L^2 norms of the Fourier components f_d are *not* squared. Thus, this is not quite the sort of L^2 condition we really want. Still, it is the *naturally arising* condition for (uniform) pointwise convergence. The case of S^1 is perhaps misleadingly simple because the exponentials $e^{\pm in\theta}$ are bounded, unlike the higher-dimensional case.

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8. *Sobolev inequalitites on* S

EDIT: more later...

To discuss differentiation of a function f on the sphere S, we might consider differentiation of the corresponding positive-homogeneous degree 0 function

$$F(x) = f(x/|x|)$$

on $\mathbb{R}^n - 0$, with respect to the usual coordinates' operators $\partial/\partial x_i$. Or we might aim at a more intrinsic notion.

[... iou ...]