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Continuous spectrum for $SL_2(\mathbb{Z})\backslash\mathfrak{H}$

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

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We recall the notion of *cuspform* for $\Gamma = SL_2(\mathbb{Z})$ acting on the upper half-plane, and prove that the orthogonal complement in $L^2(\Gamma\backslash\mathfrak{H})$ to cuspforms is spanned by *pseudo-Eisenstein series*, which in turn are expressible as *wave packets* of *Eisenstein series* E_s. Further we have a Plancherel theorem for this space. The non-trivial harmonic analysis resides in the application of the *Fourier transform* on the real line, in coordinates in which it is called a *Mellin transform*. That is, this part of the harmonic analysis of $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ reduces to harmonic analysis on a related, lower-dimensional group.

I first saw this argument in [Godement 1966].

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1. *Pseudo-Eisenstein series adjunction to constant term*

Let N be the subgroup of G consisting of upper-triangular unipotent matrices, A^+ the subgroup of diagonal matrices with positive diagonal entries, and P the parabolic subgroup consisting of all upper-triangular matrices.

[1.1] Constant terms

The **constant term** along P of a reasonable[1] function f on $\Gamma\backslash G$ is

$$(\text{constant term of } f)(g) \;=\; c_P f(g) \;=\; \int_{N\cap\Gamma\backslash N} f(ng)\, dn$$

By changing variables, $c_P f$ is left N-invariant. Since P normalizes N, the constant term also retains the left $P\cap\Gamma$-invariance of f.

[1.2] Cuspforms

A reasonable function f on $\Gamma\backslash G$ is a **cuspform** when its constant term essentially[2] vanishes:

$$f \text{ cuspform} \iff c_P f \;=\; 0$$

It is better to recast the cuspform condition in the form

$$f \text{ cuspform} \iff \int_{N\backslash G} c_P f(g)\cdot\varphi(g)\, dg \;=\; 0 \qquad (\text{for all } \varphi\in C_c^\infty((P\cap\Gamma)N\backslash G))$$

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[1] Rather than worry about conditions guaranteeing convergence of the integral for $c_P f$, or about the nature of the function $c_P f$, it suffices to consider *distributions* f on $\Gamma\backslash G$, and consider $c_P f$ as a *distribution* on $(P\cap\Gamma)N\backslash G$. Elaboration of this viewpoint is implicit in the subsequent discussion, and *explicitly* produces pseudo-Eisenstein series in a natural fashion.

[2] Again, the constant term $c_P f$ is best treated as a distribution, as we implicitly do in the sequel. It is possible, as in many sources, to treat $c_P f$ as a measurable function. In that context, *essentially* vanishing means vanishing *almost everywhere.*

That is, the cuspform condition is that the constant term vanishes *as a distribution* on $(P\cap\Gamma)N\backslash G$.

[1.3] Pseudo-Eisenstein series

Pseudo-Eisenstein series appear naturally as the solution to an *adjunction* problem, described as follows. Use the notational convention

$$\langle f, F\rangle_{H\backslash G} \;=\; \int_{H\backslash G} f\cdot F\, dg \qquad\qquad (\mathbb{C}\text{-bilinear, not hermitian})$$

even when f, F need not lie in the same space of functions, and use the same notation for extensions of integral pairings to distributions. The problem is, given φ in $C_c^\infty((P\cap\Gamma)N\backslash G)$, find $\Psi_\varphi\in C_c^\infty(\Gamma\backslash G)$ such that

$$\langle c_P f, \varphi\rangle_{(P\cap\Gamma)N\backslash G} \;=\; \langle f, \Psi_\varphi\rangle_{\Gamma\backslash G} \qquad\qquad (\text{for } f \text{ on } \Gamma\backslash G)$$

Incidentally, exhibition of such Ψ_φ will completely legitimize treatment of c_P as a map from distributions on $\Gamma\backslash G$ to distributions on $(P\cap\Gamma)N\backslash G$, even though we do not need this degree of generality.

[1.3.1] Remark: For explicit computations, it is important to understand the measure on $N\backslash G/K$ or $N\backslash G$. The measure on $\mathfrak{H}\approx G/K$ is the usual left G-invariant measure $dx\,dy/y^2$ inherited from Haar measure on $G = NA^+K$. This measure descends to dy/y^2 on $N\backslash\mathfrak{H}\approx N\backslash G/K$, *not* the measure dy/y from the Haar measure on A^+. For $\Gamma = SL_2(\mathbb{Z})$, by chance $(P\cap\Gamma)\subset NK$, so $(P\cap\Gamma)N\backslash\mathfrak{H} = N\backslash\mathfrak{H}$, and the measure is dy/y^2.

