

(March 18, 2011)

[0.0.1] Claim: On a *simple* complex Lie algebra \mathfrak{g} , up to scalars there is at most one \mathfrak{g} -invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle$.

Proof: A non-degenerate \mathfrak{g} -equivariant bilinear form B gives an \mathfrak{g} -isomorphism φ_B of \mathfrak{g} (under $\text{ad } \mathfrak{g}$) to its dual \mathfrak{g}^* (under the contragredient $\text{ad}^* \mathfrak{g}$), and vice-versa, via

$$\varphi_B(x)(y) = \langle x, y \rangle$$

The equivariance property is readily verified:

$$\varphi_B(\text{ad } z(x))(y) = B([z, x], y) = B(x, -[z, y]) = \varphi_B(x)(\text{ad } (-z)(y)) = \left((\text{ad}^* z)(\varphi_B(x)) \right)(y)$$

The non-degeneracy of B is equivalent to φ_B having trivial kernel, so by finite-dimensionality equivalent to its being an isomorphism.

Thus, given two \mathfrak{g} -invariant non-degenerate bilinear forms, B, C , the composite $\varphi = \varphi_B^{-1} \circ \varphi_C$ is a \mathfrak{g} -isomorphism $\mathfrak{g} \rightarrow \mathfrak{g}$.

That \mathfrak{g} is *simple* is equivalent to the *irreducibility* of the \mathfrak{g} -module \mathfrak{g} (under ad).

By Schur's lemma for finite-dimensional \mathfrak{g} -representations over an algebraically closed field, the only \mathfrak{g} -endomorphisms of \mathfrak{g} are scalars. Thus, φ is a scalar, so φ_B and φ_C differ by a scalar. From this, $B(x, y)$ and $C(x, y)$ differ by a uniform scalar. ///

[0.0.2] Remark: \mathfrak{g} -invariance of a bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{g} is

$$\langle [z, x], y \rangle + \langle x, [z, y] \rangle = 0 \quad (\text{for all } x, y, z \in \mathfrak{g})$$

[0.0.3] Remark: Complete reducibility of finite-dimensional \mathfrak{g} -modules is not used in the proof.

[0.0.4] Remark: The sign in

$$\varphi_B(\text{ad } z(x))(y) = \varphi_B(x)(\text{ad } (-z)(y))$$

is a special case: for *any* \mathfrak{g} -representation π on vector space V , the contragredient π^* on V^* is

$$(\pi^* z)(\lambda)(y) = \lambda(\pi(-z)(y)) \quad (\text{for } z \in \mathfrak{g}, \lambda \in V^*, y \in V)$$

That the involutive anti-automorphism $z \rightarrow -z$ appear in the contragredient is understandable and desirable for at least two reasons. For compatibility with formation of contragredients of *group* representations, appearance of this involution is necessary. In a different vein, for the contragredient to be in the same category of *left* \mathfrak{g} -modules, *some* involutory anti-automorphism is needed, to convert a *right* \mathfrak{g} -module to a *left* \mathfrak{g} -module. This is completely parallel to the fact that the construction of *contragredient* M^* of a *left* modules M over a *non-commutative* ring R is naturally a *right* R -module, by

$$(\lambda \cdot r)(m) = \lambda(r \cdot m) \quad (\text{for } \lambda \in M^*, r \in R, m \in M)$$