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# Meromorphic continuation and functional equation of $GL_2$ Eisenstein series

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Poisson summation suffices to give a general argument for meromorphic continuation and functional equation of Eisenstein series for  $GL_2$  over number fields, by integral representations of renormalized forms of the Eisenstein series.

I saw the basic idea in [Godement 1966a], and later learned that a version already appeared in [Rankin 1939]. I seem to recall that Rankin commented somewhere that his advisor Ingham already knew this device. An adelic version for  $GL_2$  appears in my old book [Garrett 1989], superficially for totally real fields, but obviously not using this feature.

Much as Iwasawa-Tate theory proves analytic continuation and functional equation for *all* zeta integrals, and delays concern about refinements and symmetry, this approach proves analytic continuation and functional equation for a large class of Eisenstein series, delaying concern for details.

Let  $\chi$  be a character on  $M_k \backslash M_{\mathbb{A}}$ , where  $M$  is the diagonal subgroup of  $G = GL_2$  over  $k$ . Let  $N$  be upper-triangular unipotent matrices. Extend  $\chi$  trivially to  $\tilde{\chi}$  on  $N_{\mathbb{A}}$ , that is,  $\tilde{\chi}$  is given by

$$\tilde{\chi} \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} = \chi_1(a) \cdot \chi_2(d) \quad (\text{for some characters } \chi_1, \chi_2 \text{ on } \mathbb{J}/k^\times)$$

Let  $\Phi$  be a Schwartz function on  $\mathbb{A}^2$ . Let  $e_2 = (0, 1)$  be the usual second basis-vector for  $\mathbb{A}^2$ , and define a vector  $\varphi_{\chi, \Phi}$  in the  $\chi^{th}$  principal series by

$$\varphi_{\chi, \Phi}(g) = \chi_1(\det g) \int_{\mathbb{J}} \chi_1 \chi_2^{-1}(t) \Phi(t \cdot e_2 \cdot g) dt$$

The required left equivariance follows from changing variables. Note the normalization: on  $P_{\mathbb{A}}$ ,

$$\varphi_{\chi, \Phi}(p) = \tilde{\chi}(p) \cdot \int_{\mathbb{J}} \chi_1 \chi_2^{-1}(t) \Phi(t \cdot e_2) dt \quad (\text{for } p \in P_{\mathbb{A}})$$

The restriction of  $\Phi$  to the second coordinate produces a Schwartz function on  $\mathbb{A}$ , and the latter integral is the Iwasawa-Tate zeta integral attached to that Schwartz function and to the idele class character  $\chi_1 \chi_2^{-1}$ . Let  $\mathbb{J}^1$  be the ideles of idele-norm 1.

Define a kind of Eisenstein series by

$$\mathcal{E}(\chi, \Phi)(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi_{\chi, \Phi}(\gamma \cdot g)$$

Let  $w_o = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  be the long Weyl element. For a character  $\chi$  on  $M$ ,

$$(\chi \circ w_o)(m) = \chi(w_o^{-1} m w_o) \quad (\text{for } m \in M_{\mathbb{A}})$$

Put

$$\tilde{\chi} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = |a| \chi_2(a) \cdot |d|^{-1} \chi_1(d) = \left| \frac{a}{d} \right| \cdot (\chi \circ w_o) \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

[0.0.1] **Theorem:** The Eisenstein series  $\mathcal{E}(\chi, \Phi)$  has a meromorphic continuation in the (complex parameter part of)  $\chi$ , with functional equation

$$\mathcal{E}(\chi, \Phi) = \mathcal{E}(\widehat{\chi}, \widehat{\Phi})$$

The Eisenstein series has poles only for  $\chi_1/\chi_2 = |\cdot|^{2s}$  for some  $s \in \mathbb{C}$ , in which case the poles are at  $s = 1, 0$ , are simple, with respective residues

$$\begin{cases} \operatorname{Res}_{s=1} \mathcal{E}(\chi, \Phi) &= \frac{\chi_1(\det g)}{|\det g|} \cdot \widehat{\Phi}(0) \cdot \frac{\kappa}{2} \\ \operatorname{Res}_{s=0} \mathcal{E}(\chi, \Phi) &= -\chi_1(\det g) \cdot \Phi(0) \cdot \frac{\kappa}{2} \end{cases} \quad (\text{when } \chi_1/\chi_2 = |\cdot|^{2s})$$

where  $\kappa$  is the residue at 1 of the completed zeta function of the groundfield, that is, the natural volume of  $\mathbb{J}^1/k^\times$ .

