Examples 05

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2016-17/real-ex-05.pdf]

For feedback on these examples, please get your write-ups to me by Monday, 06 Nov 2016.

[05.1]  Show that every vector subspace of $\mathbb{R}^n$ and/or $\mathbb{C}^n$ is (topologically) closed.

[05.2]  For a subspace $W$ of a Hilbert space $V$, show that $(W^\perp)^\perp$ is the closure of the subspace $W$ in $V$.

[05.3]  Let $T : \ell^2 \to \ell^2$ be the right shift: $T(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, z_3, \ldots)$. Determine the adjoint $T^*$.

[05.4]  Show that for $0 < x < 1$

$$\sum_{n \geq 1} \frac{\sin 2\pi nx}{n} = \pi \left( \frac{1}{2} - x \right)$$

[05.5]  Let $\varphi_n(x) = n \cdot e^{-\pi(nx)^2}$. Prove that every $f \in C^\infty_c(\mathbb{R})$ can be uniformly approximated (in sup norm) arbitrarily well as superpositions of Gaussians: given $\varepsilon > 0$, there is $\varphi \in C^\infty_c(\mathbb{R})$ and sufficiently large $n$ such that

$$\sup_{x \in \mathbb{R}} \left| f(x) - \int_{\mathbb{R}} \varphi(\xi) \cdot e^{-\pi n^2 (\xi - x)^2} d\xi \right| < \varepsilon$$

[05.6]  Without worrying too much about identifying the finite, positive constant $\int_{\mathbb{R}} \frac{(\sin x)^2}{x^2} \, dx$, prove that, for given $f \in C^\infty_c(\mathbb{R})$, given $\varepsilon > 0$, there is sufficiently large $n$ and a function $\varphi \in C^\infty_c(\mathbb{R})$ such that

$$\sup_{x \in \mathbb{R}} \left| f(x) - \int_{\mathbb{R}} \varphi(\xi) \cdot \frac{(\sin n(x - \xi))^2}{(x - \xi)^2} \, d\xi \right| < \varepsilon$$