Examples discussion 04

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04.1 Comparing $L^p$ spaces. Let $1 \leq p, p' < \infty$. When is $L^p[a, b] \subset L^{p'}[a, b]$ for finite intervals $[a, b]$ and Lebesgue measure? When is $L^p(\mathbb{R}) \subset L^{p'}(\mathbb{R})$? When is $\ell^p \subset \ell^{p'}$?

**Discussion:** Take $p < p'$. We claim that $L^p[a, b] \supset L^{p'}[a, b]$, with proper containment. The function $f$ that is $(x - a)^{-\frac{1}{p}}$ on $(a, b]$ and 0 off that interval is not in $L^{p'}$, but is in $L^p$. Given $f \in L^p[a, b]$, let $E$ be the set of $x \in [a, b]$ where $|f(x)| \geq 1$. Then $\int_a^b |f|^{p'} < \infty$ if and only if $\int_E |f|^{p'} < \infty$. On $E$, $|f|^p < |f|^{p'}$, so $\int_E |f|^p < \infty$, and then also $\int_a^b |f|^p < \infty$, so $f \in \ell^p$. //

We claim that $L^p(\mathbb{R})$ and $L^{p'}(\mathbb{R})$ are not comparable for $p \neq p'$. Take $1 \leq p < p'$. On one hand, $1/(1 + |x|)^{1/p' + \varepsilon}$ is in $L^p$ for all $\varepsilon > 0$, but not in $L^p$ for $\varepsilon$ small enough so that $\frac{1}{p} + \varepsilon < \frac{1}{p'}$. On the other hand, the function $f$ that is $-\frac{1}{p'}$ on $(0, 1]$ and 0 off that interval is not in $L^{p'}$, but is in $L^p$.

We claim that for $1 \leq p < p' < \infty$, $\ell^p \subset \ell^{p'}$, with strict containment. Indeed, $f \in \ell^{p'}$ given by $f(n) = 1/n^{p'}$ is not in $\ell^p$, but is in $\ell^p$. Let $E = \{n \in \{1, 2, \ldots\} : |f(n)| < 1\}$. Then $f \in \ell^p$ if and only if the complement of $E$ is finite, and if $\sum_{n \in E} |f(n)|^p < \infty$. Certainly $|f(n)|^p > |f(n)|^{p'}$ for $n \in E$, and the complement of $E$ is finite, so $\sum_{n \in E} |f(n)|^{p'} < \sum_{n \in E} |f(n)|^p$, and $f \in \ell^{p'}$. //

04.2 For positive real numbers $w_1, \ldots, w_n$ such that $\sum_i w_i = 1$, and for positive real numbers $a_1, \ldots, a_n$, show that

$$a_1^{w_1} \cdots a_n^{w_n} \leq w_1 a_1 + \cdots + w_n a_n$$

**Discussion:** This is a corollary of Jensen’s inequality, similar to the arithmetic-geometric mean, but with unequal weights. Namely, let $X = \{1, 2, \ldots, n\}$ with measure $\mu(i) = w_i$, and function $f(i) = \log a_i$. Then Jensen’s inequality is

$$\exp \left( \sum_{i=1}^n w_i \cdot \log a_i \right) = \sum_{i=1}^n w_i \cdot e^{\log a_i}$$

which simplifies to the assertion. //

04.3 In $\ell^2$, show that the point in the closed unit ball closest to a point $v$ not inside that ball is $v/|v|_{\ell^2}$.

**Discussion:** The minimum principle assures that there is a unique closest point $w$ in the closed unit ball $B$ to $v$, because $B$ is convex, closed, non-empty, and $v$ is not in $B$.