*Proof:* Since  $P_k$  is the isotropy subgroup in  $G_k$  of the line generated by  $e_2$ ,

$$P_k \backslash G_k \longleftrightarrow k^\times \backslash (k^2 - \{0\}) \quad \text{by} \quad P_k g \longrightarrow k^\times e_2 g$$

Thus, with

$$\begin{aligned} \Theta(\Phi, g) &= \sum_{\xi \in k^2} \Phi(\xi \cdot g) \\ \mathcal{E}(\chi, \Phi)(g) &= \sum_{\gamma \in P_k \backslash G_k} \varphi_{\chi, \Phi}(\gamma \cdot g) = \sum_{\gamma \in P_k \backslash G_k} \chi_1(\det g) \int_{\mathbb{J}} \chi_1 \chi_2^{-1}(t) \Phi(t \cdot e_2 \cdot \gamma g) dt \\ &= \sum_{\gamma \in P_k \backslash G_k} \chi_1(\det g) \int_{\mathbb{J}/k^\times} \chi_1 \chi_2^{-1}(t) \sum_{\alpha \in k^\times} \Phi(t \cdot \alpha e_2 \cdot \gamma g) dt \\ &= \chi_1(\det g) \int_{\mathbb{J}/k^\times} \chi_1 \chi_2^{-1}(t) \sum_{\gamma \in P_k \backslash G_k} \sum_{\alpha \in k^\times} \Phi(t \cdot \alpha e_2 \cdot \gamma g) dt \\ &= \chi_1(\det g) \int_{\mathbb{J}/k^\times} \chi_1 \chi_2^{-1}(t) \left( \Theta(\Phi, tg) - \Phi(0) \right) dt \end{aligned}$$

Just as Riemann did, break the integral into two parts, one over  $\mathbb{J}^+ = \{y \in \mathbb{J} : |y| \geq 1\}$  and the other over  $\mathbb{J}^- = \{y \in \mathbb{J} : |y| \leq 1\}$ :

$$\begin{aligned} &\int_{\mathbb{J}/k^\times} \chi_1 \chi_2^{-1}(t) \left( \Theta(\Phi, tg) - \Phi(0) \right) dt \\ &= \int_{\mathbb{J}^+/k^\times} \chi_1 \chi_2^{-1}(t) \left( \Theta(\Phi, tg) - \Phi(0) \right) dt + \int_{\mathbb{J}^-/k^\times} \chi_1 \chi_2^{-1}(t) \left( \Theta(\Phi, tg) - \Phi(0) \right) dt \end{aligned}$$

Estimates identical to those in Iwasawa-Tate show that the integral over  $\mathbb{J}^+/k^\times$  is *entire* in  $\chi$ .

Use Poisson summation to convert the integral over  $\mathbb{J}^-/k^\times$  into an integral over  $\mathbb{J}^+k^\times$ , with two extra, elementary terms. First, by Poisson summation, letting  $g^\theta = (g^\top)^{-1}$ ,

$$\Theta(\Phi, tg) = \sum_{\xi \in k^2} \Phi(t\xi g) = \frac{1}{|t|^2 \cdot |\det g|} \sum_{\xi \in k^2} \widehat{\Phi}(t^{-1} \xi (g^\top)^{-1}) \quad (\text{with } g^\theta = (g^\top)^{-1})$$

Then

$$\Theta(\Phi, tg) - \Phi(0) = \frac{1}{|t|^2 \cdot |\det g|} \cdot \left( \Theta(\widehat{\Phi} t^{-1} g^\theta) - \widehat{\Phi}(0) \right) + \frac{\widehat{\Phi}(0)}{|t|^2 \cdot |\det g|} - \Phi(0)$$