A canonical *expression* for the desired Ψ_φ is found by direct computation, using the left $(P\cap\Gamma)N$-invariance of φ and the left Γ-invariance of f:

$$\langle c_P f, \varphi\rangle_{(P\cap\Gamma)N\backslash G} \;=\; \int_{(P\cap\Gamma)N\backslash G} c_P f(g)\,\varphi(g)\,dg \;=\; \int_{(P\cap\Gamma)N\backslash G}\Big(\int_{N\cap\Gamma\backslash N} f(ng)\,dn\Big)\,\varphi(g)\,dg$$

$$=\; \int_{(P\cap\Gamma)N\backslash G}\Big(\int_{N\cap\Gamma\backslash N} f(ng)\,\varphi(ng)\,dn\Big)\,dg \;=\; \int_{P\cap\Gamma\backslash G} f(g)\,\varphi(g)\,dg$$

Winding up, this is

$$\int_{P\cap\Gamma\backslash G} f(g)\,\varphi(g)\,dg \;=\; \int_{\Gamma\backslash G}\sum_{\gamma\in P\cap\Gamma\backslash\Gamma} f(\gamma g)\,\varphi(\gamma g)\,dg \;=\; \int_{\Gamma\backslash G} f(g)\Big(\sum_{\gamma\in P\cap\Gamma\backslash\Gamma}\varphi(\gamma g)\Big)\,dg$$

Thus, the pseudo-Eisenstein series[3] attached to φ is

$$\Psi_\varphi(g) \;=\; \sum_{\gamma\in P\cap\Gamma\backslash\Gamma}\varphi(\gamma g)$$

While cuspforms are mysterious, these pseudo-Eisenstein series are completely *not*-mysterious.

[1.3.2] Lemma: The series for a pseudo-Eisenstein series Ψ_φ is *locally finite*, meaning that for g in a fixed compact in G, there are only finitely-many non-zero summands in $\sum_\gamma\varphi(\gamma g)$. Thus, $\Psi_\varphi\in C_c^\infty(\Gamma\backslash G)$.

Proof: Given $\varphi\in C_c^\infty((P\cap\Gamma)N\backslash G)$, let C be a compact set in G so that $(P\cap\Gamma)N\cdot C$ contains the support of φ. Fix a compact subset C_o of G in which $g\in G$ is constrained to lie. Then a summand $\varphi(\gamma g)$ is non-zero only if $\gamma g\in(P\cap\Gamma)N\cdot C$, which is to say

$$\gamma \;\in\; (P\cap\Gamma)\cdot C\cdot g^{-1}$$

[3] In 1966 Godement called these *incomplete theta series*, but more recently Moeglin-Waldspurger strengthened the precedent of calling them *pseudo-Eisenstein series*

so

$$\gamma \in \Gamma \cap (P\cap\Gamma)\cdot C\cdot C_o^{-1}$$

In the quotient $G \to (P\cap\Gamma)\backslash G$, the image of Γ is discrete. The image of the compact set $(P\cap\Gamma)N\cdot C\cdot C_o^{-1}$ under the continuous quotient map is *compact*, since $(P\cap\Gamma)\backslash(P\cap\Gamma)N$ is compact, and continuous images of compacts are compact. Thus, left modulo $P\cap\Gamma$, that intersection is the intersection of a discrete set and a compact set, so*finite*. Therefore, the series is *locally finite*, and defines a smooth function on $\Gamma\backslash G$.

To show that Ψ_φ has compact support in $\Gamma\backslash G$, proceed similarly. That is, for a summand $\varphi(\gamma g)$ to be non-zero, it must be that

$$g \in \Gamma\cdot C$$

The image $\Gamma\backslash(\Gamma\cdot C)$ is compact, being the continuous image of the compact set C under the continuous map $G\to\Gamma\backslash G$, proving the compact support. ///

[1.3.3] Corollary: The square-integrable cuspforms are the orthogonal complement of the (closed) space spanned by the pseudo-Eisenstein series in $L^2(\Gamma\backslash G)$. ///

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2. *Decomposition of pseudo-Eisenstein series: beginning*

Spectral decomposition of the data φ in Ψ_φ induces a spectral decomposition of Ψ_φ itself, in a natural manner.