Suppose $w$ is closer than $v/|v|$. Then

$$|v|^2 - 2\frac{v}{|v|} + 1 = |v - \frac{v}{|v|}|^2 > |v - w|^2 = |v|^2 - \langle v, w \rangle - \langle w, v \rangle + |w|^2 = |v|^2 - \langle v, w \rangle - \langle w, v \rangle + 1$$

Thus,

$$2|v| < \langle v, w \rangle + \langle w, v \rangle$$

Thus, the sum of the two inner products is positive, and by Cauchy-Schwarz-Bunyakowsky:

$$2|v| < \langle v, w \rangle + \langle w, v \rangle = |\langle v, w \rangle + \langle w, v \rangle| \leq 2|v| \cdot |w|$$

Thus, $1 < |w|$, which is impossible. //
[04.4] For a measurable set $E \subset [0,2\pi]$, show that

$$\lim_{n \to \infty} \int_E \cos nx \, dx = 0 = \lim_{n \to \infty} \int_E \sin nx \, dx$$

**Discussion:** This is an instance of a *Riemann-Lebesgue lemma*, namely, that Fourier coefficients of an $L^2$ function on $[0,2\pi]$ go to 0. Here, the $L^2$ function is the characteristic function of $E$, and we use sines and cosines instead of exponentials.

[04.5] One form of the *sawtooth* function is $f(x) = x - \pi$ on $[0,2\pi]$. Compute the Fourier coefficients $\hat{f}(n)$.

**Discussion:** We have the orthonormal basis $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ with $n \in \mathbb{Z}$ for the Hilbert space $L^2[0,2\pi]$. The Fourier coefficients are determined by Fourier’s formula

$$\hat{f}(n) = \int_0^{2\pi} f(x) \frac{e^{-inx}}{\sqrt{2\pi}} \, dx$$

For $n = 0$, this is 0. For $n \neq 0$, integrate by parts, to get

$$\hat{f}(n) = \left[ f(x) \cdot \frac{e^{-inx}}{\sqrt{2\pi} \cdot (-in)} \right]_0^{2\pi} - \int_0^{2\pi} \frac{e^{-inx}}{\sqrt{2\pi} \cdot (-in)} \, dx$$

$$= \left( (\pi \cdot \frac{1}{\sqrt{2\pi} \cdot (-in)}) - ( -\pi \cdot \frac{1}{\sqrt{2\pi} \cdot (-in)} ) \right) - 0 = \frac{2\pi}{\sqrt{2\pi} \cdot (-in)} = \frac{\sqrt{2\pi}}{-in}$$

The $L^2$ norm of $f$ is

$$\int_0^{2\pi} (x - \pi)^2 \, dx = \left[ \frac{(x - \pi)^3}{3} \right]_0^{2\pi} = \frac{\pi^3 - (-\pi)^3}{3} = \frac{2\pi^3}{3}$$

Thus, by Parseval,

$$\sum_{n \neq 0} \left| \frac{\sqrt{2\pi}}{-in} \right|^2 = \frac{2\pi^3}{3}$$

This simplifies first to

$$2 \sum_{n \geq 1} \frac{2\pi}{n^2} = \frac{2\pi^3}{3}$$

and then to

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

That is, Parseval applied to the sawtooth function evaluates $\zeta(2)$.  

[04.6] For fixed $y \in [0,2\pi]$, show that there is no $f_y \in L^2[0,2\pi]$ so that $\langle g, f_y \rangle = g(y)$ for all $g \in L^2[0,2\pi]$.

**Discussion:** Part of the issue here is whether $L^2$ functions truly have meaningful pointwise values at all, and we generally imagine that they do not, although such a negative fact may be hard to express formulaically. Among many approaches, one is to suppose such $f$ exists. Choose an orthonormal basis for $L^2[0,2\pi]$ consisting of the continuous functions $\psi_n(x) = e^{2\pi i nx}$, and see what the condition $\langle \psi_n, f_y \rangle = \psi_n(y)$ imposes
on the alleged $f_y$. Indeed, this condition completely determines the (complex conjugates of the) Fourier coefficients of the alleged $f_y$, so

$$f_y = \sum_{n \in \mathbb{Z}} \overline{\psi}_n(y) \cdot \psi_n$$

(with equality in an $L^2$ sense)

By Parseval,

$$|f_y|_{L^2}^2 = \sum_n |\overline{\psi}_n(y)|^2 = +\infty$$

since $|\psi_n(y)| = 1$ for all $n$. Thus, there can be no such $f_y$ in $L^2$.

