Examples discussion 05

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[This document is http://www.math.umn.edu/~garrett/m/real/examples

05.1 Show that every vector subspace of $\mathbb{R}^n$ and/or $\mathbb{C}^n$ is (topologically) closed.

Discussion: Let $v_1, \ldots, v_m$ be an orthonormal basis for the given vector subspace $W$. For a Cauchy sequence $(w_n)$ in $W$, we claim that for each $j$ the sequence $(w_n, v_j)$ is Cauchy: by Cauchy-Schwarz-Bunyakovsky,

$$|\langle w_n, v_j \rangle - \langle w_{n'}, v_j \rangle| = |\langle w_n - w_{n'}, v_j \rangle| \leq |w_n - w_{n'}| \cdot |v_j| = |w_n - w_{n'}|$$

Thus, by completeness of $\mathbb{R}$ and/or $\mathbb{C}$, that sequence has a limit $c_j$. As expected, we claim that

$$\lim_n w_n = \sum_{j=1}^{m} c_j \cdot v_j.$$ 

Indeed, using the orthonormality of the $v_j$'s,

$$\left| w_n - \sum_{j=1}^{m} c_j \cdot v_j \right|^2 = \left| w_n - \sum_{j=1}^{m} \lim_i \langle w_i, v_j \rangle \cdot v_j \right|^2 = \sum_{j=1}^{m} \left| \lim_i \langle w_n - w_i, v_j \rangle \cdot v_j \right|^2$$

$$\leq \sum_{j=1}^{m} \left| \lim_i \langle w_n - w_i, v_j \rangle \right|^2 = \lim_i \sum_{j=1}^{m} \left| \langle w_n - w_i, v_j \rangle \right|^2 \leq \lim_i \sum_{j=1}^{m} |w_n - w_i| \cdot |v_j| = \lim m \cdot |w_n - w_i|$$

Take $n_o$ large enough so that $|w_n - w_i| < \varepsilon$ for $i, n \geq n_o$. Then the latter expression is at most $m \cdot \varepsilon$. This holds for all $\varepsilon > 0$, so the limit is 0.

05.2 For a subspace $W$ of a Hilbert space $V$, show that $(W^\perp)^\perp$ is the closure of the subspace $W$ in $V$.

Discussion: Let $\lambda_x(v) = \langle v, x \rangle$ for $x, v \in V$. Then $W^\perp = \bigcap_{w \in W} \ker \lambda_w$. Similarly, $(W^\perp)^\perp = \bigcap_{x \in W^\perp} \ker \lambda_x$. From the discussion in the Riesz-Fréchet theorem, or directly via Cauchy-Schwarz-Bunyakovsky, each $\lambda_x$ is continuous, so $\ker \lambda_x = \lambda_x^{-1}(\{0\})$ is closed, since $\{0\}$ is closed. (One might check that the kernel of a linear map is a vector subspace.) An arbitrary intersection of closed sets is closed, so $(W^\perp)^\perp$ is closed.

Certainly $(W^\perp)^\perp \supset W$, because for each $w \in W$, $\langle x, w \rangle = 0$ for all $x \in W^\perp$. Thus, $(W^\perp)^\perp$ is a closed subspace, containing $W$. Being a closed subspace of a Hilbert space, $(W^\perp)^\perp$ is a Hilbert space itself. If $(W^\perp)^\perp$ were strictly larger than the topological closure $\overline{W}$ of $W$, then there would be $0 \neq y \in (W^\perp)^\perp$ orthogonal to $\overline{W}$. Then $y$ would be orthogonal to $W$ itself, so $0 \neq y \in W^\perp$, contradicting $0 \neq y \in (W^\perp)^\perp$.

05.3 Let $T : \ell^2 \to \ell^2$ be the right shift: $T(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, z_3, \ldots)$. Determine the adjoint $T^\ast$.

Discussion: The adjoint characterization is $\langle Tv, w \rangle = \langle v, T^\ast w \rangle$. That means that, for $(w_1, w_2, \ldots)$ in $\ell^2$, we want

$$\langle (z_1, z_2, \ldots), T^\ast (w_1, w_2, \ldots) \rangle = \langle T(z_1, z_2, \ldots), (w_1, w_2, \ldots) \rangle = \langle (0, z_1, z_2, \ldots), (w_1, w_2, \ldots) \rangle$$

$$= z_1w_2 + z_2w_3 + z_3w_4 + \ldots = \langle (z_1, z_2, \ldots), (w_2, w_3, \ldots) \rangle$$

Thus, we see that $T^\ast(w_1, w_2, w_3, \ldots) = (w_2, w_3, \ldots)$. That is, it is the left shift (yes, that loses the $w_1$-coordinate.

05.4 Show that for $0 < x < 1$

$$\sum_{n \geq 1} \frac{\sin 2\pi nx}{n} = \pi(\frac{1}{2} - x)$$
Proof: The non-negative real-valued-ness is of course immediate. The integral of is an approximate identity.

Let \[ \text{Claim:} \]

an approximate identity. More generally, we prove terms of approximate identities. That is, with \( \phi \) rather than reproving this general assertion in the example at hand, we simply clarify the interpretation in \( x \) and \( \xi \).

By replacing \( \xi \) by \( x/n \) in the integral, we have

Rather than reproving this general assertion in the example at hand, we simply clarify the interpretation in terms of approximate identities. That is, with \( \varphi_1(x) = e^{-\pi x^2} \), we that the sequence \( \varphi_n(x) = n \cdot \varphi_1(nx) \) is an approximate identity. More generally, we prove

\[ \text{Claim:} \] Let \( \varphi \in C^0(\mathbb{R}) \) be a non-negative \( \mathbb{R} \)-valued function, with \( \int \varphi = 1 \). Then \( \varphi_n(x) = n \cdot \varphi(nx) \) is an approximate identity.

Discussion: This is an instance of an approximate identity and the basic property of such. Namely, for an approximate identity \( \{ \varphi_n \} \) on \( \mathbb{R} \) and \( f \in C_c^0(\mathbb{R}) \), we have

By replacing \( \xi \) by \( \xi - x \) in the integral, we have

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