Then

$$\begin{aligned} \int_{\mathbb{J}^-/k^\times} \chi_1 \chi_2^{-1}(t) \left( \Theta(\Phi, tg) - \Phi(0) \right) dt &= \int_{\mathbb{J}^-/k^\times} \frac{\chi_1 \chi_2^{-1}(t)}{|t|^2 \cdot |\det g|} \cdot \left( \Theta(\widehat{\Phi}, t^{-1}g^\theta) - \widehat{\Phi}(0) \right) dt \\ &+ \frac{\widehat{\Phi}(0)}{|\det g|} \int_{\mathbb{J}^-/k^\times} |t|^{-2} \chi_1 \chi_2^{-1}(t) dt - \Phi(0) \int_{\mathbb{J}^-/k^\times} \chi_1 \chi_2^{-1}(t) dt \end{aligned}$$

The latter two integrals are elementary, as in Iwasawa-Tate:

$$\begin{aligned} \int_{\mathbb{J}^-/k^\times} |t|^{-2} \chi_1 \chi_2^{-1}(t) dt &= \begin{cases} \frac{\kappa/2}{s-1} & (\text{for } \chi_1 \chi_2^{-1} = |\cdot|^{2s}) \\ 0 & (\text{for } \chi_1 \chi_2^{-1} \text{ non-trivial on } \mathbb{J}^1) \end{cases} \\ \int_{\mathbb{J}^-/k^\times} \chi_1 \chi_2^{-1}(t) dt &= \begin{cases} \frac{\kappa/2}{s} & (\text{for } \chi_1 \chi_2^{-1} = |\cdot|^{2s}) \\ 0 & (\text{for } \chi_1 \chi_2^{-1} \text{ non-trivial on } \mathbb{J}^1) \end{cases} \end{aligned} \quad (\text{with } s = s_\chi)$$

where  $\kappa$  is the residue at 1 of the completed zeta function of the groundfield. Write  $\delta_\chi$  for 1 when  $\chi_1 \chi_2^{-1}$  is of the form  $|\cdot|^{2s}$ , and 0 otherwise. Replacing  $t$  by  $t^{-1}$  in the second non-elementary integral,

$$\begin{aligned} \mathcal{E}(\chi, \Phi)(g) &= \chi_1(\det g) \int_{\mathbb{J}^+/k^\times} \chi_1 \chi_2^{-1}(t) \left( \Theta(\Phi, tg) - \Phi(0) \right) dt \\ &+ \frac{\chi_1(\det g)}{|\det g|} \int_{\mathbb{J}^+/k^\times} |t|^2 \chi_1^{-1} \chi_2(t) \left( \Theta(\widehat{\Phi}, tg^\theta) - \widehat{\Phi}(0) \right) dt + \delta_\chi \left( \frac{\chi_1(\det g)}{|\det g|} \frac{\widehat{\Phi}(0)\kappa/2}{s_\chi - 1} - \chi_1(\det g) \frac{\Phi(0)\kappa/2}{s_\chi} \right) \end{aligned}$$

Noting that

$$\frac{\chi_1(\det g)}{|\det g|} = (|\cdot| \chi_1^{-1})(\det(g^\theta))$$

we are motivated to regroup

$$|t|^2 \chi_1^{-1} \chi_2(t) = (|\cdot| \chi_1^{-1})(t) \cdot (|\cdot| \chi_2)(t) = (|\cdot| \chi_1^{-1})(t) \cdot (|\cdot|^{-1} \chi_2^{-1})^{-1}(t)$$

to comply with the form of the integral representation of  $\mathcal{E}$ . For convenience, let  $\chi'$  be the character on  $M$  given by

$$\chi' \left( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) = (|\cdot| \chi_1^{-1})(a) \cdot (|\cdot|^{-1} \chi_2^{-1})(d)$$

with corresponding  $\chi'_1$  and  $\chi'_2$ . Then  $s_{\chi'} = 1 - s_\chi$ . Thus, so far,

$$\begin{aligned} \mathcal{E}(\chi, \Phi)(g) &= \chi_1(g) \int_{\mathbb{J}^+/k^\times} \chi_1 \chi_2^{-1} \left( \Theta(\Phi, tg) - \Phi(0) \right) dt \\ &+ \chi'_1(g^\theta) \int_{\mathbb{J}^+/k^\times} \chi'_1 \chi'^{-1}_2 \left( \Theta(\widehat{\Phi}, tg^\theta) - \widehat{\Phi}(0) \right) dt - \delta_\chi \left( \chi'_1(\det g^\theta) \frac{\widehat{\Phi}(0)\kappa/2}{s_{\chi'}} + \chi_1(\det g) \frac{\Phi(0)\kappa/2}{s_\chi} \right) \end{aligned}$$