[2.1] Fourier transform

Fourier transform $\hat{f}$ of f on the real line is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\xi x}\, dx$$

Fourier inversion for Schwartz functions is

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i\xi x}\, d\xi$$

For Schwartz functions f the integral for Fourier transform and for its inversion both converge very well. Replacing ξ by $\xi/(2\pi)$ gives the variant

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\, e^{-it\xi}\, dt\right) e^{i\xi x}\, d\xi$$

[2.2] Multiplicative coordinates

Fourier transforms on $\mathbb{R}$ put into multiplicative coordinates are Mellin transforms: for $F\in C_c^\infty(0,+\infty)$, take

$$f(x) = F(e^x)$$

Let $y=e^x$ (and r the exponentiated variable in the implied inner integral) and rewrite Fourier inversion as

$$F(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_0^{\infty} F(r)\, r^{-i\xi}\, \frac{dr}{r}\right) y^{i\xi}\, d\xi$$

The Fourier transform in these coordinates is called a *Mellin* or *Laplace* transform $\mathcal{M}F$, defined by

$$\mathcal{M}F(i\xi) = \int_0^{\infty} F(r)\, r^{-i\xi}\, \frac{dr}{r}$$

For compactly-supported F, the integral definition extends to complex s:

$$\mathcal{M}F(s) \;=\; \int_0^\infty F(r)\, r^{-s}\, \frac{dr}{r}$$

The variant Fourier inversion identity gives the Mellin inversion formula

$$F(y) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{M}F(i\xi)\; y^{i\xi}\; d\xi$$

View ξ as the imaginary part of a complex variable s, rewrite the latter integral as a complex path integral, so it becomes (since $d\xi = -i\, ds$)

$$F(y) \;=\; \frac{1}{2\pi i}\int_{0-i\infty}^{0+i\infty} \mathcal{M}F(s)\; y^s\, ds$$

[2.2.1] Remark: It is important that for $f \in C_c^\infty(\mathbb{R})$ the Fourier transform $\hat{f}(\xi)$ extends to an *entire* function in ξ, of rapid decay on horizontal lines. [4]

Then certainly the same is true for the transform $\mathcal{M}F$ of $F \in C_c^\infty(0,+\infty)$. In this case, for *any* real σ, Mellin inversion is

$$F(y) \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}F(s)\; y^s\, ds$$

For $\varphi \in C_c^\infty(0,\infty)$, the Mellin inversion formula gives

$$\varphi(y) \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)\, y^s\, ds$$

Identifying $N\backslash G/K \approx A^+$, this is

$$\varphi(g) \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)\, \chi_s\big(a(g)\big)\, ds$$

Thus, the pseudo-Eisenstein series is expressible as

$$\Psi_\varphi(g) \;=\; \sum_{\gamma\in(\Gamma\cap N)\backslash\Gamma} \varphi(\gamma g) \;=\; \frac{1}{2\pi i}\sum_{\gamma\in(\Gamma\cap N)\backslash\Gamma}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)\cdot\chi_s\big(a(\gamma g)\big)\, ds$$

Taking $\sigma = 0$ would be natural, but, with $\sigma = 0$, the double integral (sum and integral) is not absolutely convergent, and the two integrals cannot be interchanged. Subsequently, we will see that the best line is $\sigma = 1/2$, but this is not in the region of convergence. For $\sigma > 1$, elementary estimates show that the double integral *is* absolutely convergent, and (by Fubini's theorem) the two integrals can be interchanged:

$$\Psi_\varphi(g) \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s) \sum_{\gamma\in(\Gamma\cap N)\backslash\Gamma} \chi_s\big(a(\gamma g)\big)\, ds \qquad\qquad (\text{with } \sigma > 1)$$

The inner sum defines the (spherical) *Eisenstein series*

$$E_s(g) \;=\; \sum_{\gamma\in(\Gamma\cap P)\backslash\Gamma} \chi_s\big(a(\gamma g)\big) \;=\; \sum_{\gamma\in(\Gamma\cap P)\backslash\Gamma} \mathrm{Im}\,(\gamma z)^s$$

[4] This is the easy half of the Paley-Wiener theorem.

where we let Γ act as usual on the upper half-plane, and use

$$\mathrm{Im}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) \;=\; \frac{y}{|cz+d|^2} \qquad\qquad (\text{with } z = x+iy)$$

That is,

$$\Psi_\varphi(g) \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)\cdot E_s(g)\,ds \qquad\qquad (\sigma > 1)$$

[2.2.2] Remark: The decomposition should refer to $\mathcal{M}c_P(\Psi_\varphi)$, not $\mathcal{M}\varphi$, to have whatever integral formulas expressed in terms of the automorphic forms Ψ_φ themselves, not in terms of the auxiliary functions φ from which they're made.