[04.7] (In contrast to the previous example’s outcome.) Let $V$ be the complex vector space of power series $f(z) = \sum_{n \geq 0} c_n z^n$ convergent on the open unit disk $D$ in $\mathbb{C}$, having finite norm

$$|f| = \left( \int_D |f(x + iy)|^2 \, dx \, dy \right)^\frac{1}{2}$$

with hermitian inner product

$$\langle f, g \rangle = \int_D f(x + iy) \cdot \overline{g(x + iy)} \, dx \, dy$$

Show that $\langle z^m, z^n \rangle = 0$ unless $m = n$, in which case it is $\frac{\pi}{n+1}$, and that $\psi_n(z) = z^n \cdot \sqrt{\frac{n+1}{\pi}}$ is an orthonormal basis for $V$. Show that the sum $f_w(z) = \sum_{n \geq 0} \psi_n(z) \overline{\psi}_n(w)$ converges absolutely for $z, w \in D$, and that

$$\langle g(-), f_w \rangle = g(w)$$

(for $w$ in the disk)

Show that for each fixed $w \in D$, pointwise evaluation $g \rightarrow g(w)$ is a continuous linear functional on $V$.

**Discussion:** Changing to polar coordinates $z = r \cdot e^{i\theta}$,

$$\int_D z^m \cdot \pi^n \, dx \, dy = \int_0^{2\pi} e^{i(m-n)\theta} \, d\theta \cdot \int_0^1 r^{m+n+1} \, dr$$

The integral over $\theta$ is 0 unless $m = n$, in which case it is $2\pi$, and the whole becomes

$$2\pi \cdot \int_0^1 r^{2n+1} \, dr = \frac{2\pi}{2n+2} = \frac{\pi}{n+1}$$

Thus, certainly $\{1, z, z^2, \ldots\}$ is an orthogonal basis, and the corresponding orthonormal basis consists of $e_n = \sqrt{\frac{n+1}{\pi}} \cdot z^n$. To make a power series $f_w$ in $z$ such that $\langle F, f_w \rangle = F(w)$, certainly this property must hold for every monomial $F(z) = z^n$. That is,

$$\sqrt{\frac{n+1}{\pi}} \cdot w^n = e_n(w) = (e_n, f_w)$$

That is, these are the (complex conjugates of the) abstract Fourier coefficients of the alleged $f_w$ in terms of that orthonormal basis. Thus, if $f_w$ exists, it must be

$$f_w = \sum_{n \geq 0} \langle f_w, e_n \rangle \cdot e_n = \sum_{n \geq 0} \sqrt{\frac{n+1}{\pi}} \cdot \overline{e_n} = \sum_{n \geq 0} \frac{n+1}{\pi} \cdot z^n \overline{w} = \frac{1}{\pi} \int_{\mathbb{C}} z^{n+1} \overline{w} \, dz = \frac{1}{\pi} \int_{\mathbb{C}} z \overline{w} \, dz$$

since the power series does converge absolutely on the open unit disk, and power series can be differentiated termwise in their open disk of convergence. This gives

$$f_w(z) = \frac{1}{\pi} \left( \frac{1}{1 - z \overline{w}} + \frac{z \cdot \overline{w}}{(1 - z \overline{w})^2} \right) = \frac{1}{\pi} \cdot \left( \frac{1 - z \overline{w}}{(1 - z \overline{w})^2} + z \cdot \overline{w} \right) = \frac{1}{\pi} \cdot \frac{1}{(1 - z \overline{w})^2}$$