The visible symmetry under  $g \rightarrow g^\theta$ ,  $\chi \rightarrow \chi'$ ,  $\Phi \rightarrow \widehat{\Phi}$  on the right-hand side gives

$$\mathcal{E}(\chi, \Phi)(g) = \mathcal{E}(\chi', \widehat{\Phi})(g^\theta)$$

This is nice, but for  $GL_2$  we need not tolerate the transpose-inverse. Recall that

$$g^\theta = \frac{w_o^{-1} g w_o}{\det g}$$

Thus, in the region of convergence,

$$\begin{aligned} \mathcal{E}(\chi', \widehat{\Phi})(g^\theta) &= \chi_1'(\det g^\theta) \int_{\mathbb{J}/k^\times} |t|^2 \chi_1^{-1} \chi_2(t) \sum_{0 \neq \xi \in k^2} \widehat{\Phi}(t\xi g^\theta) dt \\ &= |\det \left(\frac{g}{\det g}\right)| \cdot \chi_1^{-1}(\det \left(\frac{g}{\det g}\right)) \int_{\mathbb{J}/k^\times} |t|^2 \chi_1^{-1} \chi_2(t) \sum_{0 \neq \xi \in k^2} \widehat{\Phi}(t\xi \frac{w_o^{-1} g w_o}{\det g}) dt \\ &= |\det g| \cdot \chi_2(\det g) \int_{\mathbb{J}/k^\times} |t|^2 \chi_1^{-1} \chi_2(t) \sum_{0 \neq \xi \in k^2} \widehat{\Phi}(t\xi w_o^{-1} g w_o) dt \end{aligned}$$

by replacing  $t$  by  $t \cdot \det g$ . Since  $k^2$  is stable under multiplication by  $w_o$ , this is

$$\mathcal{E}(\chi', \widehat{\Phi})(g^\theta) = |\det g| \cdot \chi_2(\det g) \int_{\mathbb{J}/k^\times} |t|^2 \chi_1^{-1} \chi_2(t) \sum_{0 \neq \xi \in k^2} \widehat{\Phi}(t\xi g w_o) dt$$

Thus, with

$$\widehat{\chi} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = |a| \chi_2(a) \cdot |d|^{-1} \chi_1(d) = \left|\frac{a}{d}\right| \cdot (\chi \circ w_o) \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

with corresponding

$$\widehat{\chi}_1(a) = |a| \chi_2(a) \quad \text{and} \quad \widehat{\chi}_2(d) = |d|^{-1} \chi_1(d)$$

we have

$$\mathcal{E}(\chi', \widehat{\Phi})(g^\theta) = \mathcal{E}(\widehat{\chi}, \widehat{\Phi} \circ w_o)(g)$$

That is, we have the functional equation

$$\mathcal{E}(\chi, \Phi) = \mathcal{E}(\widehat{\chi}, \widehat{\Phi} \circ w_o)$$

This completes the discussion. ///

[0.0.2] **Remark:** One should really prove that the half-zeta integrals are entire, as is similarly necessary in Iwasawa-Tate, and already in Riemann.

[0.0.3] **Remark:** More importantly, there remains some sorting-out of the Eisenstein series here in terms of Eisenstein series defined more directly and used in spectral theory. This sorting-out is a slightly more complicated version of the analogous refined choices of Schwartz data to get the best outcome in Iwasawa-Tate.

[Garrett 1989] P. Garrett, *Holomorphic Hilbert modular forms*, Wadsworth/Brooks-Cole, 1989.

[Godement 1966a] R. Godement, *Decomposition of  $L^2(\Gamma \backslash G)$  for  $\Gamma = SL(2, \mathbb{Z})$* , in Proc. Symp. Pure Math. **9** (1966), AMS, 211-24.

[Rankin 1939] R. Rankin, *Contributions to the theory of Ramanujan's function  $\tau(n)$  and similar arithmetic functions, I*, Proc. Cam. Phil. Soc. **35** (1939), 351-372.

[Selberg 1940] A. Selberg, *Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist*, Arch. Math. Naturvid **43** (1940), pp. 47-50.