To accomplish this, several features of the Eisenstein series E_s are necessary. We recall these before proceeding.

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3. *Recollection of facts about Eisenstein series*

We review basic features of the spherical Eisenstein series for $\Gamma = SL_2(\mathbb{Z})$. In the Iwasawa decomposition $G = N\cdot A^+\cdot K$, use coordinates on A^+

$$a_y \;=\; \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{1/y} \end{pmatrix}$$

Let $g \to a(g)$ be the A^+-component of $g\in G$. The characters

$$\chi_s \;:\; a_y \;\longrightarrow\; y^s \qquad\qquad (\text{for } s\in\mathbb{C})$$

exhaust the continuous group homomorphisms $A^+ \to \mathbb{C}^\times$.

[3.1] Adjunction

Consider only right K-invariant automorphic forms f. Because $NA^+\cap K = \{1\}$ and $A^+\cap\Gamma = \{1\}$, we can identify $(P\cap\Gamma)N\backslash G/K \approx A^+$, and use the y-coordinate on A^+.

The Eisenstein series E_s should be a function on $\Gamma\backslash\mathfrak{H}$ fitting into the adjunction

$$\langle E_s, f\rangle_{\Gamma\backslash\mathfrak{H}} \;=\; \langle y^s, c_P f\rangle_{(P\cap\Gamma)N\backslash\mathfrak{H}} \qquad\qquad (\text{for } f \text{ on } \Gamma\backslash\mathfrak{H})$$

whenever the implied integrals converge. The Eisenstein series is *determined* by this relation: with the measure of K normalized to 1,

$$\langle y^s, c_P f\rangle_{A^+} \;=\; \int_{(P\cap\Gamma)N\backslash G/K} c_P f(g)\cdot\chi_s(a(g))\,dg \;=\; \int_{(P\cap\Gamma)N\backslash G} c_P f(g)\cdot\chi_s(a(g))\,dg$$

$$=\; \int_{(P\cap\Gamma)N\backslash G}\Big(\int_{N\cap\Gamma\backslash N} f(ng)\,dn\Big)\cdot\chi_s(a(g))\,dg$$

$$=\; \int_{(P\cap\Gamma)N\backslash G}\Big(\int_{N\cap\Gamma\backslash N} f(ng)\cdot\chi_s(a(ng))\,dn\Big)\,dg \;=\; \int_{P\cap\Gamma\backslash G} f(g)\cdot\chi_s(a(g))\,dg$$

At this point, the notation becomes more transparent in upper half-plane notation. Identifying $G/K\approx\mathfrak{H}$ since everything is right K-invariant, and using $\chi_s(a(na_yk)) = \chi_s(a_y) = y^s$,

$$\langle y^s, c_P f\rangle_{A^+} \;=\; \int_{P\cap\Gamma\backslash G} f(g)\cdot\chi_s(a(g))\,dg \;=\; \int_{P\cap\Gamma\backslash\mathfrak{H}} f(z)\cdot y^s\,\frac{dx\,dy}{y^2}$$

Now wind this up:

$$\int_{P\cap\Gamma\backslash\mathfrak{H}} f(z)\cdot y^s\,\frac{dx\,dy}{y^2} \;=\; \int_{\Gamma\backslash\mathfrak{H}}\sum_{\gamma\in P\cap\Gamma\backslash\Gamma} f(\gamma z)\cdot \mathrm{Im}\,(\gamma z)^s\,\frac{dx\,dy}{y^2} \;=\; \int_{\Gamma\backslash\mathfrak{H}} f(z)\sum_{\gamma\in P\cap\Gamma\backslash\Gamma}\mathrm{Im}\,(\gamma z)^s\,\frac{dx\,dy}{y^2}$$

Thus, the adjunction gives

$$E_s(z) \;=\; \sum_{\gamma\in P\cap\Gamma\backslash\Gamma}\mathrm{Im}\,(\gamma z)^s \;=\; \sum_{\gamma\in P\cap\Gamma\backslash\Gamma}\chi_s\big(a(\gamma g)\big)$$

However, this series converges only for $\mathrm{Re}\,(s)>1$, and meromorphic continuation is necessary.

[3.2] Eisenstein series and Mellin transforms

Via the adjunction characterizing E_s, integrals against Eisenstein series are Mellin transforms of constant terms:

$$\langle E_s, f\rangle_{\Gamma\backslash\mathfrak{H}} \;=\; \int_0^\infty c_Pf(iy)\,y^s\,\frac{dy}{y^2} \;=\; \int_0^\infty c_Pf(iy)\,y^{-(1-s)}\,\frac{dy}{y} \;=\; \mathcal{M}(c_Pf)\big(1-s\big)$$

[3.3] Meromorphic continuation and functional equation

With the usual G-invariant Laplacian

$$\Delta \;=\; y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

on $\mathfrak{H}=G/K$, from

$$\Delta y^s \;=\; s(s-1)\cdot y^s$$

in the region of convergence

$$\Delta E_s \;=\; s(s-1)\cdot E_s$$

Since Δ commutes with the map $f\to c_Pf$, we see that c_PE_s is a function $u(y)$ of y satisfying the Eulerian equation

$$y^2\frac{\partial^2}{\partial\,y^2}\,u(y) \;=\; s(s-1)\,u(y)$$

For $s\neq 1/2$ this has the two linearly independent elementary solutions y^s and y^{1-s}, so, for some meromorphic functions a_s and c_s,

$$c_PE_s \;=\; a_s\,y^s + c_s\,y^{1-s}$$

In fact, a direct computation shows $a_s=1$. That is,

$$c_P\,E_s \;=\; y^s + c_s\,y^{1-s}$$

We grant that the Eisenstein series E_s has a *meromorphic continuation.* In particular, the modified function

$$\widetilde{E}_s \;=\; s(1-s)\pi^{-s}\,\Gamma(s)\,\zeta(2s)\cdot E_s$$

is *entire* and has the *functional equation*

$$\widetilde{E}_s \;=\; \widetilde{E}_{1-s}$$

Another characterization of E_s is as the unique solution to the equations

$$\Delta w \;=\; s(s-1)\cdot w \qquad\qquad \left(y\frac{\partial}{\partial y}-(1-s)\right)\,w \;=\; (2s-1)\cdot y^s \qquad\qquad (\text{on } \Gamma\backslash\mathfrak{H})$$

This characterization yields the *functional equation*

$$E_{1-s} \;=\; c_{1-s}\, E_s$$

as follows. One can check directly that E_{1-s}/c_{1-s} satisfies those equations as well, so by *uniqueness*

$$\frac{E_{1-s}}{c_{1-s}} \;=\; E_s$$

Applying the functional equation twice gives

$$c_s c_{1-s} \;=\; 1$$

Since $E_{\bar{s}} = \overline{E_s}$, also $\overline{c_s} = c_{\bar{s}}$ and

$$|c_{\frac{1}{2}+it}|^2 \;=\; 1 \qquad\qquad (\text{for } t \in \mathbb{R})$$

In particular, c_s does not vanish on the line $\mathrm{Re}(s) = \frac{1}{2}$.

[3.3.1] Remark: Just below we will see that, from the expression of pseudo-Eisenstein series in terms of Eisenstein series, poles of E_s sufficiently far to the right play a role in the decomposition of $L^2(\Gamma\backslash G)$.

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4. *Decomposition of pseudo-Eisenstein series: conclusion*

So far, we have a spectral decomposition of Ψ_φ in terms of φ:

$$\Psi_\varphi(g) \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)\cdot E_s(g)\,ds \qquad\qquad (\sigma > 1)$$

We want to rewrite this to refer only to Ψ_φ, not φ.

Granting the meromorphic continuation of the Eisenstein series, move the vertical line of integration to the left, say to the line $\sigma = 1/2$ stabilized by the functional equation of E_s:

$$\Psi_\varphi(g) \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s)\, E_s(g)\,ds \;+\; \sum_{s_o} \mathrm{res}_{s=s_o}(E_s\cdot\mathcal{M}\varphi(s))$$

The $1/2\pi i$ from the inversion formula cancels the $2\pi i$ in the residue formula.

[4.0.1] Remark: We also need to know that E_s has no pole on the line $\mathrm{Re}(s) = 1/2$.

To rewrite this in terms of Ψ_φ, on one hand, from above,

$$\langle E_s, \Psi_\varphi\rangle_{\Gamma\backslash H} \;=\; \mathcal{M}(c_P\Psi_\varphi)(1-s)$$

On the other hand, by the adjunction/unwinding property of Ψ_φ,

$$\langle E_s, \Psi_\varphi\rangle \;=\; \langle c_P E_s, \varphi\rangle_{(P\cap\Gamma)N\backslash\mathfrak{H}}$$

This is

$$\langle c_P E_s, \varphi\rangle_{(P\cap\Gamma)N\backslash\mathfrak{H}} \;=\; \langle y^s + c_s y^{1-s}, \varphi\rangle_{(P\cap\Gamma)N\backslash\mathfrak{H}} \;=\; \int_0^\infty (y^s + c_s y^{1-s})\cdot\varphi(y)\,\frac{dy}{y^2}$$

$$=\; \int_0^\infty (y^{-(1-s)} + c_s y^{-s})\cdot\varphi(y)\,\frac{dy}{y} \;=\; \mathcal{M}\varphi(1-s) + c_s\mathcal{M}\varphi(s)$$

Thus,

$$\mathcal{M}(c_P\Psi_\varphi)(1-s) \;=\; \langle E_s, \Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}} \;=\; \mathcal{M}\varphi(1-s) + c_s\mathcal{M}\varphi(s)$$

Replacing s by $1-s$, this is

$$\mathcal{M}(c_P\Psi_\varphi)(s) \;=\; \langle E_{1-s}, \Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}} \;=\; \mathcal{M}\varphi(s) + c_{1-s}\mathcal{M}\varphi(1-s)$$

The right-hand side of the latter will appear just below.

The integral part of the expression of Ψ_φ in terms of Eisenstein series can be folded in half, integrating from $\frac{1}{2}+i0$ to $\frac{1}{2}+i\infty$ rather than from $\frac{1}{2}-i\infty$ to $\frac{1}{2}+i\infty$:

$$\Psi_\varphi - (\text{residual part}) \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s)\cdot E_s(g)\,ds \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s)\,E_s + \mathcal{M}\varphi(1-s)\,E_{1-s}\,ds$$

Using the functional equation $E_{1-s} = c_{1-s}E_s$, and recognizing $\mathcal{M}(c_P\Psi_\varphi)$ as expressed above in terms of $\mathcal{M}\varphi$, this becomes

$$\frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} (\mathcal{M}\varphi(s) + c_{1-s}\mathcal{M}\varphi(1-s))\cdot E_s\,ds \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}c_P\Psi_\varphi(s)\,E_s\,ds$$

by the expression above for the Mellin transform of the constant term of Ψ_φ. That is, a pseudo-Eisenstein series is expressible as an integral of Eisenstein series E_s on the line $\mathrm{Re}(s) = 1/2$, plus a sum of residues:

$$\Psi_\varphi - (\text{residual part}) \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}c_P\Psi_\varphi(s)\cdot E_s\,ds \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \langle E_{1-s}, \Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}}\cdot E_s\,ds$$

This is the decomposition of pseudo-Eisenstein series Ψ_φ in terms of Eisenstein series E_s. ///

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5. *Residue of Eisenstein series: the constant function*

For $\Gamma = SL_2(\mathbb{Z})$, we know by various means that there is a single pole of E_s in the half-plane $\mathrm{Re}(s) \geq 1/2$, at $s = 1$, and it is simple. Indeed, we know that the residue is a *constant* function. Thus,

$$\Psi_\varphi \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \langle E_{1-s}, \Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}}\cdot E_s\,ds \;+\; \mathcal{M}\varphi(1)\cdot \mathrm{res}_{s=1}E_s$$

The coefficient $\mathcal{M}\varphi(1)$ is

$$\mathcal{M}\varphi(1) \;=\; \int_0^{+\infty} \varphi(a_y)\,y^{-1}\,\frac{dy}{y} \;=\; \int_0^{+\infty} \varphi(a_y)\,\frac{dy}{y^2} \;=\; \int_{N\backslash G} \varphi(g)\,dg$$

Rearranging,

$$\mathcal{M}\varphi(1) \;=\; \int_{N\backslash G} \varphi(g)\,dg \;=\; \int_{N\backslash G}\int_{N\cap\Gamma\backslash N} \varphi(ng)\,dn\,dg$$

$$= \int_{N\backslash G} \varphi(ng)\Big(\int_{N\cap\Gamma\backslash N} 1\,dn\Big)\,dg \;=\; \int_{N\cap\Gamma\backslash G} \varphi(g)\,dg$$

since the natural volume of $(N\cap\Gamma)\backslash N$ is 1 and φ is left N-invariant. Winding up,

$$\mathcal{M}\varphi(1) \;=\; \int_{\Gamma\backslash G} \sum_{\gamma\in N\cap\Gamma\backslash\Gamma} \varphi(g)\,dg \;=\; \int_{\Gamma\backslash G} \Psi_\varphi(g)\,dg \;=\; \langle 1, \Psi_\varphi\rangle$$

That is, $\mathcal{M}\varphi(1)$ is the inner product of Ψ_φ with the constant function 1. Constants *are* square-integrable, since the measure of $\Gamma\backslash G$ is finite, but the sense of $\langle 1, \Psi_\varphi\rangle$ does not depend upon this.

[5.0.1] Remark: The condition $\mathcal{M}\varphi(1) \neq 0$ for $\langle 1, \Psi_\varphi\rangle \neq 0$ is easily met, since $\mathcal{M}\varphi$ is entire and of rapid decay on vertical lines: if $\mathcal{M}\varphi(1) \neq 0$, then for some $0 < \ell \in \mathbb{Z}$ $(s-1)^{-\ell}\mathcal{M}\varphi(s)$ is finite and non-zero at $s = 1$. The integration by parts identity

$$\mathcal{M}\varphi(s-1) \;=\; \int_0^\infty y^{-(s-1)}\,\frac{\varphi(y)}{y}\,\frac{dy}{y} \;=\; \frac{-1}{s-1}\int_0^\infty y\frac{\partial}{\partial y}y^{-(s-1)}\cdot\frac{\varphi(y)}{y}\,\frac{dy}{y}$$

$$=\; \frac{1}{s-1}\int_0^\infty y^{-(s-1)}\cdot y\frac{\partial}{\partial y}\Big(\frac{\varphi(y)}{y}\Big)\,\frac{dy}{y}$$

Thus, the needed division by powers of $s-1$ can be achieved by taking suitable derivatives of the data entering the pseudo-Eisenstein series. In particular, constant functions are *not* orthogonal to pseudo-Eisenstein series.

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6. *Toward a Plancherel theorem*

For $\varphi \in C_c^\infty(N\backslash G)$, the pseudo-Eisenstein series Ψ_φ is expressed in terms of Eisenstein series, with complex-*bilinear* pairing $\langle , \rangle$,

$$\Psi_\varphi \;=\; \langle\frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty}\langle E_{1-s}, \Psi_\varphi\rangle\,E_s\,ds + \langle 1, \Psi_\varphi\rangle\,\mathrm{res}_{s=1}E_s$$

Then for $f \in C_c^\infty(\Gamma\backslash G)$, still with complex-bilinear pairing,

$$\langle\Psi_\varphi, f\rangle \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty}\langle E_{1-s}, \Psi_\varphi\rangle\,\langle E_s, f\rangle\,ds \;+\; \langle 1, \Psi_\varphi\rangle\,\langle\mathrm{res}_{s=1}E_s,\, f\rangle$$

Replacing f by $\overline{\Psi}_\psi$ for another $\psi \in C_c^\infty((P\cap\Gamma)N\backslash\mathfrak{H})$, using $\overline{E_s} = E_{1-s}$, and noting that the residue of the Eisenstein series is constant, this gives part of a **Plancherel theorem** on the $L^2(\Gamma\backslash\mathfrak{H})$-closure of the space of pseudo-Eisenstein series:

[6.0.1] Theorem:

$$\langle\Psi_\varphi, \overline{\Psi}_\psi\rangle \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty}\langle\Psi_\varphi, \overline{E_s}\rangle\,\langle E_s, \overline{\Psi}_\psi\rangle\,ds \;+\; \langle\Psi_\varphi, 1\rangle\,\langle 1, \overline{\Psi}_\psi\rangle\cdot\mathrm{res}_{s=1}E_s$$

[6.0.2] Remark: What is missing in this description is characterization of the functions $s \to \langle\Psi_\varphi, E_{1-s}\rangle$, or of the L^2 closure of this image.

Bibliography

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